Bifurcation in the Presence of $SO(3)$ Symmetry

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Contents

1 Introduction

In these notes we will:

1. Find all the irreducible representations of $SO(3)$.

2. Derive the dimension of fixed point subspaces of all subgroups of $SO(3)$.

3. Use the above results to solve a sample bifurcation problem on a sphere.

2 Irreducible Representation of $SO(3)$

2.1 Facts about Lie Algebra

We would require the following concepts of Lie algebra to derive the irreducible representation of $SO(3)$.

1. Lie algebra $\mathfrak{g}$ of a Lie group G can be thought of as the tangent space (to G) at the identity element of G.

2. For every $L \in \mathfrak{g}$; e^{tL} is a one-parameter subgroup in G.

3. The Lie algebra of $SO(3)$, written as $\mathfrak{so}(3)$ is three dimensional consisting of all 3×3 skew-symmetric matrices.

4. If $J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $J_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ are chosen as the basis of the Lie algebra then $[J_1, J_2] = J_3$, $[J_2, J_3] = J_1$ and $[J_3, J_1] = J_2$. Where $[A, B] = AB - BA$.

5. If a euclidean coordinate system is assigned to $\mathbb{R}^3$ then one can think of $\phi_i(\alpha) = e^{\alpha J_i}$ $(i = 1, 2, 3)$ as a rotation about each axis by angle α.

6. If $\Gamma : SO(3) \to L(\mathbb{R}^n)$ is an orthogonal representation then $L_i := \frac{d}{d\alpha}\Gamma(\phi_i(\alpha)) = D\Gamma(I)J_i$ is the basis of the Lie algebra of the representation Γ of $SO(3)$. It can be shown that [1] L_i satisfy the following commutation relation: $[L_1, L_2] = L_3$, $[L_2, L_3] = L_1$ and $[L_3, L_1] = L_2$.

Lemma 2.1 $\Gamma : SO(3) \to L(\mathbb{R}^n)$ *is irreducible iff* $\mathbb{R}^n$ *is irreducible under* $\{L_1, L_2, L_3\}$

Proof Since any rotation in $SO(3)$ can be written as a composition of rotations about three different axis (say x, y, z) we can write $\psi \in SO(3)$ as

$$\psi(\alpha_1, \alpha_2, \alpha_3) = \phi_1(\alpha_1)\phi_2(\alpha_2)\phi_3(\alpha_3)$$

$$\Rightarrow \Gamma(\psi(\alpha_1, \alpha_2, \alpha_3)) = \Gamma(\phi_1(\alpha_1))\Gamma(\phi_2(\alpha_2))\Gamma(\phi_3(\alpha_3))$$

So, Γ is irreducible iff $\mathbb{R}^n$ is irreducible under $\{\Gamma(\phi_i(\alpha))|i = 1, 2, 3, \alpha \in \mathbb{R}\}$ which is true iff $\mathbb{R}^n$ is irreducible under $\{L_i|i = 1, 2, 3\}$.

2.2 Casimir Operator

Let $\Gamma : SO(3) \to L(\mathbb{R}^n)$ be an orthogonal irreducible representation. Then L_i $(i = 1, 2, 3)$ are skew symmetric. Consider the restriction of Γ to $\{\phi_3(\alpha)|\alpha \in \mathbb{R}\}$ which is a subgroup isomorphic to $SO(2)$. So, this restriction of Γ forms an irrep of $SO(2)$. But every irrep of $SO(2)$ is either a one dimensional trivial representation where $\Gamma(\phi_3(\alpha))w = w$(or equivalently $L_3 w = 0$) or a two dimensional representation such that $\Gamma(\phi_3(\theta))$ acts on this two dimensional space (call it V_m) by rotating a vector by $k\theta$. So, on this two dimensional representation, $\Gamma(\phi_3(\alpha))$ may be written as follows:

$$\Gamma(\phi_3(\alpha)) = \begin{pmatrix} \cos m\alpha & \sin m\alpha \\ -\sin m\alpha & \cos m\alpha \end{pmatrix}$$

Then, $L_3 = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}$. Now if u and v are the orthonormal basis vectors of the V_m then $L_3 u = -mv$ and $L_3 v = mv$. Let $z = u + iv$

$$L_3 z = imz$$

Define $L_+ = L_2 + iL_1$, $L_- = -L_2 + iL_1$ and $z^+ = L_+ z$, $z^- = L_- z$. Then,

$$L_3 z^+ = L_3(L_2 + iL_1)z \tag{1}$$

$$= (L_2 L_3 - L_1 + iL_2 + iL_1 L_3)z$$

$$= i(m+1)(L_2 + iL_1)z = i(m+1)z^+$$

Where we have used the fact that $[L_1, L_2] = L_3$, etc. So, L_+ acts on an eigen vector of L_3 with eigenvalue im and gives an eigenvector z^+ of L_3 with eigenvalue $i(m+1)$. Similary, it can be shown that $L_3 z^- = i(m-1)z^-$. Let's now define the Casimir operator as follows:

$$C = -L_1^2 - L_2^2 - L_3^2$$

Remark 1. C is symmetric which implies that all it's eigenvalues are real

2. C commutes with each L_i and hence each L_i and C have common eigenvectors. Also, by Schur's lemma C must be a constant multiple of identity.

Let l be the maximum value of m in Eq. (1). Then,

$$L_3 z_l = i l z_l \Rightarrow L_+ z_l = 0$$

$$C z_l = (-L_1^2 - L_2^2 - L_3^2) z_l$$

$$= (-(L_2 - iL_1)(L_2 + iL_1) - iL_3 - L_3^2) z_l$$

$$= -iL_3 z_l - L_3 L_3 z_l = l(l+1) z_l$$

That is $C = l(l+1)I$.

Lemma 2.2 *If* $\Gamma : SO(3) \to L(\mathbb{R}^n)$ *is an orthogonal irrep. then there exists some* $u_0 \neq 0$ *such that* $L_3 u_0 = 0$.

Proof Let s be the minimum value of m appearing in Eq. (1). If we can establish that $s = 0$ then we are done.

Suppose $s \neq 0$. Since s is the minimum value, there exists a $z_s \neq 0$ such that $L3 z_s = i s z_s$ and $L_- z_s = 0$. Then it is easy to see that $C z_s = s(s-1) z_s$.

But $C = l(l+1)I$. So, $s(s-1) = l(l+1)$ for all $s < s \leq l$ which is impossible. Hence, $s = 0$ and $u_0 = z_s$. ∎

Theorem 2.3 *Let $\Gamma : SO(3) \to L(\mathbb{R}^n)$ be an irreducible representation. Then $n = 2l+1$ for some $l \in \{0, 1, 2, 3...\}$. Further, Γ is uniquely determined by l upto an isomorphism.*

Proof By previous lemma, there exists a $u_0 \neq 0$ such that $L_3 u_0 = 0$. Define $z_1 = u_1 + iv_1 := L_+ u_0$, $z_{m+1} = u_{m+1} + iv_{m+1} := L_+ z_m$ for $m \in \mathbb{N}$. Since each z_m is a distinct eigenvector of L_3, the set $\{u_0, u_1, v_1, ...u_l, v_l\}$ is linearly independent. Where l is as defined previously. Let $U_l = span\{u_0, u_1, v_1, ...u_l, v_l\}$. Clearly, U_l is a $2l + 1$ dimensional subspace of $\mathbb{R}^n$ invariant under L_3. Also, since $L_+ z_q = z_{q+1}$ and $L_- z_q = z_{q-1}$, we have

$$L_1 z_q = \frac{z_{q+1} + z_{q-1}}{2i} \quad L_2 z_q = \frac{z_{q+1} - z_{q-1}}{2}$$

where $q \in \{1, 2, 3, 4, ..., l\}$ with $z_{l+1} = z_0 = 0$. Thus, U_l is invariant w.r.t L_1, L_2 and L_3. But, by Lemma (2.1), $\mathbb{R}^n$ is irreducible wrt $\{L_1, L_2, L_3\}$. So, $U_l = \mathbb{R}^n$ and $dim\ U_l = dim\ \mathbb{R}^n = n = 2l + 1$.

Uniqueness

If Γ and $\hat{\Gamma}$ are two irreducible representations of $SO(3)$. Let $L_3 u_0 = 0$ and $\hat{L}_3 \hat{u}_0 = 0$ and $u_1, v_1, u_2, v_2, ...u_l, v_l$; $\hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2, ...\hat{u}_l, \hat{v}_l$ be defined as above. Then, $U_l = span\{u_1, v_1, u_2, v_2, ...u_l, v_l\}$ $=span\{\hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2, ...\hat{u}_l, \hat{v}_l\}$. Let A be an orthogonal transformation that takes the basis $\{u_1, v_1, u_2, v_2, ...u_l, v_l\}$ to $\{\hat{u}_1, \hat{v}_1, \hat{u}_2, \hat{v}_2, ...\hat{u}_l, \hat{v}_l\}$. It's easy to see that $L_i = A^{-1} \hat{L}_i A$. That is L_i is isomorphic to L_i. Hence $\hat{\Gamma}$ is isomorphic to Γ. ∎

Theorem 2.4 $\Gamma : SO(3) \to L(\mathbb{R}^n)$ *is an irreducible representation if and only if* $dim\{u \in \mathbb{R}^n | L_3 u = 0\} = 1$.

Proof $\Rightarrow$

Since Γ is an irreducible representation, by Lemma (2.2) there exists a $u_0 \neq 0$ such that $L_3 u_0 = 0$, i.e, $dim\{u|L_3 u = 0\} > 0$. But by the construction in the previous proof, $dim\{u|L_3 u = 0\} = 1$.

$\Leftarrow$

If Γ is a representation on $\mathbb{R}^n$, we can decompose $\mathbb{R}^n$ into it's irreducible representations:

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_j$$

where each V_i is an irreducible representation of $SO(3)$. By Lemma (2.2), there exists a $u_0^i \neq 0$ in each V_i such that $L_3 u_0^i = 0$. But $dim\{u \in \mathbb{R}^n | L_3 u_= 0\} = 1$ implies $i = 1$. That is $\mathbb{R}^n$ is irreducible under $SO(3)$. ∎

3 Spherical Harmonics

Let $u : \mathbb{R}^3 \to \mathbb{R}^3$. Consider the Laplace's equation

$$\Delta u = 0$$

With the substitution $u(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$, we have the following equations

$$r^2 R'' + 2rR' - l(l+1)R = 0 \tag{2}$$

$$\Theta'' + m^2\Theta = 0 \tag{3}$$

$$(\sin^2\phi)\Phi'' + (\sin\phi\cos\phi)\Phi' + (l(l+1)\sin^2\phi - m^2) = 0 \tag{4}$$

The solutions to the angular part $Y_{lm}(\theta, \phi) = \Theta(\theta)\Phi(\phi)$ are called *surface spherical harmonics* and are given by

$$Y_{lm}(\theta, \phi) = A\cos m\theta P_l^m(\cos\phi) + B\sin m\theta P_l^m(\cos\phi)$$

where $l \in \{0, 1, 2, 3, ...\}$, $m \in \{-l, -(l-1), ..., l-1, l\}$ and P_l^m are *associated Legendre polynomials* which are the solutions of

$$(1-x^2)y'' - 2xy' + \left(\ell[\ell+1] - \frac{m^2}{1-x^2}\right)y = 0$$

.

Definition $U_l = \{Y_l | Y_l = \sum_{m=-l}^{l} c_m Y_{lm}\}$ is called the space of spherical harmonics of order l.

Remark

1. $Y_l m$ $m \in \{-l, -(l-1), ..., l-1, l\}$ form an orthogonal basis of U_l.

2. U_l is also the set of all homogeneous polynomials in (x, y, z) on a sphere.

3.1 Maxwell-Sylvester's Theorem

Observe that $\Delta(\frac{1}{r}) \equiv 0$. So $\Delta \frac{\partial}{\partial x} \frac{1}{r}$. In fact,

$$\Delta(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z})\frac{1}{r} \equiv 0$$

for any real a, b, c. The Maxwell-Sylvester's theorem states that any spherical harmonic of order l can be written as follows [4]:

$$Y_l = C\ r^{l+1} \prod_{s=1}^{l} ((l_s \frac{\partial}{\partial x} + m_s \frac{\partial}{\partial y} + n_s \frac{\partial}{\partial z})\frac{1}{r}$$

where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and $(l_s \frac{\partial}{\partial x} + m_s \frac{\partial}{\partial y} + n_s \frac{\partial}{\partial z})$ is the directional derivative along a vector with directional cosines (l_s, m_s, n_s). Substituting $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\phi$ gives us Y_l expressed as a function of θ and ϕ. Writing the above equation in a more compact form,

$$Y_l = Cr^{l+1} \frac{\partial^l}{\partial \mathbf{v}_1 \partial \mathbf{v}_2 \partial \mathbf{v}_3 ... \partial \mathbf{v}_l} \frac{1}{r} \tag{5}$$

where $\mathbf{v}_i$ are $\mathbb{R}^3$-unit vectors with components (l_i, m_i, n_i).

Observe that if $\{\mathbf{v}_1, \mathbf{v}_2 \mathbf{v}_3, ... \mathbf{v}_l\}$ are chosen such that the set is invariant under any subgroup of $SO(3)$ then Y_l is invariant with resepect to that subgroup. It has been shown [5] that if Y_l has the symmetries of a regular polyhedron (tetrahedron, cube, icosaherdon) then Y_l can be expressed in terms of certain invariant operators.

3.2 U_l as a representation of $SO(3)$

Definition Let the representation of $\Gamma : SO(3) \to L(U_l)$ be defined as follows:

$$\Gamma(R)Y_l(\theta, \phi) := Y_l(R^{-1}(\theta, \phi)) \tag{6}$$

Theorem 3.1 $\Gamma : SO(3) \to L(U_l)$ *is an irreducible representation*

Proof Consider the set $V = \{Y_l \in U_l | L_3 Y_l = 0\}$. Then $V = \{Y_l \in U_l | \Gamma(\phi_3(\alpha))Y_l = Y_l\}$. That is, V consists of all spherical harmonics which are fixed by rotations about an axis (say, the z-axis). Where

$$\Gamma(\phi_3(\alpha))Y_l(\theta, \phi) = Y_l(\phi_3(\alpha)^{-1}(\theta, \phi)) \tag{7}$$

But by Eq. (5),

$$Y_l = Cr^{l+1} \frac{\partial^l}{\partial \mathbf{v}_1 \partial \mathbf{v}_2 \partial \mathbf{v}_3 ... \partial \mathbf{v}_l} \frac{1}{r}$$

So, for $\Gamma(\phi_3(\alpha))Y_l = Y_l$, all the vectors $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_l)$ must be parallel to the $z - axis$. Thus, V is one dimensional and by Theorem (2.4), Γ is an irreducible representation.

Remark Thus, by Theorems (2.3) and (3.1), *every irreducible representation of* $SO(3)$ *is isomorphic to* U_l *for some* $l \in \{0, 1, 2, ...\}$.

4 Subgroups of $SO(3)$

Theorem 4.1 *Every closed subgroup of* $SO(3)$ *is conjugate to one of* $SO(3)$, $O(2)$, $SO(2)$, D_n $(n \geq 2)$, Z_n $(n \geq 2)$, $\mathbb{T}$, $\mathbb{O}$, $\mathbb{I}$, I.

Proof See [2] p. 189.

Theorem 4.2 *1.* $D_n = \dot\cup\, {}^nZ_2 \,\dot\cup\, Z_n$

2. $\mathbb{O} = \dot\cup\, {}^3Z_4 \,\dot\cup\, {}^4Z_3 \,\dot\cup\, {}^6Z_2$

3. $\mathbb{T} = \dot\cup\, {}^4Z_3 \,\dot\cup\, {}^3Z_2$

4. $\mathbb{I} = \dot\cup\, {}^6Z_5 \,\dot\cup\, {}^{10}Z_3 \,\dot\cup\, {}^{15}Z_2$

Proof See [3] p. 105.

5 Fixed Point Spaces

Definition $d(\Sigma) = dim(Fix_{U_l}(\Sigma))$ is the dimension of fixed point space of U_l under the subgroup $\Sigma \in SO(3)$.

Theorem 5.1 *Let $SO(3)$ act irreducibly on U_l. The dimension of the fixed point spaces of closed subgroups of $SO(3)$, are :*

1. $d(SO(2)) = 1$
2. $d(Z_m) = 2[l/m] + 1$
3. $d(D_m) = \begin{cases} [l/m] & l \ odd \\ [l/m] + 1 & l \ even \end{cases}$
4. $d(\mathbb{T}) = 2[l/3] + [l/2] - l + 1$
5. $d(\mathbb{O}) = [l/4] + [l/3] + [l/2] - l + 1$

6. $d(\mathbb{I}) = \quad [l/5] + [l/3] + [l/2] - l + 1$

7. $d(O(2)) = \begin{cases} 0 & l \ odd \\ 1 & l \ even \end{cases}$

Proof 1. We saw in Theorem (2.3) that

$$U_l = H_0 \oplus H_1 \oplus \cdots H_k$$

where H_0 is the trivial irrep of $SO(2)$ on U_l with dimension one. The action of $\theta \in SO(2)$ on H_k is to rotate vectors in H_k by $k\theta$. Also, note that the dimension of H_k $(k \neq 0)$ is two. So,

$$Fix_{U_l}(SO(2)) = H_0 \Rightarrow d(SO(2)) = 1$$

2. Z_m will fix a non-zero vector in H_k (dim=2) iff m divides k. There are $[l/m]$ integers k between 1 and l which are multiples of m. Of course, H_0 is fixed by every Z_m. Hence (2) follows.

3. Recall, $d(\Sigma) = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} tr(\sigma)$. If $\Sigma = H_1 \dot{\cup} H_2 \dot{\cup} ... \dot{\cup} H_k$ then

$$d(\Sigma) = \frac{1}{|\Sigma|} \left[\sum_{i=1}^{k} \sum_{h \in H_i} tr(h) - (k-1) tr\ I \right]$$

$$= \frac{1}{|\Sigma|} \left[\sum_{i=1}^{k} |H_i| d(H_i) - (k-1) tr\ I \right]$$

$$= \frac{1}{|\Sigma|}\left[\sum_{i=1}^{k} |H_i|d(H_i) - (k-1)dimV\right]$$

where we have used the fact that $tr\ I$ is the dimension of the representation.

So, if $D_m = \dot{\cup}\,^m Z_2 \,\dot{\cup}\, Z_m$,

$$d(D_m) = \frac{1}{|D_m|}\left[m|Z_2|d(Z_2) + |Z_m|d(Z_2) - m\ dim\ V_l\right]$$

$$= \frac{1}{2m}\left[2m(2[l/2]+1) + m(2[l/m]+1) - m(2l+1)\right]$$

$$= 2[l/2] + [l/m] - l + 1$$

$$= \begin{cases} [l/m] & l\text{odd} \\ [l/m]+1 & l\text{even} \end{cases}$$

4. $\mathbb{T} = \dot{\cup}\,^4 Z_3 \,\dot{\cup}\, Z_2$,

$$d(\mathbb{T}) = \frac{1}{|\mathbb{T}|}\left[4|Z_3|d(Z_3) + 3|Z_2|d(Z_2) - 6\ dim\ V_l\right]$$

$$= 2[l/3] + [l/2] - l + 1$$

5. Similary, one can establish (5) and (6).

6. To find $d(O(2))$ let us first observe that

$$Fix_{V_l}(O(2)) = \bigcap_{m=1}^{\infty} Fix_{V_l}(D_m)$$

The above equation is true because a vector fixed by $O(2)$ is fixed by every D_m. Union of all D_m is dense in $O(2)$, so continuity of action of $O(2)$ implies a vector fixed by every D_m is fixed by $O(2)$. Result (7) follows by letting $m \to \infty$.

6 Bifurcation analysis of $\Delta u + f(\lambda, u) = 0$

Consider

$$\Delta u + f(\lambda, u) = 0$$

where $C^{2,\alpha} \ni u : S^2 \to \mathbb{R}^n$ and $C^2 \ni f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Also assume that $f(\lambda, 0) \equiv 0$ and f is $SO(3)$-equivariant.

Define

$$F(\lambda, u) := \Delta u + f(\lambda, u) = 0$$

Then,

$$L(\lambda, 0)\hat{u} = DF_u(\lambda, 0)\hat{u} = \Delta \hat{u} + f_u(\lambda, 0)\hat{u} = 0$$

Solutions $\hat{u}$ of the above equation exists iff

$$f_u(\lambda, 0) = l(l+1) \text{ for } l \in \{0, 1, 2, 3, ...\} \tag{8}$$

Then, $\mathcal{N}(L(\lambda_l)) = U_l$ where λ_l is the solution of Eq. (8).

Note that according to Theorem (3.1), $SO(3)$ acts irreducibly on $\mathcal{N}(L(\lambda_l))$

6.1 Solution for $l = 6$ with Octahedral Symmetry

Choose $l = 6$, then by Theorem (5.1),

$$d(\mathbb{O}) = [6/4] + [6/3] + [6/2] - l + 1 = 1$$

That is we have a one dimensional fixed point space with octahedral symmetry. So, by the equivariant branching lemma, non-trival solution branch exists locally if Crandall-Rabinowitz crossing condition is satisfied. That is, if

$$\langle \phi, L'(\lambda_6)\phi \rangle = f_{\lambda u}(\lambda_6, 0)\langle \phi, \phi \rangle \neq 0$$

$$\text{or if} f_{\lambda u}(\lambda_6, 0) \neq 0$$

where $\phi \in \mathcal{N}(L(\lambda_6))$[1]

The eigenfunctions ϕ corresponding to $l = 6$ were computed using the scheme presented in [5] and are presented below

[1]since L is self adjoint $\mathcal{N}(L^T(\lambda_6)) = \mathcal{N}(L(\lambda_6))$.

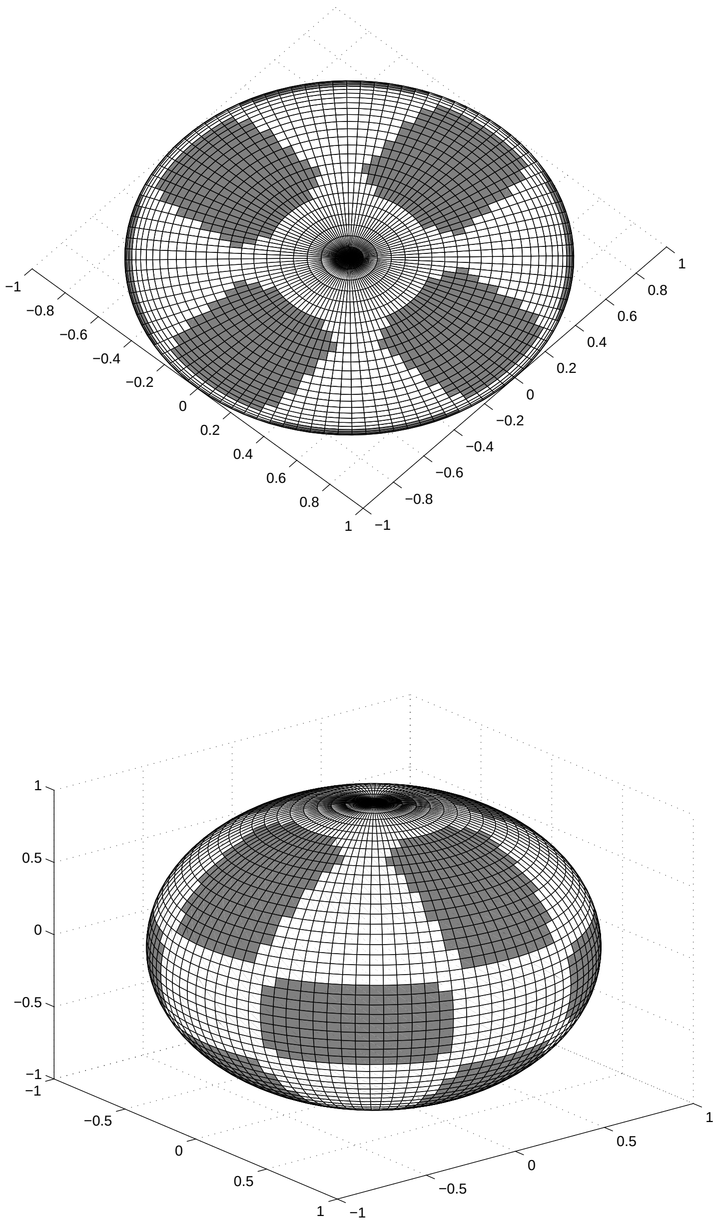

Figure 1: Eigenfunctions for $l = 6$: Dark and bright areas represent negative and positive values of ϕ respectively.

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