

# An Example for Bifurcation Analysis of Nonlinear PDEs

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## Abstract

This paper has been written as a project for the course “Topics in Bifurcation Theory” at Cornell University in Spring 2007. We outline the bifurcation analysis of a nonlinear partial differential equation (PDE) using the method of Liapunov-Schmidt. In particular we shall also sketch the necessary tools from the theory of elliptic PDE which are necessary to analyze our model problem.

## 1 Introduction

The main goal in the following paper is the bifurcation analysis of the nonlinear PDE given by:

$$\begin{cases} (\Delta + \lambda)u + u^3 = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \quad (1)$$

where  $D$  is an open, bounded and connected subset of  $\mathbb{R}^n$ ,  $\lambda$  is a parameter in  $\mathbb{R}$  and  $u : \bar{D} \rightarrow \mathbb{R}$  with  $u = u(x)$  is the function to be determined. We are trying to answer the question what happens to solutions of (1) when we vary the parameter  $\lambda$ .

It turns out that our model problem is particularly suitable for a bifurcation analysis. Let us do a heuristic calculation to get an idea what we have to deal with. So set

$$F(u, \lambda) = (\Delta + \lambda)u + u^3 \quad (2)$$

Clearly this has the ‘trivial line’  $u = 0$  as a solution and we ask for bifurcating solutions from this solution as usual. Then we can “differentiate” to get

$$DF_u(0, \lambda) = \Delta + \lambda \quad (3)$$

For the “Implicit Function Theorem” (in infinite dimensions?) to fail we need a non-trivial kernel of this mapping. Therefore we are bound to consider the following problem of finding a function  $h$  and a number  $\lambda$  such that

$$-\Delta h = \lambda h \quad (4)$$

So we need to solve the “eigenvalue problem” for the negative of the Laplace operator. Suppose we can somehow manage this and obtain an “eigenvalue”  $\lambda$  we do not know anything as yet whether it is actually real or complex, whether it is “simple” or whether the “eigenvector”  $h$  is a smooth function as for encountered in the case of Sturm-Liouville problems for ODEs.

So there are many loose ends and all originate from the fact that we are dealing with eigenvalue problems on infinite dimensional spaces and a PDE instead of an ODE. Indeed, for general nonlinear PDE we might not know anything about their solutions and their bifurcations but in this case we will see that our problem is not ‘far away’ from classical linear elliptic equations.

Nevertheless, we should first and foremost try to determine if the techniques developed for finite-dimensions so far carry over to infinite dimensions and which hypotheses are necessary for this transition. In particular we shall focus on the method of Liapunov-Schmidt and the calculations involving the bifurcation equations.

## 2 Basic Calculus in Banach Spaces

Let  $U, V$  be Banach spaces. Recall that a function  $F : U \rightarrow V$  is Frechet differentiable at  $u \in U$  if there exists a bounded linear operator  $A : U \rightarrow V$  such that

$$\lim_{h \rightarrow 0} \frac{\|F(u+h) - F(u) - Ah\|}{\|h\|} = 0 \quad (5)$$

We denote the Frechet derivative of  $F$  by  $DF$ . Suppose that we have a mapping  $F : U \times V \rightarrow \mathbb{R}$  then the classical bifurcation problem is to consider

$$F(u, v) = 0 \quad (6)$$

It turns out that the implicit function theorem still holds for a Frechet differentiable  $F$ .

**Theorem 2.1 (Implicit Function Theorem for Banach Spaces).** *Let  $(u_0, v_0) \in U \times V$  and assume that  $F$  is Frechet differentiable with continuous derivative and  $F(u_0, v_0) = 0$ . If furthermore  $D_u F$  is invertible and bounded at  $(u_0, v_0)$  then there is a neighborhood of  $(u_0, v_0)$ , say  $\tilde{U}$ , and a mapping  $f$  locally defined on  $\tilde{U}$  such that*

$$f(v_0) = u_0 \quad \text{and} \quad F(f(v), v) = 0 \quad \text{for all } v \text{ in a neighborhood of } v_0 \quad (7)$$

Furthermore all solutions of  $F(u, v) = 0$  on  $\tilde{U}$  are of the form  $(f(v), v)$ .

Recall that the last conclusion about the local uniqueness of solutions tells us in a bifurcation problem that there will be no bifurcations if the implicit function theorem works, so that we need its conditions to fail to possibly obtain bifurcating solutions. The unfortunate part in our setup is that if we are dealing with a mapping  $F$  on Banach spaces then  $D_u F = L$  is linear but might be very hard to work with. Theorems about matrices which were used to establish result in finite dimensions will no longer apply, so we need extra assumptions. One way to do this to require that  $L$  is a **Fredholm operator** which means that

- $\dim(\text{nullspace}(L)) < \infty$
- $\text{codim}(\text{range}(L)) < \infty$

We are going to adopt the shorter notations  $N(L) = \text{nullspace}(L)$  and  $R(L) = \text{range}(L)$  from now on. Furthermore let us define the **Fredholm index** of  $L$  by  $\dim(N(L)) - \text{codim}(R(L))$ . It turns out that if a Fredholm operator is of index 0 then it is possible to get results similar to finite dimensional linear operators, i.e. matrices.

Let us assume from now on that  $F$  is Frechet differentiable with derivative  $L$  and that  $L$  is a Fredholm operator of index 0. To make the notation easier we just study the general version of a bifurcation problem on a Banach space  $U$  with a single real parameter  $\lambda$ :

$$F : U \times \mathbb{R} \rightarrow Z \quad \text{and} \quad F(u_0, \lambda_0) = 0 \quad (8)$$

The extension to more complicated spaces of parameters is straightforward. We can in addition assume without loss of generality that  $u_0 = 0$ .

### 3 Liapunov-Schmidt Reduction

Recall that the key idea in the Liapunov-Schmidt method is to reduce the dimensionality of the bifurcation problem, preferably to 1 or 2 dimensions. We shall show that the assumptions about  $D_u F$  to be Fredholm suffice to carry out the reduction.

Assume as above that  $D_u F(u_0, \lambda_0) := L$  is a Fredholm operator. There exists closed complements to the nullspace and range so that we can decompose the Banach Spaces  $U$  and  $Z$ :

$$U = N(L) \oplus U_0 \tag{9}$$

$$Z = R(L) \oplus Z_0 \tag{10}$$

where we note that the two spaces  $Z_0$  and  $N(L)$  are finite by the definition of a Fredholm operator. Naturally we can define two projection maps onto these two spaces  $N(L)$  and  $Z_0$ :

$$P : U \rightarrow N(L) \tag{11}$$

$$Q : Z \rightarrow Z_0 \tag{12}$$

These two projectors are continuous. To prove this let  $u_n \rightarrow u$  and  $Pu_n \rightarrow y$  then we have to show that  $Pu = y$ . This implies that the graph of  $P$  given by

$$\Gamma(P) = \{(u, y) \in U \times N(L) \text{ such that } Pu = y\} \tag{13}$$

is closed. Now the classical **Closed Graph Theorem** states that a linear operator is continuous if and only if its graph is closed. Notice that since  $Pu_n \in N(L)$  for all  $n$  we can conclude that  $y \in N(L)$ . This means that we have  $Pu = y$  as  $P$  is a projection. Furthermore  $(I - P)u_n = u_n - Pu_n \rightarrow u - y$  and since  $U_0$  (the complement of  $N(L)$ ) is closed we deduce that  $u - y \in U_0$  because  $(I - P)u_n \in U_0$  for all  $n$ . Hence we conclude

$$0 = P(u - y) = Pu - Py = Pu - y \tag{14}$$

which gives the desired conclusion  $Pu = y$ . The argument that  $Q$  is also continuous is exactly the same. Now we are in a position to prove the main reduction

**Theorem 3.1 (Liapunov-Schmidt Reduction).** *There is a neighborhood  $\tilde{U} \times I$  in  $U \times \mathbb{R}$  such that the problem*

$$F(u, \lambda) = 0 \tag{15}$$

*is equivalent to a finite-dimensional problem for  $(u, \lambda) \in \tilde{U} \times I$ .*

*Proof.* The first key idea is to project the problem into the equivalent form

$$QF(Pu + (I - P)u, \lambda) = 0 \tag{16}$$

$$(I - Q)F(Pu + (I - P)u, \lambda) = 0 \tag{17}$$

Now define an auxillary mapping  $G$  locally given by

$$G(Pu, (I - P)u, \lambda) = (I - Q)F(Pu + (I - P)u, \lambda) \tag{18}$$

Clearly  $G(Pu_0, (I - P)u_0, \lambda_0) = F(u_0, \lambda_0) = 0$  by assumption. Also if we denote by  $D_2$  differentiation with respect to the second component of  $G$  we find

$$D_2G(Pu_0, (I - P)u_0, \lambda_0) = (I - Q)D_uF(u_0, \lambda_0) = (I - Q)L : U_0 \rightarrow R(L) \quad (19)$$

which is bijective since  $Q$  projects onto the complement of the range and  $(I - P)u$  is in the complement of the nullspace of  $L$ . This second key idea gives us that we can apply the implicit function theorem to  $G$  to conclude that  $G(Pu_0, (I - P)u_0, \lambda_0) = 0$  is equivalent to solving this problem (the problem of the second projected equation) by an implicit function  $\psi : \tilde{U} \times I \rightarrow U_0$  given by  $\psi(Pu, \lambda) = (I - P)u$ .

In particular we can use this function  $\psi$  which maps between  $N(L)$  and  $U_0$  to re-write the first projected equation  $QF(Pu + (I - P)u, \lambda) = 0$  as

$$QF(Pu + \psi(Pu, \lambda), \lambda) = 0 \quad (20)$$

which is a finite-dimensional bifurcation problem and so we have completed the reduction.  $\square$

Notice that reduction contains a function  $\psi$  which was obtained by the implicit function theorem so that we can not leave infinite dimensional spaces completely and it should not be surprising that we need calculus in Banach spaces to get explicit computational results.

## 4 Crandall-Rabinowitz and Bifurcation Formulas

Having established the reduction to a finite-dimensional problem, the same procedure as developed in the lectures can be derived for problems involving Fredholm operators. Our general assumptions are going to be the following from now on:

- $F : U \times \mathbb{R} \rightarrow Z$  is  $C^3$  (i.e. three times Frechet differentiable)
- $F(0, \lambda) = 0$  for all  $\lambda \in \mathbb{R}$  (trivial line)
- $\dim(N(D_uF(0, \lambda_0))) = 1 = \text{codim}(R(D_uF(0, \lambda_0)))$  (Fredholm of index 0)

Under these assumptions the theorem of Crandall-Rabinowitz still holds:

**Theorem 4.1 (Crandall-Rabinowitz).** *Let  $N(D_uF(0, \lambda_0)) = \text{span}\{\Phi\}$  and assume that*

$$D_{u\lambda}^2F(0, \lambda_0)\Phi \notin R(D_uF(0, \lambda_0)) \quad (21)$$

*Then there is a nontrivial  $C^1$ -curve through  $(0, \lambda_0)$  given by*

$$\{(u(s), \lambda(s)) | s \in (-\delta, \delta), (u(0), \lambda(0)) = (0, \lambda_0)\} \quad (22)$$

*and  $F(u(s), \lambda(s)) = 0$  for all  $s \in (-\delta, \delta)$ . Furthermore all solutions are of this form.*

The proof follows verbatim from the Crandall-Rabinowitz theorem presented in the course since we only need the Implicit Function Theorem and a ‘‘Fundamental Theorem of Calculus’’. Both theorems hold in Banach spaces. So we can use Crandall-Rabinowitz to verify if a nontrivial solution curve intersects the trivial line, i.e. a **bifurcation** occurs.

The next goal is to determine the local first order terms of the curve  $\lambda(s)$ . If we can find the first nonzero derivative of  $\lambda(s)$  at  $s = 0$  then we can describe the bifurcating solution (e.g. linear=transcritical, quadratic=Pitchfork, etc.).

Let  $H$  be a Hilbert space with inner product  $(\cdot, \cdot)$ . Using the fact that we can define an **adjoint**  $L^*$  to the operator  $L = D_uF(0, \lambda_0)$  in  $H$  by

$$(L^*u, v) = (u, Lv) \quad (23)$$

and  $\text{span}\{\Psi\} = N(L^*)$  the proofs developed in the lecture carry through directly to give us bifurcation formulas. Notice that we need at this step a clever choice of Hilbert space to construct an adjoint.

**Theorem 4.2 (Stationary Bifurcation Formulas).** *Let  $\Phi$  and  $\Psi$  be the nullvectors for  $D_u F(0, \lambda_0)$  and its adjoint and let  $(0, \lambda_0)$  be a bifurcation point with a nontrivial branch/curve of solutions  $\lambda(s)$  then*

$$\lambda'(0) = -\frac{1}{2} \frac{(D_{uu}^2 F(0, \lambda_0)[\Phi, \Phi], \Psi)}{(D_{u\lambda}^2 F(0, \lambda_0)\Phi, \Psi)} \quad (24)$$

Let  $Q, P$  be the projections as defined in the Liapunov-Schmitt Reduction then we also get

$$\lambda''(0) = -\frac{1}{3} \frac{(QD_{uuu}^3 F(0, \lambda_0)[\Phi, \Phi, \Phi] - 3QD_{uu}^2 F(0, \lambda_0)[\cdot, \cdot], \Psi)}{(D_{u\lambda}^2 F(0, \lambda_0)\Phi, \Psi)} \quad (25)$$

We shall not need the precise expression in the last formula for the second derivative as in our problem the second derivative  $D_{uu} F(0, \lambda_0)$  is going to vanish anyway. For a precise formulation and all details of the proofs in the general setup of Banach spaces see [Kielhöfer04].

So we are left for our model problem to determine where the bifurcation occurs which entails to find the kernel of a certain operator, namely  $D_u F(0, \lambda)$  depending on  $\lambda$ . Also we have to come up with choices for Banach spaces and a Hilbert space with an associated inner product. As a remark it should of course be noted that formally the digression to a Hilbert space is formally not necessary as we could also formulate all our results in terms of natural pairings with the dual space to our chosen Banach spaces  $U, Z$ . Unfortunately this makes explicit calculations quite difficult.

## 5 Some Elliptic PDE Theory

Let us consider the second order PDE with **Dirichlet boundary conditions** given by:

$$\begin{cases} Lu = f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \quad (26)$$

where  $D$  is an open, bounded and connected subset of  $\mathbb{R}^n$ ,  $f : D \rightarrow \mathbb{R}$  is given,  $u : \bar{D} \rightarrow \mathbb{R}$  with  $u = u(x)$  is the function to be determined and  $L$  denotes a second order differential operator having the form:

$$Lu = -\sum_{i,j=1}^n a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u \quad (27)$$

A classical example for  $L$  is the negative **Laplacian**  $-\Delta$  given by:

$$Lu = -\sum_{i=1}^n u_{x_i x_i} \quad (28)$$

which gives rise to **Laplace's equation** in (26) if  $f = 0$  or the more general formulation of **Poisson's equation** for  $f \neq 0$ . The more general theory of equation (26) should be viewed as an attempt to extend results known for these two classical equations. We shall assume from now on that we are only dealing with a **symmetric** operator  $L$ , i.e.

$$a^{ij}(x) = a^{ji}(x) \quad (i, j = 1, \dots, n) \quad (29)$$

The other major restriction is that we only consider a subset of symmetric operators. We call  $L$  **elliptic** if there exists a constant  $k > 0$  such that for almost every  $x \in U$  and all  $\xi \in \mathbb{R}^n$

$$\sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \geq k |\xi|^2 \quad (30)$$

where  $|\cdot|$  denotes the usual Euclidean norm. Algebraically this just means that the matrix of coefficients  $(a^{ij}(x))$  is positive definite for a.e.  $x$  with smallest eigenvalue greater or equal to  $k$ . This assumption on  $L$  turns out to have significant consequences. Take the initial equation  $Lu = f$  and multiply it by a 'suitable' function  $v$  and integrate by parts. this gives:

$$\int_D \left( \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \right) dx = \int_D f v dx \quad (31)$$

This motivates us to define a bilinear form

$$B[u, v] = \int_D \left( \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv \right) dx \quad (32)$$

for 'suitable' functions  $u, v$ . This suggest that  $u$  satisfying  $B[u, v] = (f, v)$  should be some kind of solution to our problem (26). To make this precise we should first define which functions are 'suitable' in our case. Recall that if  $u, v \in L^1_{loc}(D)$  then we say that  $v$  is the weak derivative of  $u$  with respect to  $x_i$  which we also write as  $u_{x_i} = v$  provided that for all **test functions**  $\phi \in C_c^\infty(D)$  we have

$$\int_D u \phi_{x_i} dx = - \int_D v \phi dx \quad (33)$$

Clearly this definition extends directly to higher derivatives. Then we can define a function space  $H^1(D)$ , called the first **Sobolev space**, composed of all functions  $u : D \rightarrow \mathbb{R}$  such that  $u \in L^2(D)$  and its weak derivative exists and belongs to  $L^2(D)$  as well. This space comes with a natural norm given by

$$\|u\|_{H^1(D)} = \left( \int_D |u|^2 dx + \sum_{i=1}^n \int_D |u_{x_i}|^2 dx \right)^{1/2} \quad (34)$$

which generalizes the usual  $L^2$  norm. From now on we shall drop the domain  $D$  if it is clear with respect to which underlying we are taking norms. We denote by  $H_0^1$  the closure of  $C_c^\infty$  in  $H^1$ .

Coming back to our PDE we define  $u \in H_0^1$  to be a **weak solution** of (26) if

$$B[u, v] = (f, v) \quad (35)$$

for all  $v \in H_0^1$ , where  $(\cdot, \cdot)$  denotes the  $L^2$  inner product. If  $B[\cdot, \cdot]$  is symmetric and therefore satisfies  $B[u, v] = B[v, u]$  for  $u, v \in L^2$  then the Riesz Representation Theorem provides the existence of a weak solution as defined above. Even in the general case it is possible to say quite a bit about the existence of weak solutions. In particular we are going to need the following:

**Theorem 5.1 (Existence of weak solutions).** *There exists an at most countable set  $S \subset \mathbb{R}$  such that the problem*

$$\begin{cases} Lu = \lambda u + f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \quad (36)$$

*has a unique weak solution for each  $f \in L^2(D)$  if and only if  $\lambda \notin S$ . Furthermore, if  $S$  is infinite then  $S$  is a nondecreasing sequence  $S = \{\lambda_k\}_{k=0}^{\infty}$  with  $\lambda_k \rightarrow \infty$ .*

*Proof.* See [Evans98]. It is interesting to note that the proof makes use of some non-basic functional analysis by using compact operators, the Fredholm Alternative and Sobolev space estimates.  $\square$

To particular importance to us will be the case when  $f = 0$  and so that we can read of from the previous result that

$$\begin{cases} Lu = \lambda u & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \quad (37)$$

has a non-trivial solution  $w \neq 0$  if and only if  $\lambda \in S$ . In this case we call  $\lambda$  an **eigenvalue** corresponding to  $L$  and  $w$  an **eigenfunction**. The special form of our operator  $L$  gave a condition on the eigenfunctions and on the **spectrum**  $S$  of  $L$  which in functional analysis notation would be denoted by  $\sigma(L)$ . This is a very special (!) case of general spectral theory of linear operators.

In particular if we take  $L = -\Delta$  we have the classical eigenvalue problem for the Laplacian and (37) is also referred to as **Helmholtz equation**. Unfortunately we have only an eigenfunction  $w$  which is a weak solution. This means that  $w \in H_0^1$  which would turn out to be very unpleasant for doing bifurcation analysis as we usually dealt only with smooth solutions to eigenvalue problems in all examples presented in the lectures.

Fortunately 'elliptic regularity theory' is going to save us here:

**Theorem 5.2 (Elliptic Regularity).** *Consider the problem*

$$\begin{cases} Lu = f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \quad (38)$$

*and assume that  $a^{ij}, b^i, c \in C^\infty(\bar{D})$  for all  $i, j$  and  $f \in C^\infty(\bar{D})$ . Also suppose that  $\partial D$  is smooth. Then any weak solution  $u$  of (38) is in  $C^\infty(\bar{D})$ .*

*Proof.* See [Evans98] or [G-T83] or any other book on elliptic PDE.  $\square$

This remarkable result tells us upon setting  $f = 0$  and  $L = -\Delta - \lambda I$  that the eigenfunctions of the Laplacian or any other elliptic operator of second order are smooth as long as the coefficient functions are.

## 6 Eigenvalues of Symmetric Elliptic Operators

In addition to the basic structure theorems of the last section we also have to look at the general eigenvalue problem of the form

$$\begin{cases} Lw = \lambda w & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \quad (39)$$

where  $D$  is open, bounded and connected and we assume that  $L$  is an elliptic differential operator of the form

$$Lu = - \sum_{i,j=1}^n a^{ij}(x)u_{x_i x_j} \quad \text{and } a^{ij} = a^{ji} \quad (40)$$

So  $L$  is formally symmetric which implies that the associated bilinear form  $B$  is also symmetric, i.e.  $B[u, v] = B[v, u]$ . In analogy to the classical result for symmetric matrices acting on finite dimensional vector spaces we get the following generalization:

**Theorem 6.1.** *Each eigenvalue  $\lambda$  in the problem (39) is real and we can order the countable set of eigenvalues*

$$0 < \lambda_0 \leq \lambda_1 \leq \dots \quad (41)$$

and furthermore there exists an  $L^2(D)$ -orthonormal basis  $w_k \in H_0^1(D)$  with  $w_k$  eigenfunctions corresponding to eigenvalues  $\lambda_k$ .

Using elliptic regularity we get that if  $a^{ij}(x)$  are smooth coefficient functions then  $w_k \in C^\infty(\bar{D})$ . We shall sketch the proof of the theorem.

*Proof.* (Sketch) It can be shown using energy estimates for elliptic equations that  $K := L^{-1}$  is a bounded, linear, compact operator from  $L^2(D)$  to itself where we recall that an operator  $T$  from a Banach space  $X$  to another Banach space  $Y$  is **compact** if for any bounded sequence  $\{v_k\}_{k=1}^\infty$  in  $X$  the sequence  $\{Tv_k\}_{k=1}^\infty$  has a convergent subsequence. Furthermore we claim that  $K : L^2(D) \rightarrow L^2(D)$  is symmetric.

Let  $f, g \in L^2(D)$  and observe that  $Kf = u$  means that  $u \in H_0^1(D)$  is a weak solution of the problem

$$\begin{cases} Lu = f & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \quad (42)$$

Similarily we get a weak solution  $v$  if  $Kg = v$ . This gives the two equalities:

$$(Kf, g) = (u, g) = B[v, u] \quad (43)$$

$$(f, Kg) = (f, v) = B[u, v] \quad (44)$$

for the  $(\cdot, \cdot)$   $L^2$  inner product. And since  $B$  is symmetric we have shown that  $K$  is also symmetric. In addition we can directly calculate

$$(Kf, f) = (u, f) = B[u, u] \geq 0 \quad (45)$$

In matrix terminology this would mean that  $K$  is symmetric and positive definite and the result about the eigenvalues would follow. It turns out that the fact about matrices also holds for compact, positive symmetric operators, more precisely:

**Fact 6.2.** *For a compact, positive, symmetric operator on a Hilbert space  $H$  the set of eigenvalues is real, positive and countable. Furthermore the eigenfunctions form an orthonormal basis of  $H$ .*

Now if  $\eta \neq 0$  is an eigenvalue for  $K$  and  $w$  a corresponding eigenfunction this means  $Kw = \eta w$  so that for  $\lambda = 1/\eta$  we have  $Lw = \lambda w$  which proves the theorem.  $\square$

Unfortunately we have not yet succeeded to get enough information for our bifurcation problem. Recall that it not only suffices to calculate the eigenvalues but there is also an assumption about their multiplicity involved in the method of Liapunov-Schmidt. We define  $\lambda_0$  to be the **principal eigenvalue** of (39). The next result shows that it is particularly easy to deal with the principal eigenvalue.

**Theorem 6.3.** *If  $u \in H_0^1(D)$  is any weak solution of*

$$\begin{cases} Lu = \lambda_0 u & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \quad (46)$$

*then  $u$  is a multiple of  $w_0$ . This means that the principal eigenvalue  $\lambda_0$  is **simple**, i.e.  $0 < \lambda_0 < \lambda_1 \leq \dots$ .*

We shall not deal with a proof of this result here. Instead let us notice that we can now claim to have all necessary information for the bifurcation analysis of our model problem.

## 7 Calculations for the Model Problem

Now that we have all machinery in place we can start the calculations for the problem

$$\begin{cases} (\Delta + \lambda)u + u^3 = 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases} \quad (47)$$

Notice that we first need to turn this problem into familiar notation and also define the relevant Banach spaces we are considering so we investigate

$$U := \{g \in C^{2,\alpha}(\bar{D}) \mid g(x) = 0 \text{ on } \partial D\} \quad (48)$$

$$F : U \times \mathbb{R} \rightarrow Z := \{g \in C^\alpha(\bar{D})\} \quad (49)$$

$$F(u, \lambda) := (\Delta + \lambda)u + u^3 = \Delta u + \lambda u + u^3 \quad (50)$$

where the two Banach spaces have the usual sup-norms given by the supremum of the function (and its respective derivatives) added to the  $\alpha$ -**Hölder seminorm** given by

$$[u]_{C^{0,\alpha}(\bar{D})} = \sup_{x,y \in D, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \quad (51)$$

so that e.g. the norm on the space  $Z$  would be given by  $\|u\| = \sup_{x \in \bar{D}} |u(x)| + [u]_{C^{0,\alpha}(\bar{D})}$ . Clearly we have the trivial solution to  $F(u, \lambda) = 0$  namely

$$F(0, \lambda) = 0 \quad (52)$$

and we shall investigate bifurcations from this “trivial line”. Note that we have to check when the implicit function theorem fails to determine when a bifurcation might occur. First we have to calculate the derivative of  $F$  on the “line”  $(0, \lambda)$ :

$$\frac{d}{d\alpha} [(\Delta + \lambda)(u + \alpha\eta) + (u + \alpha\eta)^3]_{|\alpha=0} = (\Delta + \lambda)\eta + u^3 + 3u^2\eta \quad (53)$$

Evaluating this for  $(0, \lambda)$  we immediately get the required derivative:

$$D_u F(0, \lambda)\eta = (\Delta + \lambda)\eta := L(\lambda) \quad (54)$$

where  $D_u$  obviously denotes differentiation with respect to the first variable. To determine the kernel we have to find the nullspace of the linear operator  $L(\lambda)$  and so we have to solve

$$-\Delta\eta = \lambda\eta \quad (55)$$

since we are in the Banach space  $U$  which enforces any solution of the last equation to be zero on the boundary we have precisely the eigenvalue problem for a second-order elliptic partial differential

operator as discussed in the previous section and Theorems 6.1 and Theorem 6.3 give all possible solutions. We shall only discuss bifurcations at the principal eigenvalue  $\lambda_0$ .

Since  $\lambda_0$  is simple it has a single eigenfunction  $w_0$  which we shall denote by  $\Phi = w_0$ . Note that elliptic regularity actually gives us that  $\Phi \in U$ . So we can write the nullspace  $N(L(\lambda_0))$  as

$$N(L(\lambda_0)) = \text{span}\{\Phi\} \quad (56)$$

To apply Liapunov-Schmidt we have to check whether  $L(\lambda)$  is a Fredholm operator and the following fact turns out to be true:

**Fact 7.1.** *An elliptic second-order differential operator of the form*

$$Lu = - \sum_{i,j=1}^n a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u \quad (57)$$

which satisfies the symmetry condition  $a^{ij} = a^{ji}$  is a Fredholm operator of index 0 if defined on a suitable domain.

We shall neither need nor prove this fact but it is good to know that a certain amount of generalization is possible. Instead we shall first “calculate” the adjoint of  $L(\lambda)$  formally. For this purpose let us consider the Hilbert space  $H = \{g \in L^2(D) | g(x) = 0 \text{ on } \partial D\}$  with the usual inner product for two functions  $g, h \in L^2(D)$  given by:

$$(g, h) := \int_D g(x)h(x)dx \quad (58)$$

With respect to this inner product we can use the usual integration by parts procedure to get a formal expression for the adjoint:

$$(L(\lambda)u, v) : = \int_D (L(\lambda)u)v dx = \int_D \lambda uv dx + \int_D \Delta uv dx \quad (59)$$

$$= \int_D \lambda uv dx + \int_D u(\Delta v) dx - \int_{\partial D} u \frac{\partial v}{\partial n} dx + \int_{\partial D} v \frac{\partial u}{\partial n} dx \quad (60)$$

$$= \int_D \lambda uv dx + \int_D u(\Delta v) dx = (u, L(\lambda)v) = (u, L^*(\lambda)v) \quad (61)$$

where the boundary terms vanish because we are in the Hilbert space  $H$  with zero boundary conditions. Also notice that we have indicated that  $L(\lambda)$  is **self-adjoint**, i.e.  $L = L^*$  which is not surprising as one expects the Laplacian to be a self-adjoint operator. But one should be aware that the choice of the domain makes the Laplacian self-adjoint! It is easy to construct also a non-self-adjoint version of it on another Hilbert space. Also the calculation is purely formal as we have not specified what  $u$  and  $v$  are and we should formally take them to be smooth enough to apply the Laplacian which forces a restriction on domains.

Accepting the fact that the technical steps can be made rigorous we now have:

$$N(L(\lambda)) = \text{span}\{\Phi\} = N(L^*(\lambda)) \quad (62)$$

which implies that  $\dim(N(L(\lambda))) = 1 = \text{codim}(R(L(\lambda)))$  since  $\text{codim}(R(L)) = \dim(N(L^*))$ . So that we get directly that  $L(\lambda)$  is a Fredholm operator with Fredholm index 0.

The next step is to check Crandall-Rabinowitz at the principal eigenvalue. First notice that

$$D_{u\lambda}F(0, \lambda_0)\eta = \eta \quad \Rightarrow \quad D_{u\lambda}F(0, \lambda_0) = Id \quad (63)$$

and Crandall-Rabinowitz therefore asks us to check whether  $\Phi \notin R(L(\lambda_0))$  which trivially holds since  $\Phi \in N(L(\lambda_0))$ . So Crandall-Rabinowitz applies and we get a non-trivial branch of solutions at  $\lambda_0$ . Next, we want to apply the bifurcation formulas to determine the curve  $\lambda(s)$  with near  $(0, \lambda(0)) = (0, \lambda_0)$ . The first derivative of the curve is:

$$\lambda'(0) = -\frac{1}{2} \frac{(D_{uu}^2 F(0, \lambda_0)[\Phi, \Phi], \Phi)}{(D_{u\lambda}^2 F(0, \lambda_0)\Phi, \Phi)} \quad (64)$$

where we have used already that our operator  $L(\lambda_0)$  is self-adjoint. We have to calculate the second derivative:

$$\frac{d^2}{d\alpha^2} [(\Delta + \lambda)(u + \alpha\eta) + (u + \alpha\eta)^3]_{|\alpha=0} = 2u\eta^2 \quad (65)$$

which is equal to 0 if we evaluate it at  $u = 0$  so that we can conclude that  $\lambda'(0) = 0$ . Hence we have to go to higher derivatives and calculate

$$\frac{d^3}{d\alpha^3} [(\Delta + \lambda)(u + \alpha\eta) + (u + \alpha\eta)^3]_{|\alpha=0} = 6\eta^3 \quad (66)$$

and therefore we have that

$$D_{uuu}^3 F(0, \lambda_0)[\eta, \eta, \eta] = 6\eta^3 \quad (67)$$

which means that the formula for the second derivative of our curve  $\lambda(s)$  now reads:

$$\lambda''(0) = -\frac{1}{3} \frac{(QD_{uuu}^3 F(0, \lambda_0)[\Phi, \Phi, \Phi], \Phi)}{(D_{u\lambda}^2 F(0, \lambda_0)\Phi, \Phi)} \quad (68)$$

$$= \frac{-\frac{1}{3} \int_U 6Q(\Phi^3)\Phi dx}{\int_U \Phi^2 dx} \quad (69)$$

now we note that  $Q$  projects onto the complement of the range of  $L(\lambda_0)$  and  $\Phi$  is contained in the kernel of  $L(\lambda_0)$  so that in our previous formula  $Q$  is the identity and we obtain

$$\lambda''(0) = \frac{-2 \int_U \Phi^4 dx}{\int_U \Phi^2 dx} < 0 \quad (70)$$

so that we have found that the equation exhibits a **subcritical Pitchfork bifurcation** at the principal eigenvalue  $\lambda_0$  of the negative Laplacian.

## 8 Further Reading...

The basic material we assumed on bifurcation theory is the material from the relevant course, i.e. see [Healey07]. Further reading for bifurcations of nonlinear PDE is provided by the text [Kielhöfer04] which contains the general versions and proofs of some of the results we used. The background necessary from Functional Analysis and much more is covered in [Conway94]. Basic tools for elliptic partial differential equations appear in almost any introductory text on PDEs, e.g. [Evans98], [R-R06] or [G-T83]. The model problem we used is taken from [Sattinger73] where the problem is treated using direct techniques for determining a nontrivial branch of solutions at the principal eigenvalue avoiding parts of the machinery which we used here but which is of course applicable to more general problems.

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