Hopf Bifurcations and the Center Manifold Theorem

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Abstract

The center manifold theorem is a powerful tool in dynamical systems that reduces the dimension of a system to allow easier calculations. Specifically, the stability of solutions is easily found by examining the system on an invariant manifold called the center manifold. A general introduction to the theory will comprise the first section. Later sections include the reduction of a Hopf Bifurcation to the center manifold and an accompanying example. It should be noted that no proofs will be given with the theorems mentioned. Please see Carr for proofs.

1 Introduction to the Theory[1]

1.1 Linear systems

Consider the following system of linear differential equations:

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n. \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$. The global behavior of equation (1) and the stability of the zero solution are completely determined by the eigenvalues of A. Let $\sigma(A)$ denote the spectrum of A. Then we have $\sigma(A) = \sigma_s(A) \cup \sigma_u(A) \cup \sigma_c(A)$, where

$$\begin{aligned}
\sigma_s(A) &= \{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda < 0\} \\
\sigma_u(A) &= \{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda > 0\} \\
\sigma_c(A) &= \{\lambda \in \sigma(A) \mid \operatorname{Re}\lambda = 0\}
\end{aligned}$$

Let the corresponding (generalized) eigenspaces be denoted E^s, E^u, and E^c, respectively. Its clear that orbits starting in E^s will decay exponentially, those from E^u will grow exponentially, and those from E^c neither decay or grow exponentially. In particular, this means that if $\sigma_u(A) = \emptyset$, then all orbits

[1] An introduction can be found in Carr, Kuznetsov, Wiggins, and many other good books.

originating in E^s will quickly decay to E^c. In order to determine the stability of the zero solution, the dynamics of equation (1) reduced to E^c need to be considered.

1.2 Nonlinear systems

Now suppose that we add nonlinear terms to Eq (1)

$$\dot{x} = A(\lambda)x + f(x, \lambda), \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^l. \tag{2}$$

where $f(x, \lambda) \in C^k$, $k \geq 2$, and $f(0,0) = Df(0,0) = 0$. Then $\mathbf{0}$ is still a solution to equation (2). We now must ask how the behavior of the nonlinear system (2) is related to that of the linear system (1). The answer is easy if the system if hyperbolic ($\sigma_c(A) = \emptyset$), the phase portraits are topologically equivalent by the Gröbman-Hartman theorem. The answer is not so simple if $\sigma_c(A) \neq \emptyset$. This is the topic of the center manifold (CM) theory.

In the non-hyperbolic case there exists invariant manifolds M^s, M^u and M^c analogous to the generalized eigenspaces (Warning: these manifolds are NOT necessarily linear subspaces). Proof of the existence of these manifolds will not be given here.

Since we are usually interested in the stability of the zero solution, it will be assumed that $\sigma_u(A) = \emptyset$. While this assumption is not necessary for the formulation of the center manifold theorem, $\sigma_u(A) \neq \emptyset$ guarantees that the zero solution is unstable. It can be useful to include this case since an unstable solution can undergo a secondary bifurcation and become stable.

We write $\mathbb{R}^n = E^c \oplus E^u$ and rewrite equation (2) as

$$\begin{aligned} \dot{x} &= Ax + f(x, y) \\ \dot{y} &= By + g(x, y) \end{aligned} \tag{3}$$

where $\sigma(A) = \sigma_c(A)$ and $\sigma(B) = \sigma_s(B)$. Note that we still have $f(0,0) = g(0,0) = 0$ and $Df(0,0) = Dg(0,0) = 0$. The re-use of the matrix A is clearly an abuse of notation, but its use should be clear in context. We assume $A \in \mathbb{R}^{c \times c}$ and $B \in \mathbb{R}^{s \times s}$ where $n = c + s$. We can now state the center manifold theorem.

Theorem 1 (Center Manifold Theorem)**.** *Given system (3) with the zero solution and $\sigma(A) = \sigma_c(A)$ and $\sigma(B) = \sigma_s(B)$, then there exists (locally) an invariant center manifold $M^c(0)$ that can be represented as*

$$M^c(0) = \{(x, y) \,|\, y = h(x), h(0) = Dh(0) = 0, |x| < \delta\}$$

for some sufficiently small δ. Moreover, $M^c(0) \in C^k$, same as f, g.

The notation $M^c(0)$ is used to emphasize the local nature of the center manifold. From now on it will simply be referred to as M^c.

A few notes on the CM theorem:

1. With $y = h(x)$, we can reduction of the dynamics of (2) to the CM is given by:
$$\dot{u} = Au + f(u, h(u)). \tag{4}$$
The use of the variable u is to emphasize the fact that the CM is in general not a linear subspace. Use of the variable x would not make this specific.

2. The condition that $h(0) = Dh(0) = 0$ imples that the CM is tangent to E^c at the origin.

3. The CM is C^k whenever $f, g \in C^k$ with the exception of some cases when $k = \infty$. This is due to the local nature of the CM. As $k \to \infty$, the neighborhood on which the CM is defined can shrink such that a C^∞ manifold does not exist.

4. The center manifold is NOT unique! However, in practice this non-uniqueness doesn't really pose a problem.

5. The CM has similar properties to that of E^c: M^c must contain all solutions contained in a small neighborhood of zero, including fixed points, small periodic solutions, homo- and hetero-clinic orbits.

Example

Consider the following system:
$$\dot{x} = x^2$$
$$\dot{y} = -y$$
Stable manifold: y-axis.
Center manifold: x-axis.

However, this is not the only center manifold – it is simply one CM of a one-parameter family of CMs. We can eliminate t from the system:
$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{-y}{x^2}$$
which has a solution
$$y = \alpha e^{1/x} \equiv h(x).$$
$h(x)$ is a one-parameter family of CMs (neither exponential growth nor decay). This example also exhibits the local nature of the CM as well: $\dot{x} = x^2$ grows to infinity in finite time, so the CM cannot be thought of as the trajectories that make it up.

This example should bring two questions immediately to mind:

1. *How is the CM calculated in general?*

2. *If the CM is not unique, what is the CM calculation actually calculating?*

We answer the first question by deriving a quasi-linear PDE that the center manifold must satisfy.

Assume that $y = h(x)$ is an invariant center manifold of system (2). Take the derivative of both sides:

$$\dot{y} = Dh(x)\dot{x}. \tag{5}$$

Now use $y = h(x)$ in equation (3):

$$\begin{array}{l} \dot{x} = Ax + f(x,y) \\ \dot{y} = By + g(x,y) \end{array} \Rightarrow \begin{array}{l} \dot{x} = Ax + f(x,h(x)) \\ \dot{y} = Bh(x) + g(x,h(x)) \end{array} \tag{6}$$

Substituting (6) into equation (5), we find that since $h(x)$ is an invariant manifold, it must satisfy the following:

$$Bh(x) + g(x,h(x)) = Dh(x)[Ax + f(x,h(x))] \tag{7}$$

We rewrite this in terms of an operator $\mathcal{N}$

$$\mathcal{N}(\phi(x)) \equiv D\phi(x)[Ax + f(x,\phi(x))] - B\phi(x) - g(x,\phi(x)). \tag{8}$$

We look for solutions of the form $\mathcal{N}(\phi(x)) = 0$; this is the condition every center manifold must meet. However, it seems as though this problem is actually HARDER than the original problem of finding the stability of the zero solution. So have we gained anything?? It turns out the answer is 'Yes.' The following theorem illustrates why this is the case:

Theorem 2 (Approximation Theorem)**.** *Let $\phi : \mathbb{R}^c \mapsto \mathbb{R}^s$ be a C^1 map with $\phi(0) = D\phi(0) = 0$. If $\mathcal{N}(\phi(x)) = \mathcal{O}(|x|^q)$ as $x \to 0$ (for $q > 1$), then*

$$|h(x) - \phi(x)| = \mathcal{O}(|x|^q), \qquad \text{as } x \to 0$$

In particular, this theorem allows us to use a power series to approximate the center manifold to an arbitrary degree of accuracy. This leads us to an answer to our second question from above. The approximation theorem guarantees that any two center manifolds must agree to an arbitrary order $\mathcal{O}(x^q)$. When we choose a CM, we are actually choosing a representative CM from an equivalence class of CMs that contains all of the solutions in a neighborhood of 0 (see #5 from list above).

Finally, we must address the stability of the zero solution. This leads us to the following theorem.

Theorem 3. *Assuming all of the same conditions for previous theorems, then we have the following: the zero solution of (2) is stable, asymptotically stable, or unstable if the zero solution of (4) is stable, asymptotically stable, or unstable, respectively.*

The stability theorem is the culmination of the CM theorem. Just like the linear case, when $\sigma_c(A) = 0$, the stability of the zero solution of the nonlinear system can be found by projecting the dynamics onto the CM.

2 Example of CM Reduction[2]

Find the stability of the zero solution of the following problem:

$$\dot{x} = xy + ax^3 + by^2x$$
$$\dot{y} = -y + cx^2 + dx^2y$$

Since this is a 2-dimensional system, $A = 0$ and $B = -1$. Let $\phi(x) = \mathcal{O}(x^2)$, $\phi'(x) = \mathcal{O}(x)$. Now apply $\mathcal{N}$.

$$\mathcal{N}(\phi(x)) = \phi'(x)[x\phi(x) + ax^3 + bx\phi(x)^2] + \phi(x) - cx^2 - dx^2\phi(x)$$

Collecting terms of order $\mathcal{O}(x^2)$ and ignoring higher order terms, we have

$$\phi(x) - cx^2 + \mathcal{O}(x^4) = 0 \Rightarrow h(x) = cx^2 + \mathcal{O}(x^4)$$

Therefore, on the CM the dynamics reduce to:

$$\begin{aligned}\dot{u} &= Au + f(u, h(u)) \\ &= 0 + u(cu^2) + au^3 + bu(cu^2)^2 \\ &= (a+c)u^3 + bc^2u^5\end{aligned}$$

This implies that the zero solution is stable if $a+c < 0$ and unstable if $a+c > 0$. However, if $a+c = 0$, then we need a better approximation of the CM. Suppose now that $\phi(x) = cx^2 + \psi(x)$ where $\psi(x) = \mathcal{O}(x^4)$. Then we have

$$\begin{aligned}\mathcal{N}(\phi(x)) = (2cx + \psi'(x))\left(x(cx^2 + \psi(x)) + ax^3 + bx(cx^2 + \psi(x))^2\right) \\ + (cx^2 + \psi(x)) - cx^2 - dx^2(cx^2 + \psi(x))\end{aligned}$$

Gathering all terms of $\mathcal{O}(x^4)$ or less, we have

$$\begin{aligned}0 &= 2c^2x^4 + 2acx^4 + cx^2 + \psi(x) - cx^2 - cdx^4 \\ &= 2cx^4(a+c) + \psi(x) - cdx^4 \\ \psi(x) &= cdx^4 + \mathcal{O}(x^5)\end{aligned}$$

So we have $h(x) = cx^2 + cdx^4$, and the dynamics on the CM reduce to

$$\begin{aligned}\dot{u} &= u(cu^2 + cdu^4) + au^3 + bu(cu^2 + cdu^4)^2 \\ &= cdu^5 + bc^2u^5 + \mathcal{O}(u^7) \\ &= (cd + bc^2)u^5 + \mathcal{O}(u^7)\end{aligned}$$

So if $a + c = 0$, then the zero solution is stable if $cd + bc^2 < 0$ and unstable if $cd + bc^2 > 0$. If $cd + bc^2 = 0$ then we need a higher order approximation of the CM. This is omitted.

This example also illustrates the need for the center manifold calculation. If we use the tangent space approximation (reduction to E^c), then we get

$$\dot{u} = au^3$$

which gives an incorrect answer to the stability of the zero solution.

[2]Example taken from Carr.

3 General CM Reduction for Hopf Bifurcation[3]

Center manifold reductions are useful for Hopf bifurcations. As you will see, the stability of the Hopf can easily be read off by finding the *first Lyapunov coefficient*. There is a CM projection method that involves putting the system into its eigenbasis, however, for large systems, this is undesirable due to the difficulty. I proceed with a method that does not require an eigenbasis transformation.

Consider the nonlinear system (2), and suppose that A has only two eigenvalues $\lambda_{\pm} = \pm i\omega$, and suppose that all other eigenvalues are real with $\lambda_i < 0$. Let $\Phi \in \mathbb{C}^n$ be the eigenvector corresponding to $\lambda = i\omega$, and let $\Psi \in \mathbb{C}^n$ be the eigenvector of A^* with eigenvalue $\lambda = -i\omega$. So we have the following

$$\begin{array}{rclrcl} A\Phi &=& i\omega\Phi, & A\bar{\Phi} &=& -i\omega\bar{\Phi}, \\ A^*\Psi &=& -i\omega\Psi, & A^*\bar{\Psi} &=& i\omega\bar{\Psi}. \end{array}$$

where A^* denotes the adjoint of the matrix A. Using the inner product on $\mathbb{C}^n$ we normalize the vectors Φ and Ψ such that $\langle \Psi, \Phi \rangle = \sum_i \bar{\Psi}_i \Phi_i = 1$.

Definition 1 (Hopf crossing condition)**.** The following is known as the strict crossing condition, a sufficient condition for Hopf bifurcation

$$\gamma'(0) = \langle \Psi, A'(0)\Phi \rangle \neq 0.$$

We assume the Hopf crossing condition.

Now the real eigenspace E^c corresponding to eigenvalues $\pm i\omega$ is spanned by the vectors $\{\mathrm{Re}\,\Phi, \mathrm{Im}\,\Phi\}$. Letting $E^{su} = E^s \oplus E^u$, we have by the Fredhölm Alternative that

$$y \in E^{su} \Leftrightarrow \langle \Psi, y \rangle = 0 \tag{9}$$

Notice, in particular that since $\Psi \in \mathbb{C}^n$ and $y \in \mathbb{R}^n$ this places two contraints on y, that both $\mathrm{Re}\langle \Psi, y \rangle = 0$ and $\mathrm{Im}\langle \Psi, y \rangle = 0$. Therefore, we can make the decomposition $\mathbb{R}^n = E^c \oplus E^{su}$, therefore, for all $x \in \mathbb{R}^n$ we have

$$x = (z\Phi + \bar{z}\bar{\Phi}) + y, \tag{10}$$

where z is a coordinate on E^c (recall that $z \in \mathbb{C}^1$, which is two dimensional) and $y \in E^{su}$. Using this decomposition we can write down the following:

$$\begin{array}{rcl} z &=& \langle \Psi, x \rangle \\ y &=& x - \langle \Psi, x \rangle \Phi - \langle \bar{\Psi}, x \rangle \bar{\Phi} \end{array} \tag{11}$$

Notice that this appears to be an $(n+2)$-dimensional system, however recall that there are two constraints on the y equation. Differentiating equation (11), we find that equation (2) can be put in the following form:

$$\begin{array}{rcl} \dot{z} &=& i\omega z + \langle \Psi, f(z\Phi + \bar{z}\bar{\Phi} + y, \lambda) \rangle \\ \dot{y} &=& Ay + f(z\Phi + \bar{z}\bar{\Phi} + y, \lambda) \\ && -\langle \Psi, f(z\Phi + \bar{z}\bar{\Phi} + y, \lambda) \rangle \Phi - \langle \bar{\Psi}, f(z\Phi + \bar{z}\bar{\Phi} + y, \lambda) \rangle \bar{\Phi}. \end{array} \tag{12}$$

[3]The treatment found here follows that of Kuznetsov.

While (12) is a natural way to write the system, it is not very useful computationally, so we rewrite this in a Taylor series

$$\begin{aligned}\dot{z} &= i\omega z + \tfrac{1}{2}(v_{20}z^2 + 2v_{11}z\bar{z} + v_{02}\bar{z}^2) + v_{21}z^2\bar{z} \\ &\quad +\langle v_{10}, y\rangle z + \langle v_{01}, y\rangle \bar{z} + \ldots \\ \dot{y} &= Ay + \tfrac{1}{2}(u_{20}z^2 + 2u_{11}z\bar{z} + u_{02}\bar{z}^2)\end{aligned} \tag{13}$$

where $v_{ij} \in \mathbb{C}^1$ for $i+j=2$, $v_{ij} \in \mathbb{C}^n$ for $i+j=1$, and $u_{ij} \in \mathbb{C}^n$.

Remember we want to reduce the system to its center manifold. Let the manifold have the Taylor series:

$$y = h(z, \bar{z}) = \frac{1}{2}(w_{20}z^2 + 2w_{11}z\bar{z} + w_{02}\bar{z}^2) + \mathcal{O}(|z|^3) \tag{14}$$

with $w_{ij} \in \mathbb{C}^n$. Since $y \in E^{su}$ we must have $w_{ij} \in E^{su}$.

Differentiating equation (14), using $\dot{z}$ from equation (13), and comparing terms with $\dot{y}$ from equation (13), we find the following:

$$\begin{aligned}(2i\omega - A)w_{20} &= u_{20} \\ -Aw_{11} &= u_{11} \\ (-2i\omega - A)w_{02} &= u_{02}.\end{aligned} \tag{15}$$

Its clear that we can solve for the w's since neither $\pm 2i\omega$ nor 0 are eigenvalues of A. Thus, the left hand sides are invertible. We solve for the w's and substitute them into equation (14). Substituting this expression for y into the inner products in equation (13), we collect the $z^2\bar{z}$ terms, we have the center manifold reduction

$$\begin{aligned}\dot{z} &= i\omega z + \tfrac{1}{2}(v_{20}z^2 + 2v_{11}z\bar{z} + v_{02}\bar{z}^2) \\ &+ \tfrac{1}{2}(v_{21} - 2\langle v_{10}, A^{-1}u_{11}\rangle + \langle v_{01}, (2i\omega I - A)^{-1}u_{20}\rangle)z^2\bar{z} + \ldots\end{aligned} \tag{16}$$

We write $f(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + \ldots$, with $B(x,x)$ and $C(x,x,x)$ giving the second and third order terms respectively. With more simplifications, we can write

$$\dot{z} = i\omega z + \frac{1}{2}(g_{20}z^2 + 2g_{11}z\bar{z} + g_{02}\bar{z}^2) + \frac{1}{2}g_{21}z^2\bar{z} + \ldots \tag{17}$$

where

$$g_{20} = \langle \Psi, B(\Phi, \Phi)\rangle, \quad g_{11} = \langle \Psi, B(\Phi, \bar{\Phi})\rangle \tag{18}$$

and

$$\begin{aligned}g_{21} =& \langle \Psi, C(\Phi, \Phi, \bar{\Phi})\rangle - 2\langle \Psi, B(\Phi, A^{-1}B(\Phi, \bar{\Phi}))\rangle \\ &+ \langle \Psi, B(\bar{\Phi}, (2i\omega I - A)^{-1}B(\Phi, \Phi))\rangle + \frac{1}{i\omega}\langle \Psi, B(\Phi, \Phi)\rangle\langle \Psi, B(\Phi, \bar{\Phi})\rangle \\ &- \frac{2}{i\omega}|\langle \Psi, B(\Phi, \bar{\Phi})\rangle|^2 - \frac{1}{3i\omega}|\langle \Psi, B(\bar{\Phi}, \bar{\Phi})\rangle|^2\end{aligned} \tag{19}$$

We are now in a position to state the final theorems necessary to analyze the Hopf bifurcation.

Definition 2. Let the CM reduction of equation (2) have the form:

$$\dot{z} = i\omega z + \frac{1}{2}(g_{20}z^2 + 2g_{11}z\bar{z} + g_{02}\bar{z}^2) + \frac{1}{2}g_{21}z^2\bar{z} + \dots \tag{20}$$

then the *first Lyapunov coefficient* be defined as the quantity

$$l_1(0) = \frac{1}{2\omega^2}\,\mathrm{Re}(ig_{20}g_{11} + \omega g_{21}). \tag{21}$$

Theorem 4 (Stability of Hopf Bifurcation)**.** *Let equation (2) have the center manifold reduction (20), then the Hopf bifurcation is subcritical iff* $l_1(0) < 0$ *and supercritical iff* $l_1(0) > 0$.

From equation (18), we can see that both g_{20} and g_{11} are real. The last three terms in (19) are also imaginary, so we write down the following expression for the first Lyapunov coefficient

$$\begin{aligned} l_1(0) &= \frac{1}{2\omega}\,\mathrm{Re}(g_{21}) \\ &= \frac{1}{2\omega}\,\mathrm{Re}[\langle\Psi, C(\Phi,\Phi,\bar{\Phi})\rangle - 2\langle\Psi, B(\Phi, A^{-1}B(\Phi,\bar{\Phi}))\rangle \\ &\quad + \langle\Psi, B(\bar{\Phi}, (2i\omega I - A)^{-1}B(\Phi,\Phi))\rangle]. \end{aligned} \tag{22}$$

While the value of $l_1(0)$ varies under different normalizations, the sign of $l_1(0)$ is invariant.

4 Example of CM Reduction for HB

For the example of a CM reduction for the Hopf bifurcation, I will use the system that was used in the homework: Exercise 11. Show that the system has a Hopf bifurcation at $\lambda = 0$ and show that the bifurcation is supercritical.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 2\lambda x_2 - x_1x_3 \\ \dot{x}_3 &= -x_3 + x_1^2 \end{aligned} \tag{23}$$

We put this in the form as in Section 3: $\dot{x} = A(\lambda)x + f(x)$.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -2\lambda & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1^2 \end{pmatrix}. \tag{24}$$

When $\lambda = 0$, $\omega = 1$ and we have $\lambda_\pm = \pm i$, $\lambda_3 = -1$. This indicates the possibility of a Hopf bifurcation. Finding eigenvectors of A and A^* we find

$$\Phi = \begin{pmatrix} +i \\ 1 \\ 0 \end{pmatrix} = \Psi, \quad A\Phi = i\Phi, \quad A^*\Psi = -i\Psi. \tag{25}$$

For normalization purposes, let $\Psi = \frac{1}{2}\Phi$; then we have $\langle \Psi, \Phi \rangle = 1$. Now check the Hopf crossing condition

$$\begin{aligned}
\langle \Psi, A'(0)\Phi \rangle &= \left\langle \frac{1}{2}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \right\rangle \\
&= \frac{1}{2}\left\langle \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \right\rangle \\
&= -1
\end{aligned}$$

Because $\gamma'(0) \neq 0$, there exists a Hopf bifurcation at $\lambda = 0$.

Since $f(x)$ has only quadratic terms, we see that $C(x, x, x) \equiv 0$ and

$$B(x, x) = 2f(x, x) = \begin{pmatrix} 0 \\ -2x_1x_3 \\ 2x_1^2 \end{pmatrix} \Rightarrow B(x, y) = \begin{pmatrix} 0 \\ -2x_1y_3 \\ 2x_1y_1 \end{pmatrix} \tag{26}$$

From Section 3, we know that we must calculate the first Lyapunov coefficient to find the stability of the Hopf bifurcation:

$$\begin{aligned}
l_1(0) = \; & \tfrac{1}{2\omega}\operatorname{Re}[\langle \Psi, C(\Phi, \Phi, \bar{\Phi})\rangle - 2\langle \Psi, B(\Phi, A^{-1}B(\Phi, \bar{\Phi}))\rangle \\
& + \langle \Psi, B(\bar{\Phi}, (2i\omega I - A)^{-1}B(\Phi, \Phi))\rangle].
\end{aligned} \tag{27}$$

We know that $C \equiv 0$, so the first term falls out immediately. Consider the second term:

$$\begin{aligned}
-2\langle \Psi, B(\Phi, A^{-1}B(\Phi, \bar{\Phi}))\rangle &= -2\left\langle \frac{1}{2}\begin{pmatrix} +i \\ 1 \\ 0 \end{pmatrix}, A^{-1}B\left(\begin{pmatrix} +i \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right) \right\rangle \\
&= -\left\langle \begin{pmatrix} +i \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4i \\ 0 \end{pmatrix} \right\rangle \\
&= 4i.
\end{aligned}$$

We find that the second term is imaginary, so it does not contribute to $l_1(0)$. We now consider the third and final term.

$$\begin{aligned}
\langle \Psi, B(\bar{\Phi}, (2i\omega I - A)^{-1}B(\Phi, \Phi))\rangle &= \left\langle \frac{1}{2}\begin{pmatrix} +i \\ 1 \\ 0 \end{pmatrix}, B\left(\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, (2i\omega I - A)^{-1}\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}\right) \right\rangle \\
&= \frac{1}{2}\left\langle \begin{pmatrix} +i \\ 1 \\ 0 \end{pmatrix}, \frac{4}{5}\begin{pmatrix} 0 \\ 2+i \\ 0 \end{pmatrix} \right\rangle \\
&= \frac{4}{10}(2+i)
\end{aligned}$$

Using this in (27) we find

$$\begin{aligned} l_1(0) &= \frac{1}{2}\,\mathrm{Re}\left(0 + 4i + \frac{4}{5} + \frac{2}{5}i\right) \\ &= \frac{2}{5}. \end{aligned}$$

We find that $l_1(0) > 0$, which tells us that the Hopf is supercritical.