

Bifurcation Normal Forms - A lecture for TAM

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The idea of normal forms in the study of bifurcations is similar to the idea of normal forms for matrices (eg. the Jordan Normal Form): to obtain a simple representation of the dynamical system¹. Normal forms:

1. Make bifurcations easy to represent
2. Provide a useful framework for theory building (eg. KAM theory of quasiperiodic motion relies on them extensively)

Just like in matrices, to work with normal forms in dynamical systems we need a similarity transformation:

Definition: A dynamical system $\dot{x} = \phi(x)$, where $\phi : \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ is called locally topologically equivalent at an equilibrium point x_0 to a dynamical system $\dot{y} = \psi(y)$ on the same space if there is a homeomorphism h (an continuous invertible map with a continuous inverse) such that

1. Orbits of the one vector field to the orbits of the other in a small neighborhood of x_0 .
2. $y_0 = h(x_0)$ is an equilibrium of $\psi(y)$.
3. The direction of time is preserved

The idea of a normal form is to find a polynomial which would be orbitally equivalent to a given system around a bifurcation point. Perhaps the simplest example of a normal form is the system

$$\dot{x} = \lambda + x^2$$

that has a subcritical turning point bifurcation at $(0, 0)$
[show bifurcation diagram].

The normal form idea raises several questions which we shall address:

¹This presentation is mainly based on Guckenheimer and Holmes[1] and Kuznetsov [2]

1. Can an equivalent polynomial be found, ie. does it exist?
2. Is the normal form unique?
3. Which properties of the bifurcation determine the minimal degree of such a polynomial?

Definition: Given a bifurcation, a polynomial dynamical system $\dot{x} = f(\lambda, x)$ is called a normal form of the bifurcation at $(\lambda, x) = (\lambda_0, x_0)$ if it satisfies the generic bifurcation conditions, and is orbitally equivalent to any system satisfying the same bifurcation conditions. [show brief outline]

1 Reduction to a normal form

The first step in the reduction is to obtain the center manifold. Recall,

Center manifold theorem: In a sufficiently small neighborhood around a bifurcation point, an equilibrium point will be stable iff the center manifold is stable (as long as $M^{unstable}$ is empty). Consequently, we will be assuming that the system has already been reduced (eg. Lyapunov-Schmidt) to the low-dimensional center manifold. As well, we will assume that a coordinate transformation has transposed the bifurcation point to $(0, 0)$.

Given a system $\dot{x} = f(x)$ with $f \in C^k$, our goal is to find an equivalent polynomial system with as low degree as possible, ideally linear. To do that, we need to find a diffeomorphism $x = h(y)$. Substituting:

$$\begin{aligned} Dh(y)\dot{y} &= f(h(y)) \\ \dot{y} &= (Dh(y))^{-1}f(h(y)) \end{aligned}$$

Suppose $h(y)$ is expanded in a power series about the origin. Notice that in a case of a bifurcation, the linear part of f has zero eigenvalues. As a result, the procedure is unlikely to succeed because $f(h(y))$ will be zero for a non-trivial y . Indeed, it could be shown that there will be irremovable “resonant” terms which are precisely terms of a normal form we have been looking for. More precisely, we have the following theorem, which shows what the irremovable terms would be:

Theorem (Guckenheimer and Holmes, 3.3.1) Let $L = Df(0)x$ and let H^k be the space of vectors whose coefficients are homogeneous polynomials of degree k . Define the operator $adL(Y) = [Y, L] = DLY - DY L$ where Y is a polynomial. Let the G_k be the complementary space to of adL in H_k , thus $H_k = adL(H_k) + G_k$. Then, there exists an analytic change of coordinates in a neighborhood of the origin which transforms the system $\dot{x} = f(x)$ to

$$\dot{y} = g(y) = g^{(1)}(y) + g^{(2)}(y) + \dots + g^{(r)}(y) + o(|y|^r)$$

where $L = g^{(1)}(y)$ and $g^{(k)}(y) \in G_k$. Therefore, given a dynamical system with a sufficiently smooth flow and a null space, a polynomial dynamical system equivalent to it exists. As a result, we have proven existence of normal forms. As we will see, the proof also provides an iterative procedure for constructing such a polynomial dynamical system.

Proof The idea is to use induction, where in every step we apply the transformation

$$x = h(y) = y + P(y)$$

where $P(y)$ is a homogeneous polynomial of degree 2, 3, .. Suppose that we have completed $s - 1$ and now let $x = h(y) + P(y)$ where $P(y)$ is homogeneous of degree s . Substitution into the vector field gives:

$$(I + DP(y))\dot{y} = f^{(1)}(y) + f^{(2)}(y) + \dots + f^{(s)}(y) + Df(0)P(y) + o(|y|^s)$$

The terms of degree $< s$ are unchanged by this transformation, and the only effect are on terms of degree $\geq s$. At degree s level we have:

$$f^{(s)}(y) + DLP(y) - DP(y)L = f^{(s)} + adL(P)(y)$$

By a suitable choice of P we can remove the range of $adL(P)$ from $f^{(s)}$ making $f^{(s)} + adL(P)(y) \in G^k$.

Note The above procedure does not promise to produce the unique or the simplest polynomial system having the bifurcation. This is the concept of the hypernormal form. However, it is much harder to compute and is not discussed here further.

2 Normal form of the turning point/fold bifurcation

In this section, we consider the case of a 1D system with a turning point bifurcation at $(0, 0)^2$. We will show directly that the generic conditions for the bifurcation guarantee topological equivalence to the normal form.

Recall, a system $\dot{x} = f(\lambda, x)$ with a *generic* turning point at $(0, 0)$ satisfies the following conditions:

1. $f_x(0, 0) = 0$ (necessary)
2. $f_\lambda(0, 0) \neq 0$ (generic turning point)

²Turning points are sometimes not considered bifurcations because generically, the linearization of the vector field lacks a non-trivial kernel.

3. $f_{xx}(0,0) = \pm k^2$ with $k \neq 0$ (generic turning point - the “quadratic tangentiality” condition)

The direction of the turning point was determined by

$$\tilde{\lambda}(0) = -\frac{f_{xx}(0,0)}{f_\lambda(0,0)}$$

We will show that the normal form is

$$\begin{aligned}\dot{x} &= \lambda - x^2 (\textit{supercritical}) \\ \dot{x} &= \lambda + x^2 (\textit{subcritical})\end{aligned}$$

As expected, the normal form preserves the symmetry of the vector field, unlike the pitchfork below. In general though, studying symmetries of normal forms needs to be done carefully because as low degree polynomials, normal forms have more symmetry than a generic bifurcation of an equivalent system.

Theorem (Kuznetsov, 3.2) The system $\dot{x} = f(\lambda, x)$ with $x, \lambda \in \mathfrak{R}$ and an equilibrium $f(0,0) = 0$ is locally equivalent to $\dot{y} = \lambda \pm y^2$ if the above two conditions hold.

Proof Outline:

1. Show that near the bifurcation, the flow is equivalent $\dot{y} = \beta \pm y^2 + O(y^3)$
2. Show that near the bifurcation, $\dot{y} = \beta \pm y^2 + O(y^3)$ is equivalent to $\dot{z} = \beta \pm z^2$

Taylor expanding in x about the bifurcation point:

$$f(\lambda, x) = f(\lambda) + f_x(\lambda)x + f_{xx}(\lambda)x^2 + O(x^3)$$

where $f(\lambda) \equiv f(\lambda, 0)$. The idea now is to do coordinate transformations in order to make $f(\lambda) = \beta$, eliminate f_x and to make $f_{xx} \equiv 1$.

To start, let

$$\xi = x + \delta(\lambda)$$

Getting:

$$\begin{aligned}\dot{\xi} &= f(\lambda) + f_x(\lambda)(x - \delta) + f_{xx}(\lambda)(x - \delta)^2 + O(\xi^3) \\ &= [f(\lambda) - f_x(\lambda)\delta + f_{xx}(\lambda)\delta^2 + O(\delta^3)] \\ &+ [f_x(\lambda) - 2f_{xx}(\lambda)\delta + O(\delta^2)]\xi \\ &+ [f_{xx}(\lambda) + O(\delta)]\xi^2 \\ &+ O(\xi^3)\end{aligned}$$

Considering the first order term which we want to eliminate, let

$$F(\lambda, \delta) = f_x(\lambda) - 2f_{xx}(\lambda)\delta + \delta^2\psi(\lambda, \delta)$$

where ψ is a smooth function, and set it = 0. By the quadratic tangency condition

$$\frac{\partial F(0, 0)}{\partial \delta} = -2f_{xx}(0) \neq 0$$

Therefore, by IFT, there exists $\delta = \delta(\lambda)$ smooth such that $\delta(0) = 0$ producing

$$F(\lambda, \delta(\lambda)) = 0$$

As well, using the bifurcation condition $f_x(0, 0) = 0$ have

$$\delta(\lambda) = \frac{f_{x\lambda}(0)}{2f_{xx}(0)}\lambda + O(\lambda^2)$$

As a result [we have also expanded $f_{xx}(\lambda)$],

$$\begin{aligned} \dot{\xi} &= [f_\lambda(0)\lambda + O(\lambda^2)] \\ &+ [f_{xx}(0) + O(\lambda)]\xi^2 \\ &+ O(\xi^3) \end{aligned}$$

Let now a new parameter $\mu = \mu(\lambda)$

$$\mu = f_\lambda(0)\lambda + \lambda^2\phi(\lambda)$$

with some smooth ϕ , where μ satisfies:

$$\begin{aligned} \mu(0) &= 0 \\ \mu'(0) &= f_\lambda(0) \neq 0 \end{aligned}$$

and therefore we can apply the IFT to find a function:

$$\lambda = \lambda(\mu)$$

and therefore:

$$\dot{\xi} = \mu + [f_{xx}(0) + O(\mu)]\xi^2 + O(\xi^3)$$

Now we just rescale ξ by $b(\mu) = f_{xx}(0) + \zeta(\mu)$ for ζ smooth (recall that $f_{xx}(0) \neq 0$):

$$\begin{aligned} y &= |b(\mu)|\xi \\ \beta &= |b(\mu)|\mu \end{aligned}$$

to obtain

$$\dot{y} = \beta \pm y^2 + O(y^3)$$

[We are almost done. The last step in getting the normal form is to prove that the $O(y^3)$ correction is immaterial]:

Lemma $\dot{y} = \beta \pm y^2 + O(y^3)$ is equivalent to $\dot{z} = \beta \pm z^2$ Without loss of generality, we will consider the case of a subcritical fold. [show bifurcation diagram]

Let

$$\dot{y} = F(\beta, y) = \beta + y^2 + \psi(\beta, y)$$

where $\psi = O(y^3)$. As a first step, we will find the homeomorphism h_β which maps the equilibrium points. Namely, by IFT applied to F ($F_\beta = 1$) there is a function g such that locally

$$\beta = g(y) = -y^2 + O(y^3)$$

giving two equilibrium points for each $\beta < 0$:

$$y_1 = g^+(\beta), y_2 = g^-(\beta),$$

As well, for each $\beta < 0$ we have the equilibrium points

$$x_1 = \sqrt{-\beta}, x_2 = -\sqrt{-\beta}$$

Therefore, we introduce the homeomorphism:

$$h_\beta(x) = \begin{cases} y & \beta \geq 0 \\ a(\beta) + b(\beta)y & \beta < 0 \end{cases}$$

where the coefficients are found by enforcing that

$$y_1 = h_\beta(x_1)$$

$$y_2 = h_\beta(x_2)$$

Finally, note that the vector fields are equivalent because the flow from $x_1 \rightarrow x_2$ must get mapped to the flow $y_1 \rightarrow y_2$.

3 Normal forms of the Pitchfork bifurcations

Recall, a system $\dot{x} = f(\lambda, x)$ with a *generic* pitchfork point at $(0, 0)$ satisfies the following conditions:

1. $f_x(0, 0) = 0$ (necessary for a bifurcation)
2. $f(\lambda, 0) = 0$ (necessary for a trivial line of solutions)
3. $f_{xx}(0, 0) = 0$ (guarantees local symmetry)
4. $f_{\lambda x}(0, 0) \neq 0$ (genericity condition - key in the Crandall-Rabinowitz Theorem)

Also, the direction of the super/sub-critical pitchfork is given by the sign of

$$\tilde{\lambda}(0) = -\frac{1}{3} \frac{f_{xxx}(0,0)}{f_{x\lambda}(0,0)}$$

Indeed, the normal form satisfies all of the above conditions:

$$\begin{aligned}\dot{x} &= \lambda x - x^3 \text{ (super)} \\ \dot{x} &= \lambda x + x^3 \text{ (sub)}\end{aligned}$$

Notice that the normal forms are invariant under the operation $x \rightarrow -x$, as expected for pitchforks. Likewise, it is easy to see that the bifurcation breaks the symmetry of the flow. For the *generic* transcritical bifurcation, the conditions are:

1. $f_x(0,0) = 0$ (necessary for a bifurcation)
2. $f(\lambda,0) = 0$ (trivial line of solutions)
3. $f_{xx}(0,0) \neq 0$ (transversality - guarantees asymmetry)

$$\tilde{\lambda}(0) = -\frac{1}{2} \frac{f_{xx}(0,0)}{f_{x\lambda}(0,0)} \neq 0$$

$$\dot{x} = \lambda x \mp x^2 \text{ (trans)}$$

Q. What is the status of non-generic systems exhibiting the same type of bifurcation, like the system:

$$\dot{x} = \lambda x^3 - x^5$$

A. Regardless of whether we say that this system has a pitchfork at $(0,0)$ or not, the status of the normal form is unchanged. Recall, that the normal form must be locally equivalent to any system satisfying the genericity conditions. This system does not satisfy the transversal motion inequality.

4 Normal forms of the Hopf bifurcations

For completeness, here is the normal form of the (supercritical) Hopf bifurcation at $(\lambda, x_1, x_2) = (0, 0, 0)$ is

$$\begin{cases} \dot{x}_1 = \lambda x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = x_1 + \lambda x_2 - x_1(x_1^2 + x_2^2) \end{cases}$$

The normal form makes it easy to see that the linearization has eigenvalues $\pm i$ at the bifurcation point $\lambda = 0$:

$$\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$$

As well, it is easy to see that $\lambda = 0$ is stable.
In fact, it can equally well be written in polar form:

$$\begin{cases} \dot{\rho} = \rho(\lambda - \rho^2) \\ \dot{\theta} = 1 \end{cases}$$

References

- [1] John Guckenheimer and Phillip Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Applied Mathematical Sciences v. 42. Springer-Verlag, NY, 1983.
- [2] Yu. A. Kuznetsov. *Elements of Applied Bifurcation Theory*. Applied Mathematical Sciences v. 112. Springer-Verlag, NY, 3rd edition, 2004.