Melnikov's Method as a Bifurcation Problem

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1 Introduction

Melnikov's Method provides one of the only analytic predictions of chaotic behavoir, providing a lower bound for where chaotic behavoir can occur. Melnikov's Method is usually presented as a straight perturbation problem; however, it can also be formulated as a bifurcation problem.

Consider

$$\ddot{x} + c_1 x + c_2 f(x) = \epsilon g(\lambda, x, \dot{x}, t) \tag{1}$$

and let us assume that for $\lambda = 0, \epsilon = 0$ this differential equation has a seperatrix loop in the $(x, \dot{x})$ phase. Then in general, for $\epsilon > 0$, the seperatrix loop will be broken by this perturbation. The question then is simply if the stable and unstable manifolds intersect. If they intersect, the Smale-Birkhoff theorem says the dynamical system contains a horseshoe, a necessary condition for chaotic behavoir. We should note that the existence of a horseshoe does not guarantee chaotic dynamics, one could also get so called ”transient chaos,” where by the system tends to infinity or a periodic orbit.

2 Melnikov's Method as a Perturbation Problem

To determine if the stable and unstable manifolds intersect one can proceed by straight perturbations. First we write as a two dimensional system

$$\begin{aligned} \dot{x} &= y \qquad (2) \\ \dot{y} &= -c_1 x - c_2 f(x) + \epsilon g(\lambda, x, \dot{x}, t) \qquad (3) \end{aligned}$$

And let $x^+(t), y^+(t)$, be the solution along the perturbed stable manifold, and $x^-(t)$ be the solution along the unstable manifold. We then expand both in the same manner:

$$\begin{aligned} x &= x_0 + \epsilon x_1 && (4) \\ y &= y_0 + \epsilon y_1 && (5) \\ & && (6) \end{aligned}$$

And collect powers of ϵ

$$\begin{aligned} \epsilon^0 : \dot{x}_0 &= y_0 && (7) \\ \dot{y}_0 &= -c_1 x_0 - c_2 f(x_0) && (8) \\ \epsilon^1 : \dot{x}_1 &= y_1 && (9) \\ \dot{y}_1 &= (-c_1 - c_2 \frac{\partial f}{\partial x}(x_0))x_1 + g(\lambda, x_0, y_0, t) && (10) \end{aligned}$$

Where (9) and (10) hold for both + and - superscripts.

Now consider the vector

$$(x^+(t) - x^-(t), y^+(t) - y^-(t)) \tag{11}$$

This vector connects the two perturbed manifolds. Melnikov then takes a dot product of this vector with a vector normal to the unperturbved seperatrix loop at time t. This vector is chosen to be

$$(\dot{y}(t), -\dot{x}(t)) \tag{12}$$

He then defines his key quantity $\Delta(t)$

$$\Delta(t) = (-c_1 x_0 - c_2 f(x), -y_0) \cdot (x^+ - x^-, y^+ - y^-) \tag{13}$$

which characterizes the size of the gap betweeen the stable and unstable manifolds. Thus if $\Delta(t)$ vanishes for some t, the stable and unstable manifolds intersect.

Without going into the details (see Topics in Nonlinear Dynamics by Rand, et all as well as other places), this simplies to

$$\Delta(\tau) = \int_{\infty}^{\infty} \dot{x_0} g(\lambda, x_0, y_0, t) dt \tag{14}$$

where x_0, y_0 are functions of $t - \tau$. Thus we predict the stable and unstable manifolds intersect to $O(\epsilon)$ if (14) is zero for some τ

3 Melnikov’s Method as a Bifurcation Problem

Now lets look at Melnikov’s Method as a Bifurcation Problem. Consider again

$$\ddot{x} + c_1 x + c_2 f(x) = \epsilon g(\lambda, x, \dot{x}) \tag{15}$$

under the assumption that for $\epsilon = 0$, (1) has a homoclinic orbit Γ_0.

Suppose we then expand $\lambda = \lambda_c$ as

$$\lambda_c(\epsilon) = \lambda_0 + \epsilon\lambda_1 \tag{16}$$

where $\lambda_c(\epsilon)$ is the value of λ where homoclinicity occurs for small $\epsilon > 0$. In other words where the stable and unstable manifolds intersect.

Notes:

For $\epsilon = 0, \Gamma_0(t)$ is an exact solution $\forall\ \lambda \in \mathbb{R}$. Thus if we view the problem of determining the homoclinic curve $\lambda = \lambda_c(\epsilon)$ as a bifurcation problem in (λ, ϵ) space.

We have the trivial line $\epsilon = 0$, so the constant leading term λ_c can be viewed as the bifurcation point on the $\epsilon = 0$ axis.

3.1 Setting up the Equations

Let **x** be a perturbation to the seperatrix loop $\mathbf{\Gamma}_0$, then

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{\Gamma}_0(t) + \mathbf{y}(t) && (17)\\
\mathbf{f}(x) &= (-\dot{x}, c_1 x + c_2 f(x)) && (18)\\
\mathbf{g}(x) &= (0, g(\lambda, \mathbf{x}) && (19)
\end{aligned}$$

Then we can write our original differential equation (1) as:

$$\begin{aligned}
\dot{\mathbf{y}} + \dot{\mathbf{\Gamma}}_\mathbf{0} &= -\mathbf{f}(\mathbf{\Gamma_0} + \mathbf{y}) + \epsilon\mathbf{g}(\lambda, \mathbf{\Gamma_0} = \mathbf{y}) && (20)\\
\dot{\mathbf{y}} - \mathbf{f}(\mathbf{\Gamma_0}) &= -\mathbf{f}(\mathbf{\Gamma_0} + \mathbf{y}) + \epsilon\mathbf{g}(\lambda, \mathbf{\Gamma_0} = \mathbf{y}) && (21)\\
\dot{\mathbf{y}} - \mathbf{f}(\mathbf{\Gamma_0}) &+ \mathbf{f}(\mathbf{\Gamma_0} + \mathbf{y}) - \epsilon\mathbf{g}(\lambda, \mathbf{\Gamma_0} = \mathbf{y}) = \mathbf{0} && (22)
\end{aligned}$$

Where for equation (21) we used that $\mathbf{\Gamma}_0$ satisfies the $\epsilon = 0$ equation. Finally we want to elminate phase shifting solutions $\mathbf{y}(t) = \mathbf{\Gamma}_0(t+\theta) - \mathbf{\Gamma}_0(t)$ for small θ, so we add a normalizing constraint:

$$< \dot{\mathbf{\Gamma}}_0, \mathbf{y} >= \int_{\infty}^{\infty} \dot{\mathbf{\Gamma}}_0(t)^T \mathbf{y}(t) dt = 0 \tag{23}$$

3.2 Linear Equation

Let

$$L \quad : \quad Y \longrightarrow Z \tag{24}$$

$$\mathbf{y}(\cdot) \quad \mapsto \quad \dot{\mathbf{y}}(\cdot) + \mathbf{A}(t)\mathbf{y}(\cdot) \tag{25}$$

where

$$\mathbf{A}(t) = D\mathbf{f}(\Gamma_0(t)) \tag{26}$$

$$Y = \{\mathbf{y}(\cdot) \in C^1(\mathbb{R}, \mathbb{R}^2) | \ \mathbf{y} \text{ satisfies (22) } \} \tag{27}$$

$$Z = C^0(\mathbb{R}, \mathbb{R}^2) \tag{28}$$

$$||z(\cdot)||_Z = \sup_{t \in \mathbb{R}} |z(t)| \tag{29}$$

$$||y(\cdot)||_Y = \sup_{t \in \mathbb{R}} (|\mathbf{y}(t)| + |\dot{\mathbf{y}}(t)|) \tag{30}$$

I state the following claims without proof. For a proof see ”Exponential dichotomies and trasversal homoclinic points” by Palmer

Claim: L is a bounded linear Fredholm operator of index -1

Claim: In our case where $\mathbf{y}(t) \in \mathbb{R}^2$ and $\mathbf{A}(t)$ converges exponentially to the hyperbolic linearization at the saddle equilibrium S, the normalization (23) implies that kernel $L = \{0\}$

Taking these two claims and using the fact that

$$\text{ind}(L) = \text{dim Kernel} L - \text{codim range} = -1 \tag{31}$$

we see that the a one-dimensional complement to range L is given by a bounded solution $\psi(t)$ to the adjoint equation since dim Kernel $L = 0$.

Let $\psi(t)$ be a nontrivial bounded solution of

$$L^T\psi = \dot{\psi}(t) - \mathbf{A}(t)^T\psi(t) = 0 \tag{32}$$

Then the following are true:

Claim: $\psi(\cdot)$ is orthogonal to $\mathbf{z}(\cdot) \in$ range L, i.e.

$$< \psi(\cdot), \mathbf{z}(\cdot) >= \int_{\infty}^{\infty} \psi(t)^T\mathbf{z}(t)dt = 0 \tag{33}$$

Claim: range L = $(span\{\psi(\cdot)\})^{\perp}$ with orthogonality given by (33), i.e. range$L = \{y \in Y :< \psi, y >= 0\}$

The first follows by integration by parts and the second from codim range $L = 1$ and (31). With this in hand we can find ψ.

Claim: $\psi(t) = \text{grad}_{x,\dot{x}}H(\Gamma_0(t))$ where $H(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + V(x)$ is the energy of the unperturbed Hamiltonian oscillator

Proof: By direct computation

$$\begin{aligned}\psi = grad_{x,\dot{x}}H(\Gamma_0(t)) &= (\frac{\partial V}{\partial x}, \dot{\Gamma}_0) && (34)\\ &= \begin{pmatrix} c_1\Gamma_0 + c_2 f(\Gamma_0) \\ \dot{\Gamma}_0 \end{pmatrix} && (35)\end{aligned}$$

Then

$$A^T\psi = \begin{pmatrix} 0 & c_1 + c_2 f'(\Gamma_0) \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_1\Gamma_0 c_2 f(\Gamma_0) \\ \dot{\Gamma}_0 \end{pmatrix} = \begin{pmatrix} c_1\dot{\Gamma}_0 + c_2 f'(\Gamma_0)\dot{\Gamma}_0 \\ -c_1\Gamma_0 - c_2 f(\Gamma_0)\Gamma_0 \end{pmatrix} \tag{36}$$

but

$$\psi'(t) = \begin{pmatrix} c_1\dot{\Gamma}_0 + c_2 f'(\Gamma_0)\dot{\Gamma}_0 \\ \ddot{\Gamma}_0 \end{pmatrix} = \begin{pmatrix} c_1\dot{\Gamma}_0 + c_2 f'(\Gamma_0)\dot{\Gamma}_0 \\ -c_1\Gamma_0 - c_2 f(\Gamma_0)\Gamma_0 \end{pmatrix} \tag{37}$$

Thus $\psi'(t) - A^T\psi(t) = 0$.

3.3 Nonlinear Equation using Lyapounov-Schmidt reduction

We return to the full nonlinear system and define:

$$\mathbf{F} : \mathbb{R} \times \mathbb{R} \times Y \longrightarrow Z \tag{38}$$

$$(\lambda, \epsilon, \mathbf{y}(\cdot)) \longrightarrow \mathbf{f}(\Gamma_0) - \mathbf{f}(\boldsymbol{\Gamma}_0 + \mathbf{y}) + \epsilon \mathbf{g}(\mu, \Gamma_0 + \mathbf{y}) + \mathbf{A}(t)\mathbf{y} \tag{39}$$

F must be as smooth, in Banach space as the original f and g, and we therefore want to solve:

$$L\mathbf{y} - \mathbf{F}(\lambda, \epsilon, \mathbf{y}(\cdot)) = 0 \tag{40}$$

We do this by using Lyapounov-Schmidt reduction and our function ψ. Let

$$P : Z \quad \rightarrow \quad \text{span}\, \psi(\cdot) \tag{41}$$

$$P\mathbf{z} \quad = \quad \frac{< \psi, \mathbf{z} >}{< \psi, \psi >}\psi \tag{42}$$

and its complement $Q = id - P$. Then using this projection, (41) is equivalent to

$$0 \quad = \quad QL\mathbf{y} - Q\mathbf{F} = L\mathbf{y} - Q\mathbf{F} \tag{43}$$

$$0 \quad = \quad PL\mathbf{y} - P\mathbf{F} = -P\mathbf{F} \tag{44}$$

Since $PL = 0$ and $QL = L$ by definition of ψ.

Also note that by construction

$$\mathbf{F}(\lambda, \epsilon = 0, \mathbf{y} \equiv 0) = 0 \tag{45}$$

$$D_{\mathbf{y}}\mathbf{F}(\lambda, \epsilon = 0, \mathbf{y} \equiv 0) = -D\mathbf{f}(\Gamma_0(t)) + \mathbf{A}(t) = 0 \tag{46}$$

Continuing, by the normalization, and the definitions of $\Gamma_0, \mathbf{y}, \mathbf{A}, \mathbf{F}$,

$$QL = L : Y \rightarrow QZ \text{ is invertible.}$$

Thus the standard implicit function theorem tells us then that $L\mathbf{y} - Q\mathbf{F} = 0$ has a unique solution $\mathbf{y}(\cdot) = \mathbf{y}(\lambda, \epsilon, \cdot)$ near $\mathbf{y}(t) \equiv 0$ and uniformly for bounde sets of $\lambda \in \mathbb{R}$ and small ϵ. Substituing this into (45) yields the remaining reduced equation

$$B(\lambda, \epsilon) =< \psi, \mathbf{F}(\lambda, \epsilon, \mathbf{y}(\lambda, \epsilon, \cdot) >= 0 \tag{47}$$

Geometrically, $B(\lambda, \epsilon)$ measures the distance between the separatrices. Now a few notes:

Notes:

$$\mathbf{y}(\lambda, \epsilon = 0, t) = 0 \;\; \forall \;\; \lambda, t \in \mathbb{R} \tag{48}$$
$$\mathbf{y}(\lambda, \epsilon, \cdot) = O(\epsilon) \tag{49}$$
$$B(\lambda, \epsilon = 0) = 0 \;\; \forall \;\; \lambda \in \mathbb{R} \tag{50}$$

(49) follows from uniqueness and (46), (50) follows from (49), and (51) follows from (48) and (47).

Finally then we get a necessary condition for the appearance of nontrivial solutions $\epsilon \neq 0$ from the implicit function theorem:

$$D_\epsilon B(\lambda_c, \epsilon = 0) = 0 \tag{51}$$

Computing $D_\epsilon B$ we see

$$\begin{aligned} D_\epsilon B(\lambda_c, \epsilon = 0) &= \; < \psi, D_\epsilon \mathbf{F} + D_{\mathbf{y}} \mathbf{F} \cdot D_\epsilon \mathbf{y} > & (52) \\ &= \; < \psi, D_\epsilon \mathbf{F} > & (53) \end{aligned}$$

by (47). If we then go back to the definition of **F**, we see that this is equal to

$$< \psi, \mathbf{g} >= \int_{\infty}^{\infty} \psi_2(t) g(\lambda, \Gamma_0(t)) dt \tag{54}$$

If we then let $\Gamma_0 = (x_0, \dot{x}_0)$, and use our computed $\psi_2 = \dot{x}_0$ we recover the Melnikov Function:

$$D_\epsilon B(\lambda, \epsilon = 0) = \int_{\infty}^{\infty} \dot{(x)}_0 g(\lambda, x, \dot{x}) dt = 0 \tag{55}$$

Note: $D_\epsilon B$ measures the first-order infinitesimal splitting of separatrices by the Melnikov distance B, due to the perturbation ϵ.

We could continue and obtain a sufficient condition for bifurcation of a homoclinic curve in the(λ, ϵ) space:

$$D_\epsilon B(\lambda_c, \epsilon = 0) = 0 \tag{56}$$
$$D_\lambda D_\epsilon B(\lambda_c, \epsilon = 0) \neq 0 \tag{57}$$

Which is the standard crossing condition.

4 Example

Consider duffing's equation with small dampening and forcing:

$$\ddot{x} - x + x^3 = \epsilon(\delta \cos \omega t - \gamma \dot{x}) \tag{58}$$

Then the Melnikov Function is

$$\Delta(\tau) = \int_{-\infty}^{\infty} \dot{x}_0(\delta \cos \omega t - \gamma \dot{x}_0) dt \tag{59}$$

This problem is solved in Guckenheimer and Holmes, it turns out that:

$$\Delta(\tau) = \sqrt{2}\pi\delta\omega \text{ sech } \frac{\pi\omega}{2} \sin \omega\tau + \frac{2}{3}\gamma \tag{60}$$

We are looking for values of γ and δ such that there exists a value of τ so that $\Delta(\tau) = 0$.

$$\Delta(\tau) = 0 \Rightarrow \sin \omega\tau = -\frac{2\gamma\sqrt{2} \cosh \frac{\pi\omega}{2}}{\delta \, 3 \, \pi \, \omega} \tag{61}$$

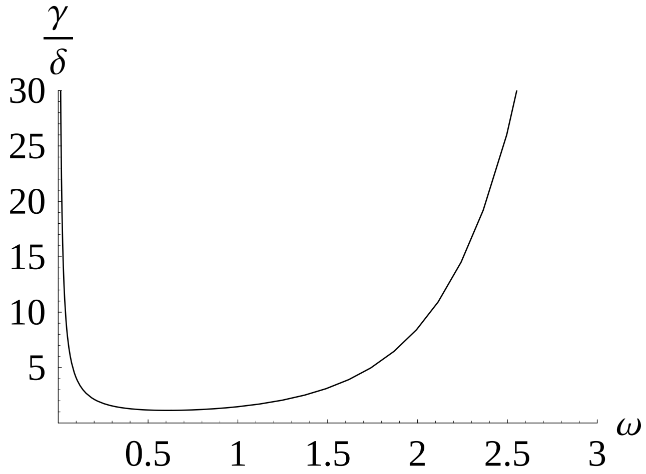
For a real value of τ we need

$$|\sin \omega\tau| \leq 1 \quad \Rightarrow \quad \frac{2\gamma\sqrt{2} \cosh \frac{\pi\omega}{2}}{\delta \, 3 \, \pi \, \omega} \leq 1 \tag{62}$$

$$\Rightarrow \quad \frac{\delta}{\gamma} \geq \frac{2\sqrt{2} \cosh \frac{\pi\omega}{2}}{3 \, \pi \, \omega} \tag{63}$$

Thus Melnikov's Method predicts the exists of a horseshoe in the Poincare map for equation (1) if

$$\frac{\delta}{\gamma} \geq (\frac{\delta}{\gamma})_{cr} = \frac{2\sqrt{2} \cosh \frac{\pi\omega}{2}}{3 \, \pi \, \omega} \tag{64}$$

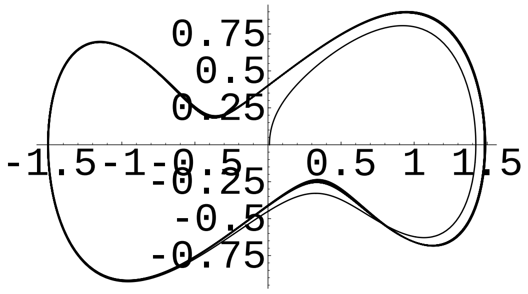


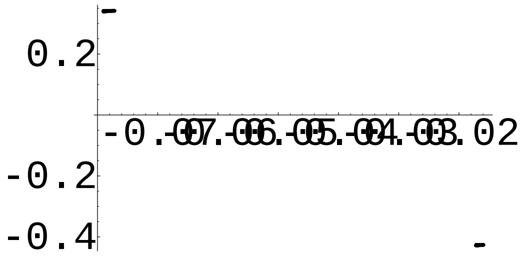
Melnikov's Method Predicts the Existence of a Horseshoe in the Poincare Map above the line.

Numerical Confirmation

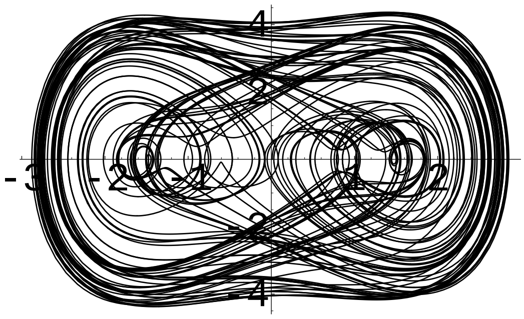
Numerically we can test our boundary. One important thing to note is that Melnikov's Method produces a lower bound for where chaos could occur. Below is the results of some numerical simulations, The first is below $(\frac{\delta}{\gamma})_{cr}$ and the second is above:

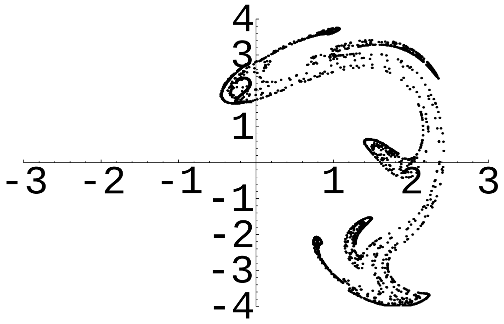
$\frac{\delta}{\gamma} < (\frac{\delta}{\gamma})_{cr}$





$\frac{\delta}{\gamma} > (\frac{\delta}{\gamma})_{cr}$





5 References

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