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Volume 1.

by

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Preface

These two volumes are the basis of the sophomore mathematics program for the students in the College of Engineering at Cornell University. It is not expected that all this material will be covered by all the students in the seven semester hours allotted to this portion of the curriculum. There is room for some selection of topics, and in particular the more difficult topics and problems are intended only for the best students.

The freshman year contains eight semester hours of mathematics - essentially the first fifteen chapters of Calculus and Analytic Geometry, by C. B. Thomas, Jr., (Addison-Wesley Publishing Co., Reading, Mass., Third Edition, 1960). In additon to this mathematical background the freshman student learns how to program in the Cornell Computing Language (CORC) and how to have his problems run in the Computing Center. Completion of most of the classical topics of differential and integral calculus in the freshman year permits a new approach to the mathematics of the sophomore year.

The sophomore program contains features that we believe will develop into significant advances in mathematical education. The arrangement of courses, syllabi, and the interdisciplinary nature of the effort are in accord with contemporary and future developments in the undergraduate mathematics education of engineers and scientists.* The dominant features include:

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*See "Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists" Report Number 5, January 1962, Committee on the Undergraduate Program in Mathematics, Mathematical Association of America.

1. The opportunity to interweave physical interpretation and engineering applications, hence stirring the students' imagination and motivation and reducing the separation between theory and practice.
2. A substantial period of time devoted to linear transformations and matrices, linear equations, quadratic forms and elementary eigenvalue theory.
3. The introduction of vector field theory in the sophomore year so that students will be prepared for the field theory approach in junior year applied science courses in fluid and solid mechanics, transport phenomena, and electromagnetism.
4. The inclusion of numerical and machine techniques in such mathematical topics as differential equations, infinite series, and linear algebra.
5. An interdisciplinary effort at both the planning and working levels whereby professional mathematicians and engineers create a course having balance between mathematical rigor and sophistication and physical insight and motivation.
6. Coordination of mathematics with the sophomore mechanics and electrical science courses.
7. A mathematics program upon which the student will be encouraged to build in subsequent years.

If a sophomore course constructed along these lines proves successful it is hoped to extend the program at a later date to include a reworking of the freshman material, so as eventually to provide a coordinated program for the first two years of engineering mathematics.

Of the many people who helped us we wish to thank particularly our colleagues Professor G. R. Livesay and Professor N. DeClaris, who were heavily involved in the initial planning; Mrs. Sandria Kerr, Mr. Neal Plotkin, Mr. Terry Gardner and Mr. Frank Moon, graduate students who served as critics and editors; Miss. Virginia Cranch, who did most of the final typing; and Mr. Harry Gleason, our most obliging printer. Our greatest thanks are due to the National Science Foundation for its generous support.

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CHAPTER 1

Differential Equations

1. Introduction

The study of differential equations originated in the middle of the sixteenth century when early scientists attempted to describe mathematically the physical laws governing their environment. As investigations in astronomy, solid and fluid mechanics, optics and heat flow deepened, the need arose for the solution of mathematical equations involving rates of change between the pertinent but elusive variables. After Newton and Leibnitz laid the foundations of calculus in the last half of the seventeenth century, a succession of physical problems of both theoretical and practical importance yielded to orderly description. The scope and depth of penetration of our understanding of the biological and physical world has continued to give rise to a wealth of problems involving relationships between functions and their derivatives. The study of these mathematical questions led to the development of that branch of mathematics known as differential equations. The mathematical study of differential equations continues to give precision, rigor and insight to the description of physical processes. Thus, for the engineer and scientist the study of differential equations is a three-fold process. The phenomenon must be described mathematically in accordance with the laws governing the process. The resulting differential equations must then be solved. Finally the results are interpreted in terms of the physical quantities being studied.

Example 1.1 A Problem in Heat Transfer

The exchange of heat between different media is of primary importance in all devices which are made to control thermal energy. Heat is transmitted by three processes, known as conduction, convection and radiation. Conduction is the process whereby energy is transmitted due to the excitation and resultant vibrational motion of matter. Convection results when matter, which has been excited by the transfer of heat, is free to move and hence carry energy by virtue of its freedom to flow. Heat transfer by radiation is the result of the propagation and absorption of electromagnetic energy between bodies. One must be able to discriminate between these mechanisms or know when not to attempt to discriminate. The laws governing the three types of heat transmission are different.

Consider the heating or cooling of a body initially at temperature T_o in an environment having constant temperature T_e. An example would be a cider jug put outside on an evening in the fall. We are immediately confronted with the problem of discrimination, for it is not obvious by which method or methods the heat is transferred. In the case of the jug in air it is not clear whether a film of air remains close to the surface and heat is conducted through this layer, or whether local convection currents are established due to density changes created by the heat flow, or whether convection due to atmospheric air flow dominates, or whether radiation is important, or perhaps a combination of effects is present. Newton (1643-1727) postulated that in such cases the rate of heat transfer, Q (BTU/hr.), between a body at temperature $T(^\circ F)$ and environment at temperature $T_e(^\circ F)$ is

$$Q = hA\ (T_e - T) \tag{1.1}$$

where A is the surface area $(ft.^2)$ through which heat flows and h a film heat-transfer coefficient which depends on the flow conditions, body shape, and physical and chemical nature of the surface. Experiment has shown that for modest temperature ranges $(T_o - T_e)$ and flow conditions, (1.1) is a valid description with conduction and convection the dominant means of heat transfer. If $T_e > T$ then heat flows to the body, while if $T_e = T$ then the body and environment are in thermal equilibrium. In the time interval from t to $t+\Delta t$ the increment of heat stored in the body is given by

$$c\rho V\ \{T(t+\Delta t) - T(t)\} \tag{1.2}$$

where c is the specific heat of the body material, ρ its density, V the volume of the body and $T(t)$ the temperature at time t. The instantaneous rate of heat storage in the body is

$$\lim_{\Delta t \to 0} c\rho V\ \frac{\{T(t+\Delta t) - T(t)\}}{\Delta t} = c\rho V T'(t) \tag{1.3}$$

or

$$c\rho V\ \frac{dT}{dt}\ . \tag{1.4}$$

From the law of conservation of energy, the rate of heat added, eq.

(1.1), must equal the rate of heat stored (1.4) so that

$$\frac{dT}{dt} = k(T_e - T) \tag{1.5}$$

where $k = hA/c\rho V$. The introduction of constants such as h in (1.1), $c\rho V$ in (1.2), or k in (1.5) is always required in analytical descriptions of physical processes. Constant h in Newton's law of cooling, (1.1), is a proportionality constant which includes several geometric and physical effects. In (1.2) the product $c\rho V$ expresses the known physical result that a material has the capacity to store heat. Constant k can be considered to be the overall rate constant of the jug, for equation (1.5) when solved for k gives the time rate of change of temperature of the jug per degree temperature difference between the jug and its environment.

A careful discussion of the mathematics involved in solving equation (1.5) is deferred until Section 5; here we give a treatment in the spirit of Thomas, Section 6-11. Put (1.5) in the form

$$\frac{dT}{T - T_e} = -k\,dt.$$

Integration gives either

$$\log(T - T_e) = -kt + C \tag{1.6}$$

or

$$\log(T_e - T) = -kt + C, \tag{1.7}$$

depending on whether $T - T_e$ is positive or negative. In our case of a cooling process, we start with the initial condition $T = T_o > T_e$

when $t = 0$, so we must use (1.6). Putting $t = 0$ gives

$$\log(T_o - T_e) = C,$$

and substituting this value for C in (1.6) gives

$$\log(T - T_e) = -kt + \log(T_o - T_e),$$

which reduces to

$$\log\frac{T - T_e}{T_o - T_e} = -kt$$

or, finally, to

$$T - T_e = (T_o - T_e)\, e^{-kt}. \qquad (1.8)$$

The reader is advised to carry through the similar procedure for the heating process, $T_o < T_e$, starting with equation (1.7). The final result should be the same, i.e. equation (1.8), the difference being that in the cooling process the two sides of (1.8) are positive, whereas in the heating process they are negative. After a sufficiently large period of time the e^{-kt} term becomes negligible, $T \longrightarrow T_e$, and thermal equilibrium is reached. We conclude that in such a cooling or heating process the temperature never "overshoots" the environmental temperature. The results are shown graphically in Figure 1.1.

If the cider jug is initially at 70°F, the night air temperature 20°F, and the cider temperature 60°F after one hour, we can determine when the student should set his alarm clock in order that the cider be rescued. The rate constant k is determined from the knowledge of the temperature after one hour. Hence, (1.8) becomes

$$(60 - 20) = (70 - 20)\, e^{-k},$$

which when solved for k gives

$$k = 0.223\ {}^\circ\text{F/hr. per }{}^\circ\text{F}.$$

Thus, (1.8) is

$$T - 20 = 50e^{-0.223t}.$$

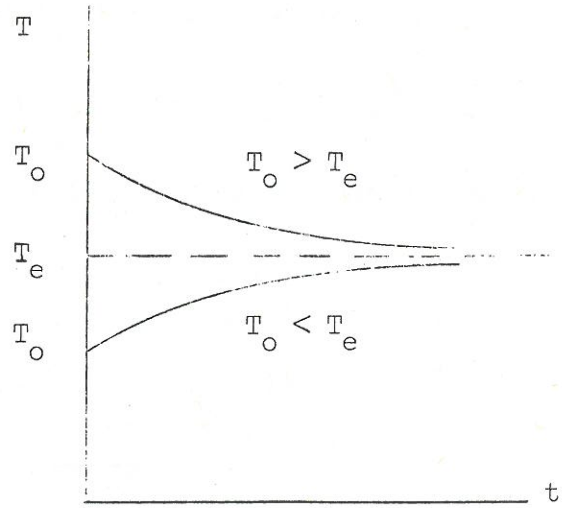

Figure 1.1

If we assume that the freezing point of cider is 32°F, then we can determine the time t required for T to achieve this value:

$$(32 - 20) = 50\, e^{-0.223t} \quad \text{and} \quad t = 6.40 \text{ hrs.}$$

Example 1.2. Light Absorption in the Ocean

When man attempts to explore underwater regions he is faced with the difficult problem of visibility. Incident light radiation is absorbed and scattered by a thin layer of water in direct proportion to both the amount of incident radiation and the thickness of the layer. The proportionality coefficient c_1 depends on the properties of the water layer, including its turbidity, so that it might vary with depth. Many regions of the ocean support luxuriant plant and animal life, especially in those strata where the intensity of light is high. Assume that experiment shows that the amount of light absorbed by a layer of plants and animals is proportional to both the thickness of the layer and some power p of the intensity of light incident on the layer. Both the proportionality constant c_2 and power p must be determined from experiment. The problem is to find at what depth vision is no longer possible without artificial means.

Figure 1.2 shows a layer of ocean at depth x (ft.) below the surface. At the surface the incident radiation is $I(0)=I_0$. The law of conservation of energy when applied to the layer of thickness Δx must equate the amount of incoming radiation $I(x)$ to that leaving, $I(x+\Delta x)$, plus the radiation absorbed or scattered by the water, plants and animals. According to the processes previously discussed this gives

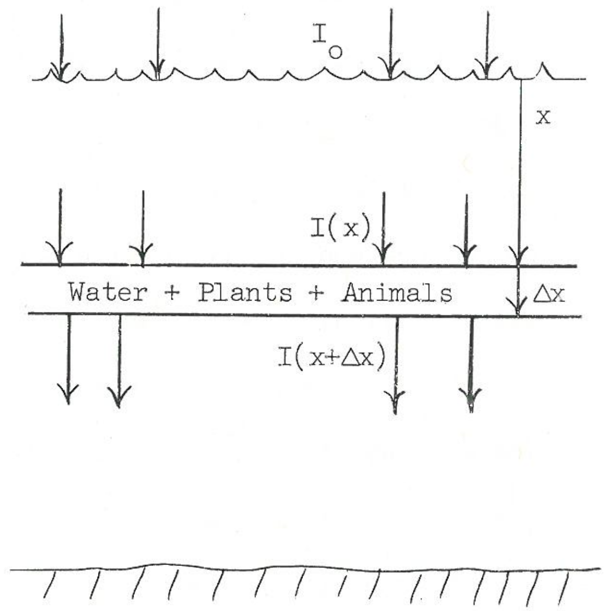

Figure 1.2

$$(1.9) \qquad I(x) = I(x+\Delta x) + c_1(\bar{x})I(x)\Delta x + c_2 I^p(x)\Delta x$$

where $\bar{x}$ is a depth in the layer between x and $x+\Delta x$ such that the term $c_1(\bar{x})I(x)\Delta x$ is the intensity absorbed or scattered by the water in this layer. This equation can be rewritten as

$$(1.10) \qquad \frac{I(x+\Delta x) - I(x)}{\Delta x} = -c_1(\bar{x})I(x) - c_2 I^p(x).$$

We will, of course, take the limit of (1.10) as $\Delta x \longrightarrow 0$ and thereby arrive at the differential equation governing the light intensity at depth x. However, in doing this we should be cognizant of any tacit assumptions made. We wish to emphasize that we are constructing a mathematical model of a situation which contains both physical and biological processes. The energy absorption previously described assumed a continuous composition of the medium which is, in fact, discrete. The assumption gives an excellent model for the water alone which contains a large number* of water molecules per cubic foot, a slightly questionable model when turbidity is included (for the nature and density of the sediments would affect the result), and a decidedly questionable model for the plants and animals, which on the scale of water molecule size are discrete and free to cluster and move about. The scientist takes one of two viewpoints when confronted with this common dilemma in describing nature. He abandons the original problem and concentrates on a detailed study of radiation absorption and scattering by liquids, plants and animals, or he performs his experiments on samples and under conditions representative of the actual conditions, recognizing that there are limitations to the validity of his model and accuracy of his results. We will adopt the second viewpoint, take the limit of (1.10) as $\Delta x \longrightarrow 0$ even though we may cut through fish in the process, and arrive at the differential equation

$$\frac{dI}{dx} = -c_1(x)I - c_2 I^p \qquad (1.11)$$

with initial condition $I(0)=I_o$. Equation (1.11) does not possess

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*About 9.45×10^{26}

the simplicity of (1.5), in which the variables were separable and the equation easily integrated.

Some special cases are interesting to examine, for they shed light on the properties of the solution to the original problem. If $c_2=0$ there are no plants or animals present and (1.11) can be put in the form

$$\frac{dI}{I} = -c_1(x)dx,$$

which integrates to

$$\log I = -F(x) + C, \tag{1.12}$$

where

$$F(x) = \int c_1(x)dx$$

is an indefinite integral of $c_1(x)$. (Since I is never negative we have only one form of (1.12) to consider.) By the same procedure as was used in Example 1.1 equation (1.12) can be reduced to

$$I = I_o e^{F(0)-F(x)}. \tag{1.13}$$

The intensity decreases with depth; the exact rate of decay depends on the form of F(x). Thus, at a certain depth the threshold of visibility is reached. If the absorption coefficient c_1 is constant, then the intensity varies as $I_o e^{-c_1 x}$. The term $c_2 I^p$ can only reduce the intensity so that this will alter the decay curve to something other than an exponential and cause the threshold of visibility to occur at a shallower depth. If c_1 is constant the original equation (1.11) can be put in the form

$$(1.14) \qquad \int_{I_o}^{I} \frac{dI}{c_1 I + c_2 I^p} = - \int_{o}^{x} dx$$

but this is of limited help because without a change of variable the left-hand side can be integrated in closed form only for special values of p. For example, with $p=2$, equation (1.14) can be integrated in closed form. Using $c_1=0.80\text{x}10^{-2}$, $c_2=0.10\text{x}10^{-2}$, $I_o=1$ and $x=225$ ft. integration yields a value $I=0.15$. Even if (1.14) cannot be integrated in closed form it can be integrated numerically and the relation between I and x determined.

For the general equation (1.14), it is not obvious at this stage that a solution can be found. The fact that there is a unique solution is guaranteed by Theorem 2.1.

2. The Fundamental Theorem of The First Order Equation - Euler's Method

An equation of the form

$$(2.1) \qquad \frac{dy}{dx} = f(x,y)$$

is called a first order differential equation. **A** function $y(x)$ is called a solution of equation (2.1) if, upon substituting $y(x)$ for y, (2.1) becomes an identity. For example, in the cooling problem, the function $T = T_e+(T_o-T_e)e^{-kt}$ is a solution of the differential equation $\frac{dT}{dt} = k(T_e-T)$. Differential equation (2.1) together with the initial condition $y(x_o) = y_o$ is known as the first order propagation (or first order initial value) problem, because as we shall see it defines a state of affairs whereby the solution can be generated by a "marching forward process" from the initial value y_o. Physically, such problems arise when mass or energy is transmitted by the law of transmission (differential equation) from the initial state.

Theorem 2.1* Let $f(x,y)$ and $\frac{\partial f}{\partial y}$ be continuous in a domain D of the xy-plane. Let (x_o,y_o) be a point in D. Then there is a number $H > 0$ such that on the interval $|x-x_o| < H$ there exists one and only one function $y(x)$ which is a solution of the differential equation (2.1) and satisfies the initial condition $y(x_o) = y_o$.

This theorem when applied to equation (1.11) of the light absorption problem assures us that there is a unique function $I(x)$ satisfying (1.11) and the initial condition $I(0)=I_o$. However, finding this solution is a different matter. In general, (2.1) cannot be manipulated in such a way that a closed form solution can be found. In such cases one resorts to numerical methods.

We seek a means of solution for the class of propagation problems represented by

$$\frac{dy}{dx} = f(x,y) \tag{2.1}$$

with initial condition

$$y(x_o) = y_o \, . \tag{2.2}$$

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*For a proof of Theorem 2.1 see R.P. Agnew, Differential Equations, McGraw-Hill Book Co., New York, 1960, Chapter 15; or W.S. Kaplan, Ordinary Differential Equations, Addison-Wesley Publishing Co., Reading, Mass., 1958, Chapter 12.

Instead of seeking a closed form solution which could only be obtained for special forms of $f(x,y)$ (cf. Sections 5 and 6) we will develop a method due to Euler (1707-1783) in which we find approximate values y_s of the solution for a discrete, successive set of points x_s. Once we overcome our prejudice for closed form solutions in terms of elementary functions, the appeal of a discrete method is great, especially if it can be adapted to a high speed computing machine. Euler's method consists of <u>approximating</u> the derivative dy/dx by the difference quotient

$$(2.3) \quad \left(\frac{dy}{dx}\right)_{s-1} = \frac{y_s - y_{s-1}}{x_s - x_{s-1}}$$

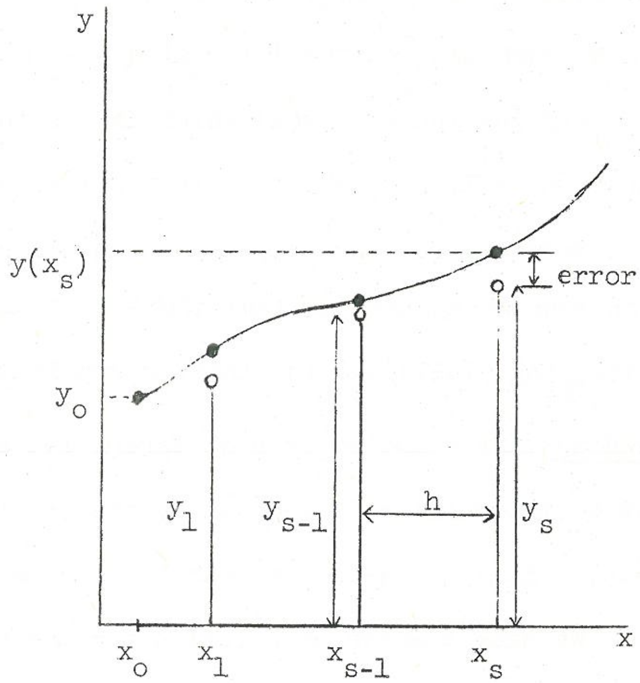

Figure 2.1

and if this approximate expression for dy/dx is used in (2.1), that equation gives

$$(2.4) \qquad y_s = y_{s-1} + h\, f(x_{s-1}, y_{s-1})$$

where $h = x_s - x_{s-1}$; see Figure 2.1. Equation (2.4) is called a <u>recursion formula</u>. The process of generating successive values for y_s begins by using the initial value, from (2.2), in (2.4) to yield

$$y_1 = y_o + h\, f(x_o, y_o)$$

and then continues with the use of recursion formula (2.4). Function $f(x,y)$ may be given empirically or in tabular form rather than as an analytical expression. The simplicity of the method is such that the approximate equations (2.3) and (2.4) might be viewed with suspicion and skepticism for they imply certain errors inherent in the process. However, it can be shown that as $h \longrightarrow 0$ the approximate solution converges to the true solution. One should be warned however that as $h \longrightarrow 0$ the number of computations increases as $1/h$ and consequently the round-off error can become unacceptably large.

Example 2.1. A Cooling Problem

Although the cooling problem of Example 1.1 resulted in the closed form solution (1.8) , it is instructive to illustrate the Euler method in a case where the exact solution is known. The jug initially at 70° F is placed in an environment of 20° F and cools with a rate constant $k = 0.223$. In this case equation (2.1) has the form given by (1.5) which is

$$\frac{dT}{dt} \doteq -0.223(T-20).$$

Using (2.3) to approximate the derivative dT/dt, recursion formula (2.4) becomes

s	t_s	T_s	Error
0	0	70	0
1	0.20	67.77	0.048
2	0.40	65.64	0.093
3	0.60	63.60	0.134
4	0.80	61.65	0.181
5	1.00	59.79	0.215
10	2.00	51.67	0.335
15	3.00	45.20	0.416
20	4.00	40.10	0.437
25	5.00	35.97	0.430
30	6.00	32.73	0.393
31	6.20	32.15	0.393
32	6.40	31.61	0.387

Table 2.1

$$T_s = T_{s-1} - 0.223h(T_{s-1} - 20).$$

We know that the time for the jug to reach 32° F is over 6 hours so that a choice of $h = 0.20$ hr. is a relatively small time increment. With $T_o = 70$ the first step with the recursion formula gives

$$T_1 = 70 - 0.223(0.20)(70 - 20) = 67.77.$$

Equation (1.8) gives the exact result

$$T_1 = 20 + (70 - 20)e^{-0.223(0.20)} = 67.82.$$

The results are tabulated in Table 2.1 which gives the temperature T_s calculated by the Euler method at time t_s. The last column of Table 2.1 gives the difference between the exact result and that obtained from the Euler method; hence, the last column is the error. It is seen that $T_s = 32°$ F occurs when t_s is approximately 6.25 hrs.; the exact value for the time is 6.40 hrs. so the Euler method predicts the time to freezing within about 2.3%. The error in the temperature at a given time t_s is seen to never exceed about 1.2%.

Example 2.2. Light Absorption in the Ocean

Euler's method will now be used to obtain a solution to the ocean visibility problem formulated in Example 1.2. Assume experiment shows that for the process of light absorption by the plants and animals $p=2.0$ and $c_2=0.10 \times 10^{-2}$. The proportionality coefficient c_1 varies with depth according to the values given in Table 2.2; it rises from a surface value of 0.75×10^{-2} to 0.87×10^{-2} at depths greater than 190 ft. Equation (1.14) becomes

$$(2.5) \quad \frac{dI}{dx} = -c_1(x)I - 0.10 \times 10^{-2} I^2$$

and cannot be solved in a closed analytical form. If the Euler method is used, recursion formula (2.4) gives

$$I_s = I_{s-1} - c_1 h I_{s-1} - 0.10 \times 10^{-2} h I_{s-1}^2 .$$

If the surface light intensity is taken as 1 so that $I(0) = 1$ and with depth increments of 5 ft. so that $h = 5$, the first value is given by $I_1 = 1 - (0.75 \times 10^{-2})(1)(5) - (0.10 \times 10^{-2})(1)^2(5) = 0.9575$. If I_1 is

s	Depth ft.	$c_1(x) \times 10^2$	I_s
0	0	0.75	1.000
1	5	0.75	0.958
2	10	0.75	0.918
3	15	0.75	0.879
4	20	0.75	0.842
5	25	0.76	0.806
6	30	0.76	0.772
7	35	0.76	0.740
8	40	0.76	0.709
9	45	0.77	0.679
10	50	0.77	0.651
11	55	0.77	0.624
12	60	0.77	0.598
13	70	0.79	0.547
14	80	0.81	0.500
15	90	0.83	0.456
16	100	0.83	0.416
26	200	0.87	0.164
27	210	0.87	0.149
28	220	0.87	0.136
29	230	0.87	0.124

Table 2.2

rounded off to three decimal places then $I_1 = 0.958$ and $I_2 = 0.958 - (0.75\times10^{-2})(0.958)(5) - (0.10\times10^{-2})(0.958)^2(5) = 0.918$. If the process is repeated the I_s values in Table 2.2 are generated. This calculation shows that if the threshold of visibility is at an intensity of 0.15 this value occurs at a depth of 210 ft. If c_1 is constant, (2.5) can be integrated in closed form; with $c_1 = 0.80\times10^{-2}$ the solution gives a threshold intensity of 0.15 at a depth of 223 feet. The Euler method is simple, direct, and amenable to repetitive numerical computation. It is flexible, for the increment size h is not necessarily fixed. Table 2.2 shows that smaller increments were taken in the region where the slope was largest. Further, it is seen that $c_1(x)$ may be obtained empirically at discrete points. The nature of the approximation of the derivative, (2.4), suggests that if h is chosen sufficiently small and the slope does not change rapidly then the error incurred is small. This intuition is reinforced when the error can be determined exactly in similar problems and is found to be small.

Although no theory of error propagation will be presented here, the various sources of error will be briefly described. Identify the exact solution by $y(x)$ so that the exact propagation problem is given by

(2.6) $$\frac{dy}{dx} = f(x,y) \quad , \; x > x_0 .$$

If the exact solution y has a continuous second derivative, then by the Extended Mean Value Theorem (Thomas, Section 3-9) we have

$$y(x_s) = y(x_{s-1}) + h\,f(x_{s-1}, y_{s-1}) + \frac{h^2}{2}\frac{d^2y(\xi_s)}{dx^2} \tag{2.7}$$

where $x_{s-1} < \xi_s < x_s$ but the exact location of ξ_s is unknown. If we <u>assume</u> that the approximate and exact values are equal at step s-1, $y(x_{s-1}) = y_{s-1}$, then the error in the Euler method is the difference between (2.4) and (2.7),

$$y(x_s) - y_s = \frac{h^2}{2}\frac{d^2y(\xi_s)}{dx^2} \quad . \tag{2.8}$$

Thus, the Euler method can be considered as employing the first two terms of (2.7) so that an error is introduced due to omitting the remainder term. The error (2.8) is therefore called the <u>truncation error</u>. It is proportional to the second derivative and hence related to the curvature, which is a measure of the change of slope.

<u>Example 2.3.</u> Truncation Error

In the cooling problem of Examples 1.1 and 2.1 the temperature is given by (1.8) as

$$T = 20 + 50\,e^{-0.223t};$$

hence

$$\frac{dT}{dt} = -11.15\,e^{-0.223t}$$

and

$$\frac{d^2T}{dt^2} = 2.486\,e^{-0.223t} .$$

In order to evaluate the truncation error when s=1, equation (2.8) gives

$$\frac{h^2}{2}\frac{d^2T(\xi_1)}{dt^2} = \frac{(0.20)^2}{2}(2.486)\ e^{-0.223\xi_1}$$

where $0 \leqq \xi_1 \leqq 0.20$. Although the exact ξ_1 is unknown, if we use $\xi_1 = 0$ we clearly get a bound on the error. Thus,

$$\left|\frac{h^2}{2}\frac{d^2T(\xi_1)}{dt^2}\right| < \frac{(0.20)^2}{2}(2.486) = 0.050.$$

This estimate of the truncation error should be compared with the total error given in Table 2.1 when s=1.

Another error arises in any numerical computation because the arithmetic is rarely carried out exactly; numbers are rounded off at each step. For instance, in Example 2.2 a round-off error of .0005 was introduced at s=1 by taking I_1=0.958 instead of 0.9575. The magnitude of round-off errors can be controlled by selecting the number of decimal places to be carried, but one must be alert to the possible serious accumulation of such errors in a large computation.

Finally, any "marching forward process" for solving differential equations involves an inherited error. This arises from the fact that in the step from x_{s-1} to x_s the error in y_s, as computed from (2.4), is due not only to the truncation and round-off errors inherent in this formula but also to the error already present in y_{s-1}. Thus the total error in y_s builds up in a complicated way as s increases.*

The truncation error in Euler's method, given in (2.8) is of order h^2. If h is decreased by a factor of 2 the truncation error is decreased by a factor of 4; on the other hand, twice as many steps are needed to cover a given range of values of x, so the effect of the inherited error is more pronounced. The combined effect can be shown to be a gain in accuracy, and in fact one can prove that as h approaches zero the approximate solution approaches the true solution. The round-off error steadily grows, however, and in actual calculation the ultimate accuracy is limited by the number of decimal places carried.

Euler's method is the simplest of a large number of marching forward processes. In computing practice, other methods such as those of Runge-Kutta, Milne, or Adams are generally used. They have a smaller truncation error than the Euler method and for the same step size achieve greater accuracy. Thus bigger, and therefore fewer, steps can be taken, and machine time saved accordingly.

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*For a discussion of the inherited error in Euler's method see S.H. Crandall, Engineering Analysis, McGraw-Hill Book Co., New York, 1956, Section 3-8.

Problems

2.1 In Example 1.1 what assumption was made about the conductivity of the body in order that its thermal state be characterized by one temperature?

2.2 The cathode heating element of a vacuum tube supplies heat at a constant rate r BTU/min. while the tube is in an environment having constant temperature T_E. Assuming Newton's law of cooling, determine the differential equation which governs the tube temperature. What temperature does the tube approach after a long period of time? Ans. $T \longrightarrow T_E + \frac{r}{hA}$.

2.3 A fundamental problem in the entrainment of small solid or liquid particles set in a stream is to determine the particle velocity as a function of time. Assume aerosol particles with zero initial velocities are deposited in an airstream having constant velocity V. It is known from a postulate first made by Stokes, and experimentally verified, that the drag force exerted on such particles is proportional to the relative velocity of the stream and particle. If the particles are of mass m, determine an expression for their velocity as a function of time. In addition to the stated assumptions, give an additional pertinent assumption made in deriving the differential equation. Find the position of the particles as a function of time. Ans. $v = V(1 - e^{-\frac{k}{m}t})$.

2.4 In the cooling of glass it is important to know the temperature - time behavior so that properties can be controlled. Assume the cooling obeys Newton's law of cooling but that the surface properties change with temperature so that the film heat transfer coefficient h is proportional to $(T - T_e)^{1/4}$ where T is the glass temperature and T_e the temperature of the environment. Let k be the overall rate constant for the cooling process.

(a) Derive the differential equation which describes the cooling and give the required dimensions of k. Ans. $\frac{dT}{dt} = -k(T-T_e)^{5/4}$.

(b) Find the solution to the differential equation and show that it can be put in the form

$$\frac{T - T_e}{T_o - T_e} = \frac{1}{[1 + \frac{kt}{4}(T_o - T_e)^{1/4}]^4}$$

where T_o is the initial glass temperature, i.e.: when $t = 0$, $T = T_o$.

2.5 Alpha particles are emitted from radium as it decomposes. If we assume that the rate at which the amount of radium decreases is proportional to the amount present, determine an expression for the amount of radium at any time. Initially, there are R_o grams of radium. Ans. $R = R_o e^{-kt}$.

2.6 Surfaces containing electrical charges are known to discharge due to "leakage." If the rate of discharge is proportional to the amount of charge, determine the charge as a function of time.

2.7 Bathyscaph devices descending in the ocean pass through regions having different temperatures. The problem is to determine the temperature of the bathyscaph as a function of time, t. Assume there is no internal heat generated and the gondola loses heat according to Newton's law of cooling with rate constant $k = 0.04$ per min. It is known from depth-time information that the ocean environment temperature is given by the following relation

$$T_E = 60° \text{ F}, \qquad 0 \leq t \leq 10 \text{ min.}$$

$$T_E = 60\ e^{-0.018(t-10)}, \qquad 10 \leq t \leq 20 \text{ min.}$$

$$T_E = 45\ e^{-0.005(t-20)}, \qquad t > 20 \text{ min.}$$

(a) Write the differential equations governing the rate of change of temperature for each period of time. Write the recursion relations for an Euler step-by-step solution for each time period in which you cannot integrate the differential equation.

(b) If initially the bathyscaph temperature is 70° F., determine its temperature at $t = 40$ min. by a hand (and head) calculation. Use $h = 5$ min.

(c) Write the CORC program for the calculation of part (b) and from computer results find the temperature at $t = 40$ min. Use $h = 1$ min.

2.8 The decrease with time in the number of aerosol particles per unit volume is due to the following two effects. There is a decrease resulting from surface loss which is proportional to the number per unit volume present; this process has a rate constant $b = 0.25$ per minute. In addition, coagulation occurs at a rate proportional to the 1.5 power of the number per unit volume present. The coagulation rate constant is $c = 2.5 \times 10^{-4}$ $\left[(\text{no. per cubic cm.})^{-1/2}/\text{min.}\right]$

(a) Find the differential equation governing the rate of change of aerosol particles per unit volume. Determine the recursion formula for an Euler step-by-step solution. Ans. $\frac{dN}{dt} = -.25N - (2.5 \times 10^{-4})\ N^{1.5}$

$N_s = N_{s-1} - .25hN_{s-1} - (2.5 \times 10^{-4})h(N_{s-1})^{1.5}$.

(b) If initially there are 3×10^6 aerosol particles per cubic centimeter, determine the number per c.c. after 20 minutes. Use $h = 1$ min.

2.9 A simple time delay network consists of a capacitor which is charged through resistor R connected in series to constant voltage E. Initially no charge is on the capacitor. When switch S is closed, the rise of current is opposed by the capacitor of capacitance C. The voltage drop across a capacitor is q/C where q is the instantaneous charge on the capacitor. With S closed, determine the differential equation governing charge q and find its solution. Also find the current-time relation and sketch the $i(t)$ and $q(t)$ curves. Note: $i = \frac{dq}{dt}$.

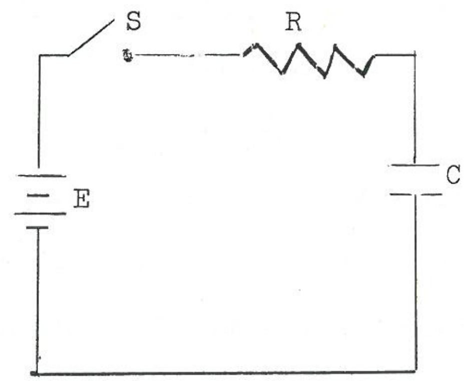

Figure 2.2

2.10 A radioactive waste liquid is stored in a 10,000 gallon shielded tank until its radioactivity level decays to half its initial lethal value in 30 days. The tank is then flushed with fresh water which enters the tank at a rate of 100 gal./min. and the mixture leaves the tank at the same rate. Derive a differential equation for the amount, x lbs., of radioactive material in the tank at any time t after flushing has begun which includes both the flushing and decay processes. Integrate the equation to find the amount of radioactive material at any time. Determine how long it takes to reduce the amount to half that when flushing begins; make the same calculation neglecting the decay process during flushing. Ans.

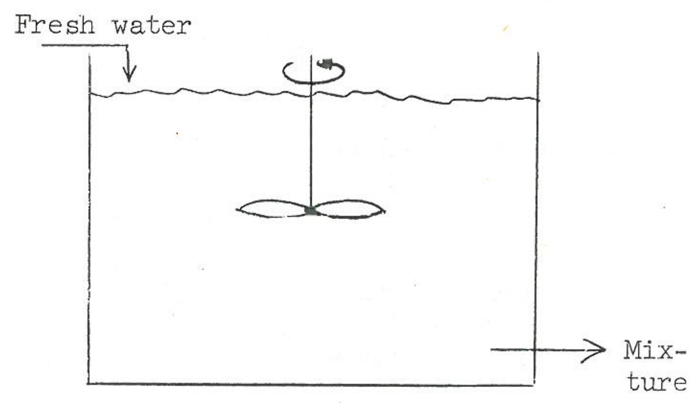

Figure 2.3

$$x = x(0)\, e^{-\left(\frac{1}{100} + \frac{\log 2}{43,200}\right)t} .$$

3. Systems of First Order Simultaneous Equations.

In the class of problems considered in previous sections, the state of the system was defined by one dependent variable y: a situation which is called one-dimensional. In many circumstances, more than one dependent variable occurs as a result of coupling between the objects comprising the system. The coupling results from a transfer of mass or energy between the several parts. In such cases n quantities $y_1(x)$, $y_2(x), \ldots, y_n(x)$ are needed to define the state of the system which is then called an n-dimensional system. We shall consider the mathematical form which is a generalization of the one-dimensional propagation problem. Given

(3.1)
$$\begin{aligned}
\frac{dy_1}{dx} &= f_1(x, y_1, y_2, \ldots, y_n), \\
\frac{dy_2}{dx} &= f_2(x, y_1, y_2, \ldots, y_n), \\
&\cdots\cdots\cdots\cdots \\
\frac{dy_n}{dx} &= f_n(x, y_1, y_2, \ldots, y_n),
\end{aligned}$$

with initial values

(3.2) $$y_1(x_o),\ y_2(x_o),\ \ldots\ , y_n(x_o),\ .$$

determine the state of the system $y_1(x),\ y_2(x),\ \ldots\ , y_n(x)$ for $x > x_o$. In order to avoid writing out the expressions (3.1) and (3.2) we adopt the notation

(3.1) $$\frac{dy_j}{dx} = f_j(x, y_1, y_2, \ldots, y_n),$$

with initial values

(3.2) $$y_j(x_o) = y_{j,o},$$

where $j = 1, 2, \ldots, n$. The theorem assuring existence and uniqueness of a solution has the following form.

<u>Theorem 3.1.</u> For $j = 1,2,\ldots,n$ and $k = 1,2,\ldots,n$ let $f_j(x,y_1,y_2,\ldots,y_n)$ and $\frac{\partial f_j}{\partial y_k}$ be continuous in a domain D of the $(x,y_1,y_2,\ldots,y_n)$ space. Let $(x_0,y_{1,0},y_{2,0},\ldots,y_{n,0})$ be a point in D. Then there is a number $H > 0$ such that on the interval $|x - x_0| < H$ there exists one and only one set of functions $y_1(x),y_2(x),\ldots,y_n(x)$ which are a solution of the system of differential equations (3.1) and satisfy the initial conditions (3.2).

The solution of equations (3.1) is in general complicated by the coupling of variables so that the equations do not stand alone as in the one-dimensional case. Only in a restricted class of problems can we obtain elementary, closed-form solutions. However, the Euler step-by-step method is easily generalized to the n-dimensional case. As in the one-dimensional case, approximate each derivative by

$$\frac{dy_j}{dx} = \frac{y_{j,s} - y_{j,s-1}}{h} . \tag{3.3}$$

When (3.3) is substituted into (3.1) and solved for $y_{j,s}$ we get

$$\begin{aligned} y_{1,s} &= y_{1,s-1} + hf_1(x_{s-1},y_{1,s-1},y_{2,s-1},\ldots,y_{n,s-1}), \\ y_{2,s} &= y_{2,s-1} + hf_2(x_{s-1},y_{1,s-1},y_{2,s-1},\ldots,y_{n,s-1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_{n,s} &= y_{n,s-1} + hf_n(x_{s-1},y_{1,s-1},y_{2,s-1},\ldots,y_{n,s-1}). \end{aligned} \tag{3.4}$$

The step-by-step method is initiated by starting at the initial point $(y_1(x_0),y_2(x_0),\ldots,y_n(x_0))$. The right-hand side of (3.4) is then known and the left-hand side gives a new point $(x_0 + h, y_{1,1}, y_{2,1},\ldots,y_{n,1})$.

Thus, equations (3.4) for the first step are

$$y_{1,1} = y_{1,0} + hf_1(x_0, y_{1,0}, \ldots, y_{n,0}),$$

$$y_{2,1} = y_{2,0} + hf_2(x_0, y_{1,0}, \ldots, y_{n,0}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_{n,1} = y_{n,0} + hf_n(x_0, y_{1,0}, \ldots, y_{n,0}).$$

In this manner numerical values of y_j are obtained at discrete points. Modern computing departments have established programs that one can use to solve say one hundred equations (3.4) so that one need not write the program each time.

Example 3.1. The Parachutist Spin Problem

Consider a problem which arises when a parachutist falls from a high altitude. He is accelerated downward by the attraction of gravity, but the air flow about his body creates drag forces which impede his fall. Due to an inevitable lack of geometric symmetry these drag forces exert moments on him which cause rotation about his center of gravity. The speed of rotation may increase and exceed the 150 RPM which is fatal to most men. Assume the man is released from rest at 85,000 feet. During the descent he remains in a supine position. Figure 3.1(a) shows his instantaneous downward velocity V(ft./sec.), and spin velocity ω(rad./sec.), while end view (b) is a free body diagram showing the force of gravity W, drag force D, and drag moment M. Newton's dynamical laws for linear and rotational motion are

$$W - D = \frac{W}{g}\frac{dV}{dt} \tag{3.5}$$

and

(3.6) $$M = I \frac{d\omega}{dt}$$

where I is the moment of inertia of the man about a vertical axis through his center of gravity. Wind tunnel tests on representative dummies with parachute show that drag force D, even in the presence of rotation, is proportional to the square of the instantaneous velocity V and approximately equal to

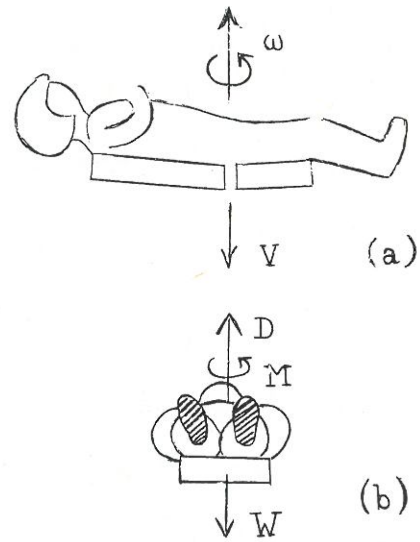

Figure 3.1

(3.7) $$D = 0.692 \times 10^{-2} V^2 \text{ lbs.}$$

These tests also show that the size of moment M is not influenced by the rotation but that it depends on velocity V according to the curve shown in Figure 3.2. The moment increases from zero to a maximum and then decreases. We will assume a "standard man" possesses the following properties: W = 200 lbs. and I = 12 slug-ft^2. In this mathematical model we will assume that D and M do not depend on altitude(air density changes

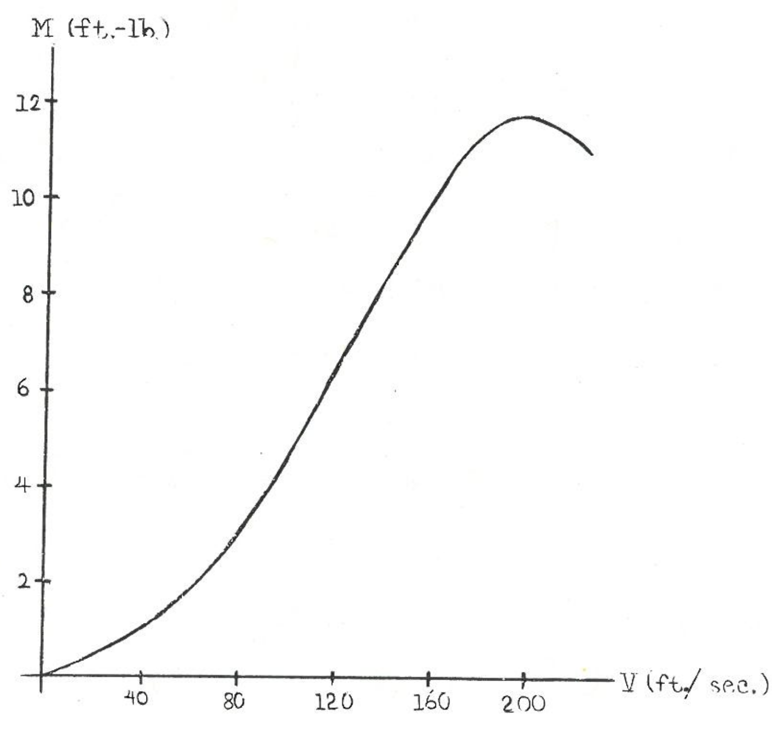

Figure 3.2

are neglected) and W is constant. Equations (3.5) and (3.6) become

(3.8) $$\frac{dV}{dt} = 32.2 - 0.11x10^{-2}V^2,$$

and

(3.9) $$\frac{d\omega}{dt} = \frac{1}{12} M(V),$$

with initial conditions $V(0) = 0$, $\omega(0) = 0$. Equation (3.8) shows that in the early stage the V^2 term is negligible compared to the acceleration of gravity, $g = 32.2$ ft./sec.^{2}; however, as the velocity increases the squared term increases. If the fall is sufficiently long the drag force term, $0.11x10^{-2}V^2$ builds up to just cancel the force of gravity and the parachutist descends with essentially a constant velocity - the terminal velocity. Equation (3.9) is coupled through V with (3.8) and the equations are a two-dimensional special case of the general theory. We can take $y_1 = V$, $f_1 = 32.2 - 0.11x10^{-2}V^2$, $y_2 = \omega$, $f_2 = \frac{1}{12} M(V)$ and $x = t$. The total time to fall 85,000 ft. is the order of magnitude of 500 sec. and the acceleration is largest in the early stages of descent, thus causing the largest velocity changes during this period. Hence, time increments h should be small compared to the total time of fall and sufficiently small so that large velocity changes do not occur in any increment h. During the initial stages of fall a value of $h = 0.5$ sec. meets these conditions without necessitating a great number of steps in the calculation. Table 3.1 shows that after the initial period h can be increased to 1.0 without changing the velocity increments appreciably. Further, h can be increased to 2.0 as the constant terminal velocity is approached. The choice for h depends on the problem and the desired accuracy. It is important to realize that h is adjustable and can be altered to fit the needs of the problem. Equations (3.4) of the step-by-step method yield

(3.10) $$V_s = V_{s-1} + 32.2h - 0.11x10^{-2}V_{s-1}^{\ 2}h,$$

and

(3.11) $$\omega_s = \omega_{s-1} + \frac{h}{12} M(V_{s-1}) .$$

The first step is

$$V_1 = V(0) + 32.2(.5) - 0.11x10^{-2}V^2(0)(.5) = 16.1,$$

$$\omega_1 = \omega(0) + \frac{(.5)}{12} M(V(0)) = 0$$

because $V(0) = \omega(0) = M(V(0)) = 0$. The second step yields

$$V_2 = V_1 + 32.2(.5) - 0.11x10^{-2}V_1^2(.5) = 32.1,$$

$$\omega_2 = \omega_1 + \frac{(.5)}{12} M(16.1) = \frac{.5}{12} (0.12) = .005.$$

Table 3.1 gives the step-by-step values and the curves of Figure 3.3 are drawn through these discrete points. The angular velocity is seen to exceed the fatal threshold of 150 RPM after about 21 sec. has elapsed. Thus, methods must be devised to reduce and control the angular velocity. The terminal velocity of 170.2 ft./sec. (which is easily obtained from (3.8) with $\frac{dV}{dt} = 0$) is essentially reached after only 16 sec. of fall.

s	Time (sec.)	h=Δt	V_s (ft/sec)	M_s (ft-lb)	ω_s (RPM)
0	0		0	0	0
1	0.5	0.5	16.1	0.12	0
2	1.0	0.5	32.1	0.47	0.047
3	1.5	0.5	47.6	1.00	0.235
4	2.0	0.5	62.4	1.77	0.633
5	2.5	0.5	76.3	2.65	1.34
6	3.0	0.5	89.2	3.60	2.39
7	4.0	1.0	112.6	5.50	5.25
8	5.0	1.0	130.7	7.20	9.55
9	6.0	1.0	144	8.40	15.4
10	8.0	2.0	162	10.2	28.7
11	10.0	2.0	168	10.6	45.6
12	12.0	2.0	169.8	10.7	62.5
13	14.0	2.0	170	10.8	79.5
14	16.0	2.0	170.2	10.8	96.6
15	18.0	2.0	↓	↓	123
16	20.0	2.0	↓	↓	141
17	22.0	2.0	↓	↓	158

Table 3.1

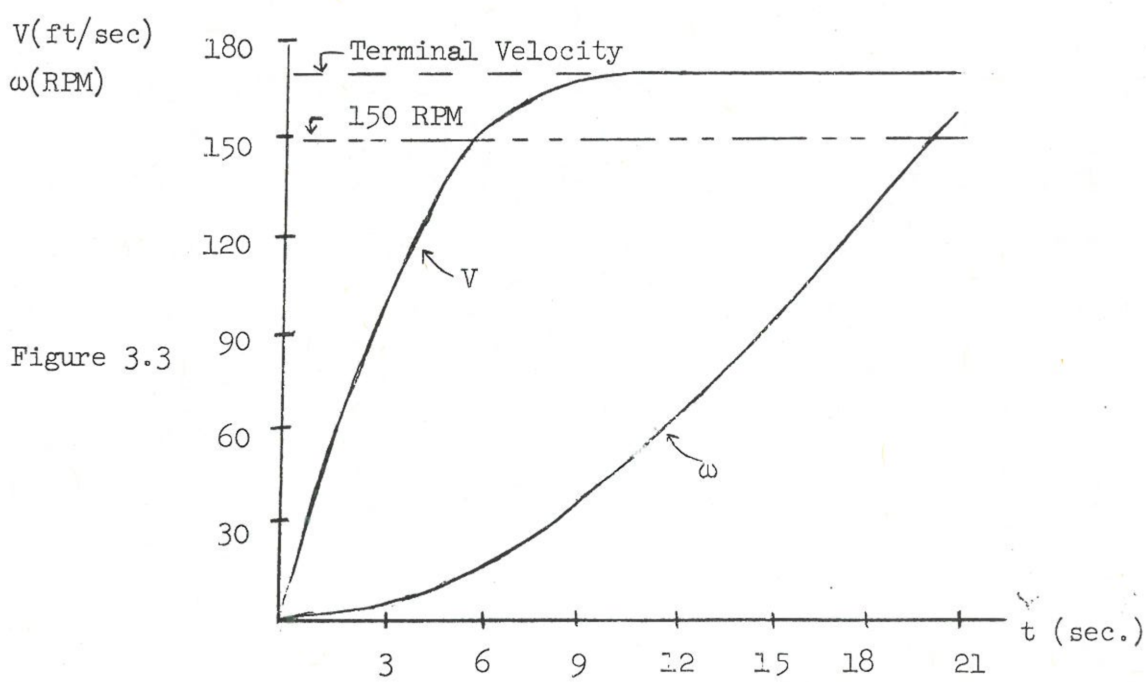

Figure 3.3

Problems

3.1 In the parachutist spin problem the effect of air density change with altitude on the velocity drag force, D, was neglected. The density is known to vary with altitude z ft. above the Earth as

$$\rho = \rho_o e^{-bz}$$

where $\rho_o = 0.0034$ slug/ft.^{3} and $b = 1/22{,}000$ ft.$^{-1}$ This causes the drag force to vary with z as

$$D = 0.692 \times 10^{-2} e^{-bz} V^2.$$

(a) Determine the system of differential equations for velocity V and altitude z from which one could in principle find the time variations of these quantities. (Hint: $\frac{dz}{dt} = -V$).

(b) If the parameters and initial conditions are the same as in Example 3.1, set up the recursion formulas for an Euler step-by-step solution of the differential equations. Do the first two steps to find V_1, z_1 and V_2, z_2 using $h = 1$ sec.

(c) Write the CORC program for the calculation of part (b). From the computer results determine the maximum value of V and the time to fall 85,000 ft.

3.2 In the design of long-range ballistic missiles one must be able to predict the motion in the re-entry phase of flight. The missile enters the "outer reaches" of the atmosphere, $H = 40$ miles, with a velocity of 20,000 ft./sec. Assume the body weighs 2000 lb. and descends vertically from the 40 mile altitude. Gravity exerts a downward force and at the high velocities considered here, the drag force, D lbs., is given by

$$D = C_D \rho V^2$$

where $C_D = 40$, ρ is the air density given by the exponential variation in Problem 3.1 and V is the instantaneous downward velocity.

(a) Determine the system of differential equations for velocity V and altitude z from which one could find the time variation of these quantities. (Hint: $\frac{dz}{dt} = -V$).

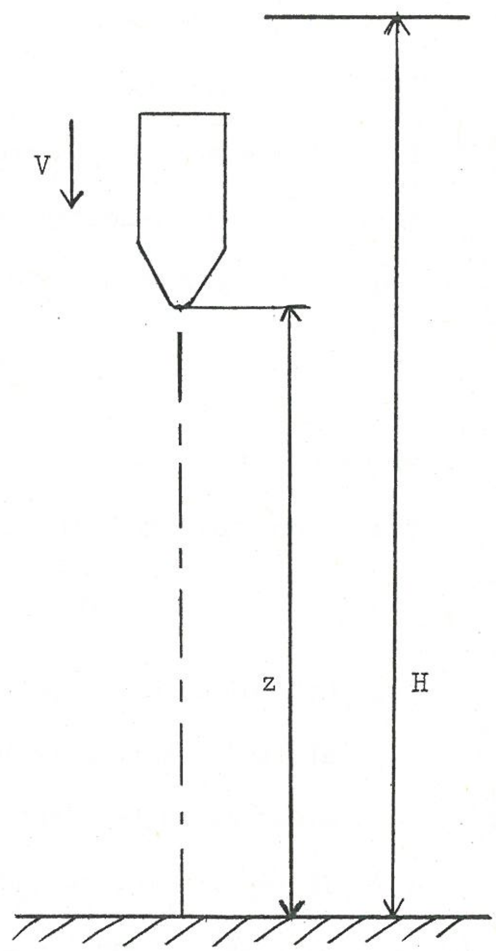

Figure 3.4

(b) Write the recursion relations for an Euler step-by-step solution of the differential equations. Do the first two steps to find V_1, z_1 and V_2, z_2 using h = 1 sec.

(c) Write the CORC program for the calculation of part (b). From the computer results find the velocity and time at z = 40,000 ft.

(d) Determine the non-dimensional deceleration, $\frac{dV}{dt}/g$, versus altitude, z, curve for the 40 miles to 40,000 ft. region.

3.3 The two-loop electric circuit shown has voltage E(t) applied across terminals a, b. The inductances L_1, L_2 and resistances R_1, R_2 are constant. If the voltage drop across an inductor is $L\frac{di}{dt}$ and across a resistor iR, write the Kirchhoff voltage law for each loop thus deriving a system of simultaneous differential equations for currents i_1 and i_2. Write the recursion formulas which you would use to obtain an Euler step-by-step solution for the system of differential equations. Ans. $\frac{di_1}{dt} = \frac{1}{L_1}(E - i_1R_1 + i_2R_1)$ *

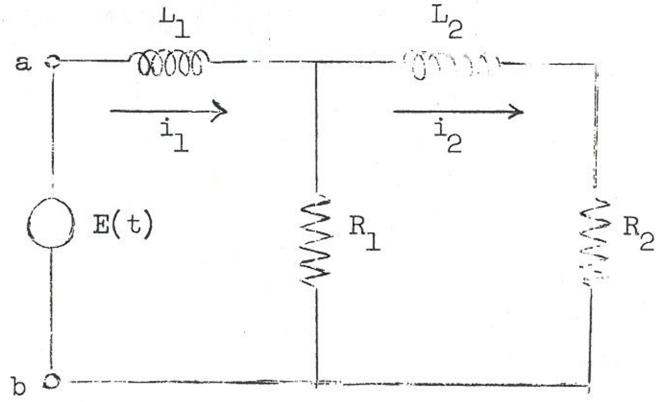

Figure 3.5

3.4 Processes which depend on more than one variable in the following manner are called second order processes. The rate of change of variable u is proportional to variable x when variable y is held constant and proportional to y when x is constant. Consequently we can write

$$\frac{du}{dt} = kxy$$

where k is the rate constant for the process. Rate $\frac{du}{dt}$ is said to <u>depend jointly on x and y</u>. In general, there will be other equations relating the variables so that a system of equations results.

We wish to construct a mathematical model for an epidemic in a community of total size N. An infection spreads between the number of susceptibles x and the number of infectives y. In addition, z persons

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*Answer, continued. $\frac{di_2}{dt} = \frac{1}{L_2}(i_1R_1 - i_2R_2 - i_2R_1)$; $i_{1,s} = i_{1,s-1} + \frac{h}{L_1}(E(t_{s-1}) - i_{1,s-1}R_1 + i_{2,s-1}R_2)$; $i_{2,s} = i_{2,s-1} + \frac{h}{L_2}(i_{1,s-1}R_1 - i_{2,s-1}R_2 - i_{2,s-1}R_1)$.

are removed from the infectious process due to immunity upon recovery from the infection. If the population size is constant, then $x + y + z = N$. The rate of decrease of susceptibles x is known to depend jointly on the number of susceptibles x and the number of infectives y, with an infection rate constant b. Hence,

$$\frac{dx}{dt} = - bxy.$$

In addition to this process between x and y, the rate of increase of immune persons z is proportional to the number of infectives y, with a rate constant c.

(a) Determine the other two differential equations which give the rates of change for y and z.

(b) Write the recursion formulas for an Euler step-by-step solution to this problem. Ans. $x_s = x_{s-1} - hbx_{s-1}y_{s-1}$,

$$y_s = y_{s-1} + hbx_{s-1}y_{s-1} - hcy_{s-1},$$

$$z_s = z_{s-1} + hcy_{s-1}.$$

3.5 Consider a set of constant-volume consecutive chemical reactions. In one reaction chemical A forms chemical B at a rate r_1; the forward rate constant r_1 is the number of lbs. of B formed per unit time per lb. of A. This reaction is reversible so that B becomes A with a reverse rate constant r_2. Thus we can write

(1) $$A \underset{r_2}{\overset{r_1}{\rightleftharpoons}} B.$$

Further, another reaction occurs in which B becomes chemical C at a rate r_3, but this reaction is also reversible so that C becomes B with a rate r_4. Thus

(2) $$B \underset{r_4}{\overset{r_3}{\rightleftharpoons}} C.$$

Initially, when the chemicals are mixed the amounts present are $N_a(0) = 2$ lb., $N_b(0) = 1$ lb. and $N_c(0) = 0$.

If N_a is the amount (lbs.) of A at any time, then conservation of mass requires that

$$\frac{dN_a}{dt} = r_2 N_b - r_1 N_a.$$

(a) Determine the other two rate equations involving N_b and N_c.

(b) Add the three differential equations together and show that the result is consistent with the overall conservation of mass for all three chemicals.

(c) Set up the recursion formulas for an Euler method solution to this problem.

3.6 Let N_1 and N_2 be the populations of two species of animals. Volterra's Competition Equations are

$$\frac{dN_1}{dt} = aN_1 - bN_1N_2,$$

$$\frac{dN_2}{dt} = cN_2 - dN_1N_2.$$

Explain the rationale behind these equations. In particular note that if the coefficients a, b, c, d are positive this corresponds to two competing species, while if d is negative this corresponds to the second species preying on the first. Explain why the term aN_1^2 would be more appropriate than the term aN_1 as given, if the individuals of the first species were widely dispersed, rather than living in compact groups.

3.7 In electromagnetic relay and switching devices, a movable armature is actuated by being part of a magnetic circuit having flux ϕ which is established by passing current i through a coil. A constant source of voltage E is in series with switch S and the coil, which has resistance R. When switch S is closed, the current rise in the coil circuit is opposed by the induced voltage $\frac{d}{dt}(N\phi)$, where N is the number of coil turns. Kirchoff's law for the voltage drop around the coil circuit gives

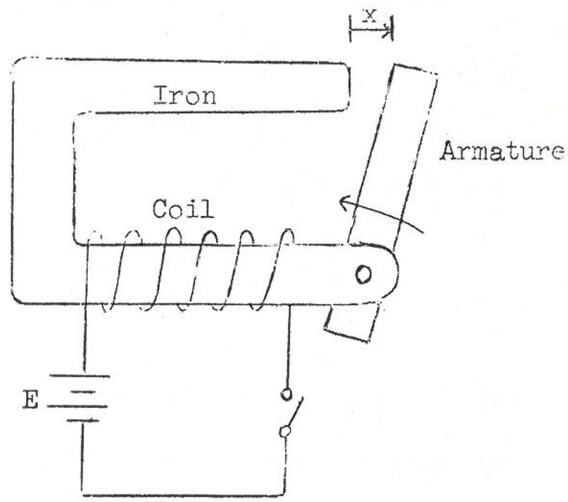


Figure 3.6

$$E = Ri + \frac{d(N\phi)}{dt} .$$

Assume that an empirical equation connecting current and flux, $i = i(\phi)$, is known. The armature motion is coupled to the magnetic circuit by the flux ϕ at the gap which exerts a force $F = k\phi^2/(c+x)^2$ on the armature. x is the gap opening; k and c are constants.

(a) Write the system of three first-order differential equations which govern this electromechanical system. Introduce any inertia or length parameters of the armature as needed; x is small compared to the armature length.

(b) Set up the recursion formulas for an Euler method solution and indicate how you would find the time required to close the armature.

4. Higher Order Differential Equations

The order of a differential equation is the order of the highest derivative of the unknown function which appears in the equation. For example, the differential equations

$$\frac{dy}{dx} + y^2 e^x = 0, \tag{4.1}$$

$$5\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = \sin x, \tag{4.2}$$

$$(1+x^2)\frac{d^4y}{dx^4} + y^3\frac{dy}{dx} = 0, \tag{4.3}$$

are of order 1, 2, and 4, respectively.

Thus, a differential equation of order n is an equation of the form

$$F(x,y,y',\ldots,y^{(n)}) = 0, \tag{4.4}$$

where $y^{(n)}$ denotes $\frac{d^ny}{dx^n}$. The function y is called the dependent variable or the unknown function and x is called the independent variable. A function $y = \varphi(x)$ is called a solution of (4.4) if, when $\varphi(x)$ is substituted for y in (4.4), that equation is satisfied for all x.

In most cases equation (4.4) defines $y^{(n)}$ as a function of the variables $\{x,y,y',\ldots,y^{(n-1)}\}$ and thus (4.4) can be expressed in the form

$$\frac{d^ny}{dx^n} = f(x,y,y',\ldots,y^{(n-1)}). \tag{4.5}$$

This is called "solving for the highest order derivative."

For example, (4.1), (4.2), (4.3) can be written in the forms

$$\frac{dy}{dx} = -y^2 e^x ,$$

$$\frac{d^2y}{dx^2} = \frac{1}{5}(\sin x - 2y' - 3y),$$

$$\frac{d^4y}{dx^4} = - \frac{y^3y'}{1+x^2} .$$

We shall assume, in what follows, that the equation is in the form (4.5).

Many problems call for finding a function $y(x)$ that satisfies not only a differential equation of the form (4.5), but, in addition, meets the requirement that, when x is set equal to a certain value, say x_0, the function y and its first (n-1) derivatives $y', \ldots, y^{(n-1)}$ take on specified values. That is $y(x)$ is required to satisfy the <u>n</u> conditions

$$\begin{aligned} y(x_0) &= c_0, \\ y'(x_0) &= c_1, \\ &\vdots \\ y^{(n-1)}(x_0) &= c_{n-1}, \end{aligned} \tag{4.6}$$

where $c_0, c_1, \ldots, c_{n-1}$ are specified constants. Equations (4.6) are called <u>initial conditions</u>.

The differential equation (4.5), together with the initial conditions (4.6), is called an <u>initial value problem</u>.

<u>Example 4.1.</u> Newton's Second Law of Motion (F = ma), applied to a particle moving along the x-axis, is expressed by the second order differential equation

(4.7) $$m\frac{d^2x}{dt^2} = F(t,x,x'),$$

where $F(t,x,x')$ is the force in the x-direction. In this problem, t is the independent variable and x the dependent variable. The initial values

(4.8) $$x(t_0) = x_0, \qquad x'(t_0) = v_0$$

specify the position and velocity at time $t = t_0$.

The initial value problem (4.5), (4.6) for an nth-order differential equation can be put in the form of the initial value problem (3.1), (3.2) for a system of n first-order equations in n unknowns. This may be seen by making the change of variables

(4.9) $$\begin{aligned} y_1 &= y, \\ y_2 &= \frac{dy}{dx}, \\ &\vdots \\ y_n &= \frac{d^{n-1}y}{dx^{n-1}} . \end{aligned}$$

The nth- order equation (4.5) and the equations (4.9) can then be expressed in the form

(4.10) $$\begin{aligned} \frac{dy_1}{dx} &= y_2, \\ \frac{dy_2}{dx} &= y_3, \\ &\vdots \end{aligned}$$

$$\frac{dy_{n-1}}{dx} = y_n,$$

$$\frac{dy_n}{dx} = f(x, y_1, y_2, \ldots, y_n),$$

which is a special case of (3.1). Similarly the initial conditions (4.6) become, under the change of variable (4.9),

$$
\begin{aligned}
y_1(x_0) &= c_0, \\
y_2(x_0) &= c_1, \\
&\vdots \\
y_n(x_0) &= c_{n-1},
\end{aligned}
\tag{4.11}
$$

which is a special case of (3.2).

Thus Theorem 3.1 can be applied to the initial value problem (4.5), (4.6) with the following result.

Theorem 4.1. Suppose that the function $f(x, y, y', \ldots, y^{(n-1)})$ in equation (4.5) is continuous in x and has a continuous partial derivative with respect to each of its last n arguments in a domain D of (n+1)-space. Let $(x_0, c_0, c_1, \ldots, c_{n-1})$ be a point in D. Then there is a number $H > 0$ such that on the interval $|x - x_0| < H$ there exists one and only one function $y(x)$ satisfying the conditions (4.5), (4.6).

Similarly by writing the problem (4.5), (4.6) in the form (4.10), (4.11) the Euler method of solution described in Section 3 may be applied.

Example 4.2. Physical laws are frequently expressed in terms of differential equations involving derivatives higher than the first. The acceleration in Newton's Law of Motion, $F = ma$, is an example of a second derivative. Thus, in the parachutist problem, Example 3.1, if u is the position of the man downward from his starting altitude then Newton's Law gives

$$W - c\left(\frac{du}{dt}\right)^2 = \frac{W}{g}\frac{d^2u}{dt^2}, \tag{4.12}$$

where $c\left(\frac{du}{dt}\right)^2$ is the drag force and $\frac{d^2u}{dt^2}$ the acceleration.

Equation (4.12) can be written as

$$\frac{d^2u}{dt^2} = g - \frac{gc}{W}\left(\frac{du}{dt}\right)^2, \tag{4.13}$$

which is a second order differential equation having the form of (4.5). In order to write this as a system of first order equations we let

$$\frac{du}{dt} = V, \tag{4.14}$$

and (4.13) becomes

$$\frac{dV}{dt} = g - \frac{gc}{W}V^2. \tag{4.15}$$

Thus, (4.14) and (4.15) are a system of first order equations of the form (4.10). Equation (4.15) is of course identical to (3.8) which we solved numerically by the Euler method. With the values of V_s given in Table 3.1 equation (4.14) is easily solved by the Euler method for

$$u_s = u_{s-1} + hV_{s-1}.$$

Thus, the distance dropped can be found.

It should be noted that the above remarks apply to the initial value problem (4.5), (4.6). If the initial conditions (4.6) are replaced by other conditions, in particular by conditions at more than one point, then the existence and uniqueness theorem (Theorem 4.1) may not hold. The Euler method of solution will also run into difficulties. To illustrate these remarks consider the second order differential equation

(4.16) $$y'' + k^2 y = 0 \qquad (k \neq 0).$$

As we shall see later all the solutions to this equation are of the form

(4.17) $$y = A \cos kx + B \sin kx,$$

where A and B are arbitrary constants. If we require the initial value conditions $y(0) = a$, $y'(0) = b$ then (in agreement with Theorem 4.1) there exists the unique solution

$$y = a \cos kx + \frac{b}{k} \sin kx.$$

If, on the other hand, we require the <u>two-point boundary conditions</u>

$$y(0) = a\ , \quad y(1) = b\ ,$$

then we find upon substitution of these conditions into solution (4.17) that there is

(i) a unique solution $y = a \cos kx + \dfrac{b - a \cos k}{\sin k} \sin kx$ if $\sin k \neq 0$;

(ii) no solution if $\sin k = 0$ and $b \neq a \cos k$;

(iii) infinitely many solutions $y = a \cos kx + B \sin kx$, where B is arbitrary, if $\sin k = 0$ and $b = a \cos k$.

The point of this discussion is to show that the existence and uniqueness situation is considerably more complicated for the two-point boundary conditions, than it is for the initial value problem.

We shall be concerned mostly with the initial value problem. (See Problem 1.7 of Chapter 2 for a numerical solution of a certain type of two-point boundary value problem.)

Problems

4.1 A mathematical model of a loudspeaker consists of a central mass M attached to a very light cone which resists longitudinal motion with spring constant k. The system has inherent damping which exerts a retarding force on M proportional to the velocity of M. Coil C is coupled electromagnetically with M and drives M with a force proportional to the voltage E(t) across the coil. Determine the system of differential equations governing the velocity and displacement of M. Find the recursion formulas for an Euler method solution of the differential equations. Ans. $x_{1,s} = x_{1,s-1} + hx_{2,s-1}$; $x_{2,s} = x_{2,s-1} + \frac{h}{M}(cE(t_{s-1}) - kx_{1,s-1} - ax_{2,s-1})$.

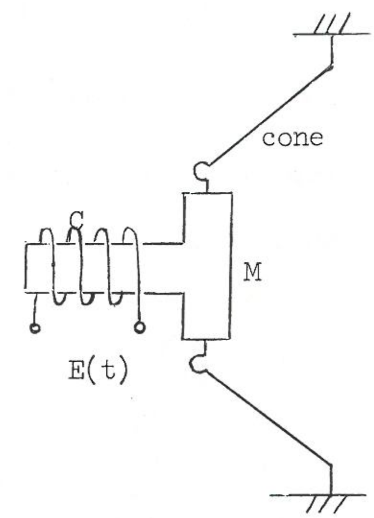


Figure 4.1

4.2 In the docking of a large ship of mass M the engines are essentially stopped while tugs T exert the forces necessary for maneuvering. Water resists the sidewise motion of the ship with a force which is proportional to the velocity. Assume that the ship has an initial sidewise velocity $V(0) = V_0$ when the tugs come in contact with it and exert a constant opposing force F.

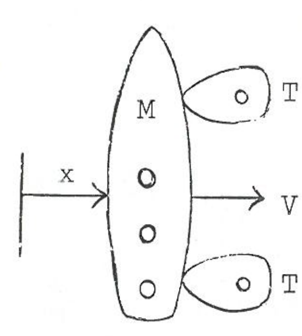


Figure 4.2

Determine the system of differential equations for the sidewise velocity V and displacement x. Solve these equations and thus find V and x as functions of time.

4.3 The differential equation for the flow of heat in a circular cooling fin is $\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - c(T-T_e) = 0$, where T is the temperature at radius r, T_e the constant environmental temperature and c a constant. Write this second order equation as a system of first order equations and determine the recursion formulas for the Euler method of solution.

4.4 Find a numerical solution to the differential equation $\frac{d^2y}{dx^2} = x^2 + 2xy + y$, which passes through the point x = 0 with y = 1 and slope 0.5. Find the solution over the range x = 0 to x = 1 using increments in x of 0.2.

Ans.

x	y_1	y_2
0.0	1.0	.5
0.2	1.1	.7
0.4	1.24	1.016
0.6	1.443	1.494
0.8	1.742	2.201
1.0	2.182	3.235

4.5 Show that the angular displacement θ of the pendulum shown in Figure 4.3 satisfies the initial value problem

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta,$$

with $\theta(0) = \theta_0$ and $\frac{d\theta}{dt}(0) = \omega_0$. Write the recursion formulas

Figure 4.3

necessary for computing $\theta(t)$ by Euler's method and carry out the first three steps if $\theta_0 = 0.2$ rad., $\omega_0 = 0$, $L = 3$ ft. and $h = .2$ sec.

4.6 **A** column with ends free to rotate carrying constant axial load P when deflected sideways an amount y satisfies the differential equation

$$\frac{d^2y}{dx^2} + \frac{P}{EI}\, y = 0.$$

EI is the constant bending stiffness of the column. If the ends are not allowed to deflect sideways then the two-point boundary conditions are

$$y(0) = y(L) = 0.$$

Show that the solutions of the differential equation and boundary conditions depend upon whether $\sin\left[\sqrt{\frac{P}{EI}}\right]L$ is or is not zero. Find the solutions under these conditions and determine their physical significance.

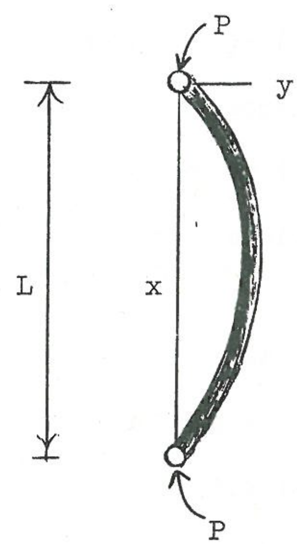


Figure 4.4

5. Special Integrable Forms.

As the discussion in Section 1 shows, the solution of a problem arising in the physical world involves three stages: the construction of a mathematical model of the problem, the solution of the corresponding mathematical problem, and the interpretation of the results. These three stages are not independent, and in particular the techniques to be used in stage two must depend on the information supplied by stage one and the results desired in stage three. Nevertheless, in developing the mathematical techniques it is generally wisest to concentrate on the mathematics. This is what we propose to do in this section and Section 6, leaving the complications involved in applications to Section 7.

Theorem 2.1 and the numerical methods described in Section 2 apply to a very general class of differential equations. There are, however, some special cases that can be solved much more easily, and in ways capable of giving information not obtainable by using the Euler method. We shall consider the most important of these cases.

Case I. The indefinite integral. If $f(x,y)$ in equation (2.1) is independent of y the equation becomes

$$\frac{dy}{dx} = f(x). \tag{5.1}$$

y is therefore the indefinite integral of $f(x)$,

$$y = \int f(x)\, dx + C, \tag{5.2}$$

and to get y in explicit form we have at our disposal all the techniques of integration. The constant C can, of course, be determined by initial conditions; if no initial conditions are given, (5.2) is called the general solution of (5.1).

Note. The term "general solution" has been much misused in discussions of differential equations. We shall use it in this chapter merely to refer to the equation containing a constant arising from an indefinite integration. For a critical discussion see R.P. Agnew, Differential Equations (2nd. ed.), McGraw-Hill, 1960.

Example 5.1. $\frac{dy}{dx} = xe^{-ax}, \qquad y(0) = 0.$

By integrating by parts or using tables we get the general solution

$$y = \int xe^{-ax}\,dx + C$$
$$= -\frac{1}{a}e^{-ax}(x + \frac{1}{a}) + C.$$

Putting $x = 0$ and $y = 0$ gives

$$0 = -\frac{1}{a}(\frac{1}{a}) + C, \qquad C = \frac{1}{a^2},$$

so that the desired solution is

$$y = \frac{1}{a^2}(1 - e^{-ax}(1+ax)).$$

Example 5.2. Find the general solution of

$$\frac{dy}{dx} = \frac{2}{1-x^2}.$$

Using partial fractions gives

$$y = \int \left(\frac{1}{1 - x} + \frac{1}{1 + x}\right) dx + C$$

$$= - \log|1 - x| + \log|1 + x| + C$$

or

$$y = \log\left|\frac{1 + x}{1 - x}\right| + C.$$

This second example illustrates a need for caution. The graph of the solution, shown in Figure 5.1, consists of three disconnected pieces. To see just what is happening from the mathematical viewpoint we shall state the theorem that takes the place of Theorem 2.1 in our present special case.

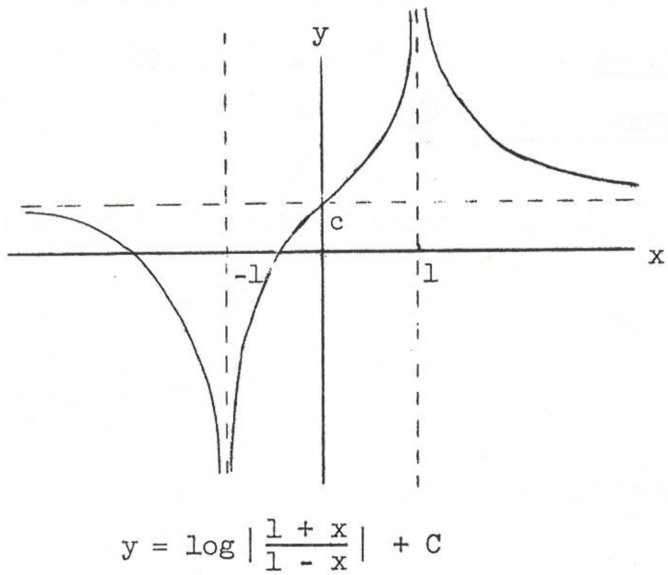


Figure 5.1

<u>Theorem 5.1</u>. Let $f(x)$ be a continuous function on the interval I: $a < x < b$, let x_o be a number in I, and let y_o be any number. Then the equation $\frac{dy}{dx} = f(x)$ with $y(x_o) = y_o$ has the unique solution

(5.3) $$y(x) = y_o + \int_{x_o}^{x} f(u)du$$

for x in I.

This theorem is merely one form of the Fundamental Theorem of Calculus (cf. Thomas, p. 215).

Apply Theorem 5.1 to Example 5.2 with $x_o = 2$, $y_o = 3$. Since $f(x) = \frac{2}{1-x^2}$ is discontinuous at $x = 1$ and $x = -1$ we take for I the interval $1 < x < \infty$, this being the largest interval that contains x_o $(= 2)$ but no point of discontinuity. Then

$$y = 3 + \int_2^x \frac{2}{1-u^2}\,du$$

$$= 3 + \Big[- \log|1-u| + \log|1+u| \Big]_2^x$$

$$= 3 + \Big[- \log|1-x| + \log|1+x| \Big] - \Big[- \log|1-2| + \log|1+2| \Big].$$

Now for x in I, i.e. for $1 < x < \infty$, we have $|1-x| = x-1$, $|1+x| = 1+x$, and so

$$y = 3 - \log(x-1) + \log(1+x) - \log 3$$

or

$$y = 3 + \log \frac{1 + x}{3(x-1)} \quad \text{for} \quad 1 < x < \infty.$$

The same result is obtained if the initial condition $x_o = 2$, $y_o = 3$ is used to evaluate C in the solution of Example 5.2.

In general, Theorem 5.1 offers us no help in extending our solution past a discontinuity. This does not mean that we can never do this, but we must use other devices. (See Problem 5.7).

The form of the solution given in (5.3) is often more convenient than (5.2). If, for example, we are given $y(x_o) = y_o$ and wish to find $y(x_1)$ we can express the answer directly as

$$y(x_1) = y_o + \int_{x_o}^{x_1} f(u)\,du,$$

and the definite integral appearing here may be easier to evaluate than the indefinite integral of (5.2).

Example 5.3. Find $y(3)$ if $y(1) = 2$ and $\frac{dy}{dx} = \sqrt{1 + x^3}$.

Here

$$y(3) = 2 + \int_1^3 \sqrt{1 + u^3}\, du.$$

Since the indefinite integral $\int \sqrt{1 + u^3}\, du$ is not expressible in terms of elementary functions we must find other means of evaluating the definite integral. Two well-known numerical methods are the Trapezoidal Rule and Simpson's Rule, defined, in general, by the formulas

$$\int_a^b f(x)dx \cong h[\tfrac{1}{2}f(x_o) + f(x_1) + f(x_2) + \dots + f(x_{n-1}) + \tfrac{1}{2}f(x_n)]$$

and

$$\int_a^b f(x)dx \cong \frac{h}{3}[f(x_o) + 4\, f(x_1) + 2\, f(x_2) + \dots + 4\, f(x_{n-1}) + f(x_n)],$$

(See Thomas, Sections 4-10 and 7-11.) Here $h = (b-a)/n$, $x_o = a$, $x_n = b$, $x_i = x_o + ih$. In the second formula (Simpson's Rule) n must be even, and the coefficients not shown are alternately 4 and 2. If we take $h = .2$ the two formulas give the respective approximations 8.2210 and 8.2206. For comparison, Euler's Method gives 7.833.

If we want not only $y(3)$ but a whole table of values $y(x)$ we can use one of these Rules to replace the Euler Method. Writing (5.3) with x_{s-1} and x_s for x_o and x gives

$$y(x_s) = y(x_{s-1}) + \int_{x_{s-1}}^{x_s} f(u)du,$$

and the Trapezoidal Rule gives the approximation

$$y_s = y_{s-1} + \frac{h}{2}[\, f(x_{s-1}) + f(x_s)]$$

to be used instead of (2.4). In a similar manner, equation (5.3) can be written as

$$y(x_{2s}) = y(x_{2s-2}) + \int_{x_{2s-2}}^{x_{2s}} f(u)du,$$

and Simpson's Rule yields

$$y_{2s} = y_{2s-2} + \frac{h}{3}[f(x_{2s-2}) + 4\, f(x_{2s-1}) + f(x_{2s})].$$

For a given value of h Simpson's Rule generally gives more accurate values of $y(x)$ but at intervals of $2h$ instead of h. Either process is readily programmed for an automatic computer.

Case II. Variables separable. If equation (2.1) can be reduced to the form

(5.4) $$g(y)\frac{dy}{dx} = f(x)$$

we are said to have "separated the variables." An equation with variables separated can also be solved (in a sense to be made precise later) by indefinite integration. For let $F(x)$ and $G(y)$ be indefinite integrals of $f(x)$ and $g(y)$; that is, $F(x)$ and $G(y)$ are functions such that

$$\frac{d}{dx} F(x) = f(x), \qquad \frac{d}{dy} G(y) = g(y).$$

Suppose that $y(x)$ is any solution of (5.4). Then $G(y)$ can be considered as a function of x, and, by the chain rule for differentiation,

$$\frac{dG}{dx} = \frac{dG}{dy}\frac{dy}{dx} = g(y)\frac{dy}{dx} = f(x) = \frac{dF}{dx} .$$

Hence $G(y)$ and $F(x)$ have the same derivative with respect to x and so differ by at most a constant, and we have

(5.5) $$G(y) = F(x) + C.$$

Thus any solution $y(x)$ of (5.4) satisfies (5.5), and by working backwards it is easy to show that any solution $y(x)$ of (5.5) satisfies (5.4). Since (5.5) involves no derivatives it is said to be obtained from (5.4) by "integrating."

The mechanics of integrating an equation with variables separated can be simplified by writing (5.4) in the differential form (in which the variables are truly "separated")

(5.6) $$g(y)dy = f(x)dx$$

and then "integrating both sides,"

$$\int g(y)dy = \int f(x)dx,$$

to give (5.5).

<u>Example 5.4.</u> $\frac{dy}{dx} = -\frac{x}{y}$.

Separating the variables gives

(5.7) $$y\,dy = -x\,dx,$$

which integrates to

(5.8) $$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C.$$

This equation has two solutions,

$$y = \sqrt{2C - x^2},$$

$$y = -\sqrt{2C - x^2},$$

each of which is defined in the interval $I: -\sqrt{2C} < x < \sqrt{2C}$. Obviously we must have $C > 0$ in order for (5.8) to have a solution. The value of C and the appropriate solution are determined by the initial conditions. If, for instance, we want the trajectory through $(-1, -1)$ we obtain from (5.8)

$$\frac{1}{2} = -\frac{1}{2} + C, \qquad C = 1,$$

and the appropriate solution is

$$y = -\sqrt{2 - x^2}.$$

As in Case I the definite integral can be used to avoid explicit introduction of the constant of integration. Thus from (5.7) we get

$$\int_{-1}^{y} y\,dy = \int_{-1}^{x} -x\,dx$$

or

$$\frac{1}{2}y^2 - \frac{1}{2} = -\frac{1}{2}x^2 + \frac{1}{2},$$

giving the same result as before. If we want the value y_1 of y when $x = \frac{1}{2}$ we can use

$$\int_{-1}^{y_1} y\, dy = \int_{-1}^{1/2} -x\, dx ,$$

giving $\frac{1}{2} y_1^2 - \frac{1}{2} = -\frac{1}{8} + \frac{1}{2}$, or $y_1^2 = 7/4$. Here it is not so obvious that $y_1 = -\sqrt{7/4}$ rather than $\sqrt{7/4}$, and in general we must be very careful in applying this procedure to see that in passing from x_o to x_1 and from y_o to y_1 there is no loss of continuity of any of the functions involved. If we took $y_1 = \sqrt{7/4}$ the passage of y from -1 to y_1 would require y to go through 0, which is a point of discontinuity for the original function $-x/y$. Hence we must take $y_1 = -\sqrt{7/4}$.

<u>Example 5.5.</u> $\frac{dy}{dx} = 2xy$.

Here we get

$$\frac{1}{y}\, dy = 2x\, dx,$$

which integrates to

$$\log |y| = x^2 + C,$$

and gives the two solutions

$$y = e^C e^{x^2},$$

$$y = -e^C e^{x^2},$$

each holding for all values of x. Since e^C can have any positive value and $-e^C$ any negative value we can lump the two solutions together in the form

$$y = A\,e^{x^2}, \tag{5.9}$$

where A has any value except 0. Now we note that our original equation has the obvious solution $y = 0$. This was lost when we divided by y in passing to the separated form. We can thus conclude that the solutions to our equation are given by (5.9) for <u>any</u> value of the constant A.

This example is readily generalized to show that the equation

$$\frac{dy}{dx} = f(x)y$$

has the general solution

$$y = A\,e^{F(x)},$$

where $F(x)$ is any indefinite integral of $f(x)$. If $f(x)$ has discontinuities the same precautions must be taken as in Case I.

<u>Example 5.6.</u> $\dfrac{dy}{dx} = \dfrac{(x+1)(y^2+1)}{x(y+1)}, \qquad y(1) = 2.$

Separating the variables and integrating gives

$$\frac{1}{2}\log|y^2+1| + \arctan y = x + \log|x| + C.$$

Note that since y^2+1 is never negative we can dispense with the absolute value signs in the first term. Putting in the initial conditions gives

$$\frac{1}{2}\log 5 + \arctan 2 = 1 + \log|1| + C = 1 + C.$$

Since we used the determination $|x| = x$ for $x = 1$ in the above process, we must, as in Example 5.2, continue to use this determination, and so

$y(x)$ is the solution of

$$\frac{1}{2}\log(y^2+1) + \arctan y = x + \log x + \frac{1}{2}\log 5 + \arctan 2 - 1$$

which assumes the value 2 when $x = 1$. To get this solution in any explicit form is a major task. No analytic solution seems possible. A numerical approach would be to assign a succession of values to x and solve the resulting equations by Newton's method (Thomas, Section 9-3). This is a tedious job, and certainly more difficult than a direct numerical integration of the original differential equation.

Case III. Second order equations. In many cases second-order equations can be reduced to first-order by introducing a new variable $v = \frac{dy}{dx}$ and using either

$$\frac{d^2y}{dx^2} = \frac{dv}{dx}$$

or

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy} .$$

Example 5.7. Let us take the general case of the linear motion of the parachutist in Example 3.1, writing equation (3.8) in the form

$$m\frac{d^2y}{dt^2} = mg - a\left(\frac{dy}{dt}\right)^2 , \tag{5.10}$$

y being the distance dropped. We ask two questions: (a) What does the velocity do as $t \longrightarrow \infty$; (b) What is the velocity after dropping a distance h?

(a) Here we want a relation between v and t, so we put (5.10) in the form

$$\frac{dv}{dt} = g - \frac{a}{m} v^2 ,$$

and separating variables gives

$$\frac{dv}{g - \frac{a}{m} v^2} = dt.$$

To put this in a more convenient form we multiply both sides by $\frac{a}{m}$; then, setting $c = \sqrt{\frac{gm}{a}}$, we get

$$\frac{dv}{c^2 - v^2} = \frac{a}{m} dt .$$

Our initial conditions are $t = 0$, $v = 0$, so

$$\int_0^v \frac{dv}{c^2 - v^2} = \int_0^t \frac{a}{m} dt ,$$

or

$$\frac{1}{2c} \log \left| \frac{c+v}{c-v} \right| \Big]_0^v = \frac{a}{m} t.$$

Since $\left| \frac{c+v}{c-v} \right| = \frac{c+v}{c-v}$ when $v = 0$, we must use this same determination throughout. We therefore get, in succession,

$$\frac{1}{2c} \log \frac{c+v}{c-v} = \frac{a}{m} t,$$

$$\log \frac{c+v}{c-v} = \frac{2ac}{m} t,$$

$$\frac{c+v}{c-v} = e^{\frac{2ac}{m}t} ,$$

$$\frac{c-v}{c+v} = e^{-\frac{2ac}{m} t} .$$

In this last expression the right hand side approaches 0 as $t \longrightarrow \infty$, and hence so must the left hand side. That is

$$\lim_{t \longrightarrow \infty} v = c = \sqrt{\frac{gm}{a}} .$$

This is the constant <u>terminal velocity</u> of the falling body. It should be observed that once we know the physical problem has such a limiting velocity, it can be obtained directly from (5.10) by solving for the velocity at which the acceleration is zero.

(b) In this case we want a relation between v and y, so we write (5.10) as

$$m\,v\,\frac{dv}{dy} = mg - av^2 . \tag{5.11}$$

With the same value of c as before this becomes

$$\frac{v\,dv}{v^2 - c^2} = -\frac{a}{m}\,dy .$$

The initial conditions are $y = 0$, $v = 0$, and we want the value of v when $y = h$, so

$$\int_0^v \frac{v\,dv}{v^2 - c^2} = \int_0^h -\frac{a}{m}\,dy,$$

or

$$\frac{1}{2}\log|v^2-c^2|\Big]_0^v = -\frac{a}{m}h.$$

For $v = 0$, $|v^2 - c^2| = c^2 - v^2$, so we get

$$\log(c^2 - v^2) - \log c^2 = -\frac{2a}{m}h,$$

$$\log(1-\frac{v^2}{c^2}) = -\frac{2ah}{m},$$

$$1 - \frac{v^2}{c^2} = e^{-\frac{2ah}{m}},$$

$$v^2 = c^2(1 - e^{-\frac{2ah}{m}}).$$

The right-hand side of (5.11) is always positive and hence the acceleration is greater than zero so that v increases from its initial value $v(0) = 0$. Thus, we would choose the positive value for v.

Many other special forms of differential equations, susceptible of analytic solution by a variety of devices, can be found in books on differential equations. (See, in particular, R.P. Agnew, loc. cit., Chap. 5, and W. Kaplan, loc. cit., Chapter 2.) These devices have become less important with the widespread use of computers and numerical methods of solution but are still very useful in certain cases.

Problems

5.1 Find a general solution of each of the following equations.

(a) $\frac{dy}{dx} = a \cos nx.$ Ans. $y = \frac{a}{n} \sin nx + C.$

(b) $\frac{dy}{dx} = \frac{\sin 3x}{\cos^2 3x}.$

(c) $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$ Ans. $y = \arcsin x + C.$

(d) $\frac{dx}{dt} = \frac{1}{t - t^2}.$

(e) $\frac{dz}{dy} = y^2 e^{ay}.$ Ans. $z = \frac{y^2 e^{ay}}{a} - \frac{2ye^{ay}}{a^2} + \frac{2e^{ay}}{a^3} + C.$

(f) $\frac{dx}{dy} = \frac{1}{1 - \sqrt{y}}.$

(g) $\frac{du}{dv} = \frac{e^{-v}}{1 - e^{-v}}.$ Ans. $u = \log|1-e^{-v}| + C.$

(h) $\frac{dy}{dx} = x^2 \log(1 - x).$

5.2 Find the solution of each of the following equations satisfying the given initial conditions. In each case specify the interval in which the solution is valid.

(a) $\frac{dy}{dx} = \sin^2 ax,$ $y(0) = 0.$ Ans. $y = \frac{x}{2} - \frac{\sin 2ax}{4a}$, $(-\infty, +\infty)$.

(b) $\frac{dy}{dx} = \frac{4x}{x^2 + 4},$ $y(1) = 1.$

(c) $\frac{dy}{dx} = \frac{4x}{x^2 - 4},$ $y(1) = 1.$ Ans. $y = 1 + 2\log\frac{4-x^2}{3}$, $(-2,2)$.

(d) $\frac{dy}{dx} = \frac{4}{\sqrt{4 - x^2}},$ $y(1) = 1.$

(e) $\frac{dy}{dt} = \frac{e^{at} - e^{-at}}{e^{at} + e^{-at}},$ $y(0) = 0.$ $a > 0.$ Ans. $y = \frac{1}{a}\log\frac{e^{at} + e^{-at}}{2}$; $(-\infty,\infty)$.

5.3 Integrate each of the following equations and find an explicit solution $y(x)$ when possible.

(a) $\frac{dy}{dx} = x^2 y$. Ans. $y = Ce^{x^3/3}$.

(b) $\frac{dy}{dx} = \frac{ay}{x}$.

(c) $\frac{dy}{dx} = a\frac{y^m}{x^n}$, $m \neq 1$, $n \neq 1$. Ans. $y = (\frac{a(1-m)}{1-n} x^{1-n} + C)^{1/1-m}$.

(d) $\frac{dy}{dx} = \cos x \cos y$. Ans. $y = \frac{\pi}{2} - 2 \arctan Ce^{-\sin x}$.

(e) $\frac{dy}{dx} = e^{x+y}$. Ans. $y = -\log(C-e^x)$.

(f) $\frac{dv}{dt} = 1 + \sqrt{v}$.

(g) $\frac{dy}{dt} = \sqrt{1 - e^{-2by}}$, $b > 0$. Ans. $t = \frac{1}{b} \log (e^{by} + \sqrt{e^{2by} - 1}) + C$, $y = \frac{1}{b} \log [\frac{1}{2}(e^{bt-bC} + e^{bC-bt})]$.

5.4 Find the solution of the corresponding problem in 5.3 subject to the given conditions.

(a) $y(1) = 2$, (b) $y(2) = 1$, (c) $y(0) = 1$,

(d) $y(0) = 0$, (e) $\lim_{x \to -\infty} y(x) = 0$, (f) $v(0) = 0$,

(g) $y(0) = 0$.

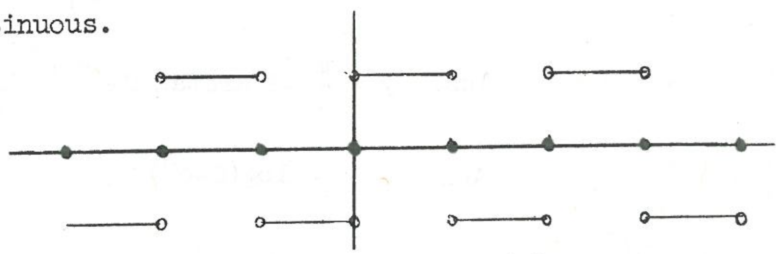
5.5 Find the general solution of $\frac{d^2f}{dr^2} + \frac{n}{r}\frac{df}{dr} = 0$ for the cases $n = 3$ and $n = 2$. (cf. Chapter 8). Ans. $f = \frac{C_1}{r} + C_2$ for $n = 2$.

5.6 Find the general solution of $\frac{d^2y}{dx^2} + a^2 y = 0$. Ans. $y = A \sin(ax + B)$.

5.7 Let $f(x)$ be the "square wave" function defined for all x by

$$f(x) = \begin{cases} 0 & \text{if } x = k, \text{ an integer,} \\ a & \text{if } 2k < x < 2k+1, \\ -a & \text{if } 2k-1 < x < 2k. \end{cases}$$

Show that there is a unique function $y(x)$, continuous for all x, satisfying $y(0) = 0$, and such that $\frac{dy}{dx} = f(x)$ for any value of x at which $f(x)$ is continuous.



Square wave function $y = f(x)$

5.8 Write a CORC program to tabulate the function $y(x) = \int_a^x f(x)dx$ for $x = a(h)b$, with $b = a + nh$, using the Trapezoidal Rule. Use your program with $f(x) = \sqrt{1 + \sqrt{1 + e^x}}$, $a = 0$, $b = 10$, $n = 20$.

5.9 Write a CORC program to evaluate a definite integral $\int_a^b f(x)dx$ by Simpson's Rule. Let the evaluation of $f(x)$ for a given value of x be made a subroutine. Use the program to evaluate $\int_0^{10} \sqrt{1 + \sqrt{1 + e^x}}\, dx$ correct to five decimal places, starting with $n = 10$ and successively doubling n until two results are obtained that agree to the desired accuracy. **Ans.** 46.67407 with $n = 80$.

5.10 (a) In Example 5.7(a) we obtained the relation

$$\frac{c + v}{c - v} = e^{2kt}, \qquad k = \frac{ac}{m},$$

connecting v and t. Solve this for v to get

$$v = \frac{dy}{dt} = c\,\frac{e^{kt} - e^{-kt}}{e^{kt} + e^{-kt}}$$

and integrate, with initial conditions $t = 0$, $y = 0$, to get

$$y = \frac{m}{a}\log\frac{e^{(ac/m)t} + e^{-(ac/m)t}}{2}.$$

(b) Taking the relation

$$v = \frac{dy}{dt} = c\sqrt{1 - e^{-(2a/m)y}}$$

from part (b) of the same example, integrate (cf. Problem 5.3(g)) and solve for y to get the same result as above.

5.11 One of the devices for solving differential equations is the replacement of the dependent variable y by a simple function of a new dependent variable v and the independent variable x. The success of the method depends on a clever choice of this function, but certain standard choices, in particular $y = xv$, are often helpful.

(a) By putting $y = xv$, so that $\frac{dy}{dx} = x\frac{dv}{dx} + v$, integrate $\frac{dy}{dx} = \frac{y + x}{y - x}$. Ans. $y^2 - 2xy - x^2 = C$.

(b) Integrate $\frac{dy}{dx} = \frac{x + y}{x - y}$. Answer, in polar coordinates, $r = Ae^{\theta}$.

6. The First Order Linear Equation.

An important special type of differential equation amenable to so-called closed form solution is the first order linear equation

(6.1) $$\frac{dy}{dx} = f(x,y),$$

where

(6.2) $$f(x,y) = q(x) - p(x)y.$$

Eq. (6.1) can be written as

(6.3) $$\frac{dy}{dx} + p(x)y = q(x)$$

where $p(x)$ and $q(x)$ are continuous functions over some range of x. Differential equation (6.3) is called linear because the left side is a linear function of y and y'. In general the equation is not separable.

The method of attack is to multiply the equation by a function of x, $R(x)$, such that the terms involving y and y' constitute the derivative with respect to x of a function of x and y. The resulting equation can then be treated in the same manner as an equation of the form $\frac{dz}{dx} = f(x)$.

If we multiply equation (6.3) by $R(x)$ we get

(6.4) $$R\frac{dy}{dx} + Rpy = Rq.$$

We wish to choose R so that (6.4) can be written as

(6.5) $$\frac{d}{dx}(Ry) = Rq.$$

Since

$$\frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx}$$

this equals the left side of (6.4) if

(6.6) $$\frac{dR}{dx} = Rp.$$

By the remark following Example 5.5 we see that the general solution of (6.6) is $R = Ae^{P(x)}$, where

(6.7) $$P(x) = \int p(x)dx.$$

Since R can be <u>any</u> solution of (6.6) we might as well take $A = 1$, so that

(6.8) $$R = e^{P(x)}.$$

For this value of R, (6.4) reduces to (6.5), which can then be integrated to give

$$Ry = \int Rq\, dx + C$$

or

$$y = R^{-1}\int Rq\, dx + CR^{-1}.$$

Written out explicitly in terms of the function $P(x)$ defined by (6.7) this is

(6.9) $$y = e^{-P(x)}\int e^{P(x)}q(x)dx + Ce^{-P(x)}.$$

Note. Since multiplication by $R(x)$ enables us to integrate the equation, $R(x)$ is called an <u>integrating factor</u>. Integrating factors, similar to R, but often involving both x and y, are sometimes helpful in solving equations other than linear equations. Extended discussions of this topic can be found in texts on differential equations.

Example 6.1.

$$(1 + x^2)\,\frac{dy}{dx} + xy = x(1 + x^2)^2$$

Putting the equation in standard form

$$\text{(6.10)} \qquad \frac{dy}{dx} + \frac{x}{1+x^2}\,y = x(1+x^2)$$

gives

$$p(x) = \frac{x}{1+x^2}, \qquad q(x) = x(1+x^2).$$

Then $P(x) = \int \frac{x}{1+x^2}\,dx = \frac{1}{2}\log(1+x^2) = \log(1+x^2)^{1/2}$. Rather than trying to remember the form of (6.9) and substituting this value of $P(x)$ into it, it is better to retrace the steps from which (6.9) was obtained. Thus we first get

$$R(x) = e^{P(x)} = (1+x^2)^{1/2},$$

and multiplying (6.10) by this integrating factor gives

$$\text{(6.11)} \qquad (1+x^2)^{1/2}\,\frac{dy}{dx} + x(1+x^2)^{-1/2}\,y = x(1+x^2)^{3/2}\,.$$

The left hand side of this equation should now be

$$\frac{d}{dx}\,(Ry) = \frac{d}{dx}\,[(1+x^2)^{1/2}y]$$

and differentiation of this should be carried out to serve as a check on the work to this point. Integration of (6.11) gives

$$(1+x^2)^{1/2} y = \int x(1+x^2)^{3/2}\, dx + C$$

$$= \frac{1}{5}\,(1+x^2)^{5/2} + C.$$

Hence the solution is

$$y = \frac{1}{5}\,(1+x^2)^2 + C(1+x^2)^{-1/2}.$$

Example 6.2. Find the solution of

$$\frac{dy}{dt} + py = E \sin \omega t$$

that satisfies the initial condition $y(0) = 0$; p, E and ω being constants. The integrating factor is e^{pt}, hence

$$e^{pt} \frac{dy}{dt} + pe^{pt} y = \frac{d}{dt}\,(e^{pt} y) = Ee^{pt} \sin \omega t,$$

$$e^{pt} y = E \int e^{pt} \sin \omega t \, dt + C.$$

Evaluating the integral (using tables) and dividing by e^{pt} gives

$$y = Ce^{-pt} + \frac{E}{p^2+\omega^2}\,(p \sin \omega t - \omega \cos \omega t).$$

To satisfy the given initial condition we must have

$$0 = C + \frac{E}{p^2+\omega^2}\,(-\omega),$$

or

$$C = \frac{E\omega}{p^2+\omega^2}.$$

The desired solution is therefore

$$y = \frac{E}{p^2+\omega^2}(\omega e^{-pt} + p \sin \omega t - \omega \cos \omega t).$$

Linear differential equations have many special properties that make them extremely useful in a wide variety of applications. To investigate some of these properties let us return to the general solution (6.9), writing it in the form

$$y(x) = Cy_a(x) + y_b(x), \tag{6.12}$$

where

$$y_a(x) = e^{-P(x)}, \qquad P(x) = \int p(x)dx, \tag{6.13}$$

$$y_b(x) = e^{-P(x)} \int e^{P(x)} q(x)dx. \tag{6.14}$$

Of the three parts, C, y_a, and y_b, that make up the solution (6.12) we note the following:

(i) $y_a(x)$ is a solution of the equation

$$\frac{dy}{dx} + p(x)y = 0. \tag{6.15}$$

Equation (6.15) is said to be the <u>homogeneous part</u> of (6.3), and $y_a(x)$ is called a <u>complementary solution</u> of (6.3).

(ii) $y_b(x)$ is a particular solution of (6.3).

(iii) The integration constant C depends on the initial condition $y(x_o)$.

In view of (i) and (ii) we might wonder if $y_a(x)$ and $y_b(x)$ obtained from (6.13) and (6.14) are special forms of the solution or whether any solutions of (6.15) and (6.3) would do as well. That the latter is indeed the case is expressed in the following theorem.

Theorem 6.1. Let $y_a(x)$ be a solution, not identically zero, of $y' + py = 0$, and $y_b(x)$ a solution of $y' + py = q$. Then any solution $y_c(x)$ of $y' + py = q$ is expressible in the form $y_c = Cy_a + y_b$, C being a constant.

Proof. We are given that

$$y_c' + py_c = q$$

and

$$y_b' + py_b = q.$$

Subtracting the second equation from the first gives, since the difference of the derivatives of two functions is equal to the derivative of their difference,

$$(y_c - y_b)' + p(y_c - y_b) = 0.$$

This says that $y_c - y_b$ is a solution of the equation $y' + py = 0$, and is therefore of the form

$$y_c - y_b = A_b e^{-P(x)}.$$

Also, y_a is a solution of the same equation, so that

$$y_a = A_a e^{-P(x)},$$

and $A_a \neq 0$ since $y_a(x) \not\equiv 0$. From these two equations we get

$$y_c - y_b = \frac{A_b}{A_a} y_a$$

or, putting $C = A_b/A_a$,

$$y_c = Cy_a + y_b$$

as was desired.

To give physical interpretation to these properties we can regard the equation as a process, or an abstract "black box", that has the function $q(x)$ as input, and as output the solution $y(x)$. The structure of the black box determines the function $p(x)$. In most of these processes the independent variable x is time, but the input and the output can be almost anything - rotation of a shaft, electric current, flow of fluid, air pressure, man hours of work, price of a commodity, etc. Of course, most of the useful black boxes such as microphones, television sets, or models of the national economy, involve much more than a simple

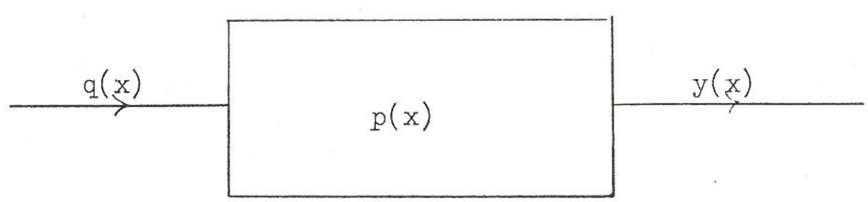


Figure 6.1

first order differential equation. Our present work is only the first step in the study of black boxes.

In these terms Theorem 6.1 tells us that any output or <u>response</u> to an input $q(x)$ is expressible in terms of the following quantities:

(i) A response $y_a(x)$ to zero input. This depends only on $p(x)$, i.e. on the structure of the black box.

(ii) A particular response $y_b(x)$ to the input $q(x)$. This depends on $p(x)$ and $q(x)$ but not on any initial condition.

(iii) A constant C which is obtained from an initial condition $y(x_o)$.

Most of our examples and problems will be confined to the case when $p(x)$ is a constant. In terms of our general physical picture with x representing time it means that the structure of the black box does not change with time.

If $p(x) = p$, a constant, the complementary solution is Ce^{-px}, so our chief concern is in getting a particular solution. In the following examples we solve this problem for three important types of input, and additional types are considered in problems. Finally, for any input not included in these cases we can always go back to equation (6.14) for a particular solution, and use numerical techniques, if necessary, to evaluate the integral.

<u>Example 6.3</u>. <u>$q(x)$ is a constant E</u>. Find a particular solution to

$$\frac{dy}{dx} + py = E, \quad \text{where} \quad p \neq 0. \tag{6.16}$$

Inspection of the left-hand side shows that it is a constant if y is a constant. Hence, take y_b to be

$$y_b = A,$$

where the unknown constant A is determined by substitution in the differential equation,

$$pA = E \quad \text{or} \quad A = E/p.$$

Hence, a particular solution is

$$y_b = E/p$$

and the total solution to (6.16) is sum of y_b and the complementary solution. Hence

$$y = Ce^{-px} + E/p.$$

<u>Example 6.4.</u> <u>$q(x)$ is of exponential form.</u> Find a particular solution to

$$\frac{dy}{dx} + py = Ee^{rx}. \tag{6.17}$$

The exponential function has the property that its derivative has the same form; thus, if we take y_b of exponential form

$$y_b = Ae^{rx}$$

a solution can be obtained by merely adjusting the constant A. Substituting y_b in the differential equation we get

$$(r+p)Ae^{rx} = Ee^{rx}$$

which is satisfied for all x if

$$A = E/(r+p).$$

Hence a particular solution is

$$y_b = \frac{E}{r+p} e^{rx},$$

provided $r \neq -p$.

In the case $r = -p$, we can see at once that Ae^{-px} cannot be a particular solution of

(6.18) $$\frac{dy}{dx} + py = Ee^{-px},$$

for Ce^{-px} is the complementary solution and so

$$\frac{d}{dx}(Ae^{-px}) + p(Ae^{-px}) = 0,$$

regardless of the value of **A**. We leave it to the reader to show that

$$y_b = Exe^{-px}$$

is a particular solution of (6.18), so the general solution is

$$y = Ce^{-px} + Exe^{-px}.$$

Example 6.5. q(x) is sinusoidal. Find a particular solution of

(6.19) $$\frac{dy}{dx} + py = E \sin \omega x.$$

If we now take y_b to be a sine function of the same frequency ω as $q(x)$, we see that the $\frac{dy_b}{dx}$ term will give a cosine function and (6.19) cannot be satisfied identically. To get around this difficulty we try a solution with both sine and cosine terms,

(6.20) $$y_b = A \sin \omega x + B \cos \omega x$$

and attempt to find the two constants A and B by substitution of (6.20) into (6.19). We get

$$[pB + \omega A] \cos \omega x + [pA - \omega B] \sin \omega x = E \sin \omega x.$$

With two constants available this relation can be satisfied for all x by equating coefficients of $\sin \omega x$ and $\cos \omega x$ on each side of the equation; thus,

$$pA - \omega B = E, \qquad (6.21)$$
$$\omega A + pB = 0.$$

Solving these equations simultaneously gives

$$A = \frac{pE}{p^2+\omega^2}, \qquad B = \frac{-\omega E}{p^2+\omega^2}, \qquad (6.22)$$

so that a particular solution is

$$y_b = \frac{E}{p^2+\omega^2}(p \sin \omega x - \omega \cos \omega x). \qquad (6.23)$$

Note that (6.19) is the same as the equation in Example 6.2, and that the same particular solution was obtained in the two cases.

The same trial solution (6.20) would be used if the input in (6.19) involved $\cos \omega x$ instead of $\sin \omega x$. In fact, even if terms of both types were present in the input, that is, if

$$q(x) = E \sin \omega x + F \cos \omega x, \qquad (6.24)$$

we would not have to treat the two terms separately but could carry both in the same computation. The only change would be to replace the second equation of (6.21) by $\omega A + pB = F$, leading to appropriate changes in (6.22) and (6.23).

The common technique in these three examples has been to assume a particular solution of a certain form and to adjust constants to give us the precise solution. This procedure has been dignified by the name, <u>the method of undetermined coefficients</u>.

If $q(x)$ involves more than one functional form, as in (6.24) for example, it is frequently advantageous to seek particular solutions corresponding to each functional form, and then combine these solutions to form the total particular solution. This approach is successful because it is based on an important property of linear differential equations, known as the Principle of Superposition.

<u>Theorem 6.2</u>. If $y_1, y_2, \dots, y_n$ are responses to the inputs $q_1, q_2, \dots, q_n$ respectively, then $y_1 + y_2 + \dots + y_n$ is a response to the input $q_1 + q_2 + \dots + q_n$.

<u>Proof</u>. We are given that

$$y_1' + py_1 = q_1,$$

$$y_2' + py_2 = q_2,$$

$$\dots$$

$$y_n' + py_n = q_n.$$

To secure the desired result we have only to add these equations and use the well-known additive property of the derivative to get

$$(y_1+y_2+ \dots +y_n)' + p(y_1+y_2+ \dots +y_n) = q_1+q_2+ \dots +q_n.$$

The Superposition Principle given by Theorem 6.2 enables us to add the outputs y_i corresponding to inputs q_i and thus construct the total output due to a sum of inputs. This principle is often referred to by the phrase "the sum of solutions is a solution for the sum of the inputs".

Example 6.6. Find a particular solution of

$$\frac{dy}{dx} + 2y = 6 + e^{-x} + \sin x - 2\cos x.$$

We consider the input as the sum of three parts, 6, e^{-x}, and $\sin x - 2\cos x$, and apply the method of undetermined coefficients to each of these separately.

For the input 6 we assume a particular solution $y = A$ of the equation $\frac{dy}{dx} + 2y = 6$. Substituting gives $2A = 6$, or $A = 3$, and so this partial particular solution is $y_1 = 3$.

For the input e^{-x} we assume $y = Ae^{-x}$. Then $\frac{dy}{dx} + 2y = e^{-x}$ gives $-Ae^{-x} + 2Ae^{-x} = e^{-x}$, or $A = 1$. This partial solution is then $y_2 = e^{-x}$.

For the input $\sin x - 2\cos x$ we choose

$$y = A\sin x + B\cos x,$$

and get

$$A\cos x - B\sin x + 2A\sin x + 2B\cos x = \sin x - 2\cos x.$$

Equating the coefficients of $\sin x$ and of $\cos x$ gives

$$2A - B = 1, \qquad A + 2B = -2,$$

which have the solution $A = 0$, $B = -1$. This solution is therefore $y_3 = -\cos x$.

By the Principle of Superposition the particular solution of our original problem is $y_b = y_1 + y_2 + y_3$ so that

$$y_b = 3 + e^{-x} - \cos x.$$

The general solution is just this plus Ce^{-2x}.

Problems

6.1 Find the general solution of each of the following equations.

(a) $\frac{dy}{dx} + \frac{2}{x} y = 6.$ Ans. $y = 2x + \frac{C}{x^2}.$

(b) $\frac{dy}{dx} + y = e^{-x}.$

(c) $\frac{dy}{dx} + xy = x.$ Ans. $y = 1 + Ce^{-x^2/2}.$

(d) $\frac{dy}{dx} + y \tan x = \sin 2x.$

(e) $\frac{dy}{dx} + \frac{x-1}{x} y = x + e^{-x}.$ Ans. $y = x + \frac{x\log|x|}{e^x} + \frac{Cx}{e^x}.$

6.2 Find the general solution of each of the following equations.

(a) $\frac{dy}{dx} + y = 2 + e^x.$ Ans. $y = 2 + \frac{e^x}{2} + \frac{C}{e^x}.$

(b) $\frac{dy}{dx} + 2y = e^{2x} + e^{-2x}.$

(c) $\frac{dy}{dx} - y = \sin 2x + 2 \cos 2x.$ Ans. $y = \frac{1}{5}(3 \sin 2x - 4 \cos 2x) + Ce^x.$

(d) $\frac{dy}{dx} + 2y = \sin x + \sin 2x + \sin 3x.$

6.3 An equation of the type

$$\frac{dy}{dx} + p(x)y = q(x)y^n,$$

is called a Bernoulli equation.

(a) Show that a substitution of the type $v = y^{1-n}$ will reduce the Bernoulli equation to the linear equation

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)q(x).$$

(b) Solve

(i) $\frac{dy}{dx} = y + xy^2$,

(ii) $\frac{dy}{dx} = y + x\sqrt{y}$.

6.4 We wish to consider the most general form of $q(x)$ for which the method of undetermined coefficients will work. The criterion is as follows: if $q(x)$ and all its derivatives $q'(x)$, $q''(x)$, ... involve terms of only a finite number of types, then there is a particular solution involving terms of these types; with one exception, namely if $q(x)$ involves a term of the type $x^n e^{-px}$, $n \geqq 0$, then the particular solution may involve a term of type $x^{n+1} e^{-px}$.

For example, let $q(x) = x \sin \omega x$. Then

$$q'(x) = \sin \omega x + \omega x \cos \omega x$$

involves terms of the type $\sin \omega x$ and $x \cos \omega x$. Derivatives of these involve $\cos \omega x$ and $x \sin \omega x$, $\cos \omega x$ being the only new type. The next derivative involves nothing new and hence neither do any further derivatives. We therefore take a solution of the type

$$y = a \sin \omega x + b \cos \omega x + cx \sin \omega x + dx \cos \omega x.$$

Substituting in $y' + py = x \sin \omega x$ and collecting terms gives

$$(pa - \omega b + c) \sin \omega x + (\omega a + pb + d) \cos \omega x$$
$$+ (pc - \omega d)\, x \sin \omega x + (\omega c + pd)\, x \cos \omega x = x \sin \omega x.$$

This equation will be true for all x if

$$pa - \omega b + c = 0,$$

$$\omega a + pb + d = 0,$$

$$pc - \omega d = 1,$$

$$\omega c + pd = 0,$$

and so solution of these four simultaneous equations for a, b, c, d will give the particular solution.

Find a particular solution of each of the following equations.

(a) $y' + 3y = 8e^{-x} \sin 2x.$ Ans. $y = 2e^{-x}(\sin 2x - \cos 2x).$

(b) $y' - 2y = xe^{x}.$

(c) $y' - 2y = xe^{2x}.$ Ans. $y = \frac{1}{2}x^2 e^{2x}.$

(d) $y' + y = x^2 - 2x - 3.$

(e) $y' + 2y = 2x \sin x - 3x \cos x.$

Ans. $y = \frac{1}{5} x \sin x - \frac{8}{5} x \cos x + \frac{6}{25} \sin x + \frac{17}{25} \cos x.$

6.5 Find the solution to the differential equation

$$\frac{dy}{dx} - y = e^{x}\sqrt{1 + x^3} \qquad \text{with } y(1) = 0.$$

The presence of the $\sqrt{1 + x^3}$ factor in $q(x)$ makes it impossible to apply the method of undetermined coefficients or its general form given in Problem 6.4 because successive derivatives lead to new functional forms. Apply the integrating factor method of solution to show that

$$ye^{-x} = \int_1^x \sqrt{1 + x^3}\, dx.$$

The right-hand side is the integral of Example 5.3 which cannot be integrated in closed form, but a numerical solution could be obtained using the Trapezoidal or Simpson's Rule.

7. Applications

The formulation of an appropriate mathematical model is often not an obvious or trivial thing and the scientist or engineer is frequently confronted with two alternatives. He can develop a mathematical description of a phenomenon which includes as few assumptions as possible. The resulting model may be accurate, but so complicated analytically that only an approximate solution is obtainable. Alternatively, he can construct (knowningly or unknowingly) a cruder model which leads to equations which can be solved exactly. The scientist or engineer must develop the threefold ability to derive a suitable mathematical model, solve the analytical problem, and draw conclusions or useful information from the result.

Example 7.1. Poisoning the Sea

The disposal of waste material is becoming a serious problem, for the nature and quantities of material may create hazards for man. For several years concentrated radioactive waste material has been disposed of by simply putting it in drums and dropping them overboard in about 50 fathoms (1 fathom equals 6 feet) depth of ocean. As a relative of the dolphin family you may be concerned about the possible poisoning effects if the drums break open upon striking bottom. Those responsible for this operation claim that there is "absolutely no danger." At the request of irate oceangraphers, towing experiments have been performed to determine the force required to tow drums at different orientations through water. These results show that the orientation has little effect and the drag force is directly proportional to the first power of the velocity, with a coefficient of proportionality $c = 8x10^{-2}$ lb./(ft./sec.) Further, impact experiments show that drums do not break if they strike rock with a velocity less than 40 ft./sec. Determine whether the oceanographers and dolphins have cause for concern.

Figure 7.1, a free body diagram, shows the force of gravity less the buoyant force pulling the drum down, and the drag force resisting the motion. Newton's Law gives

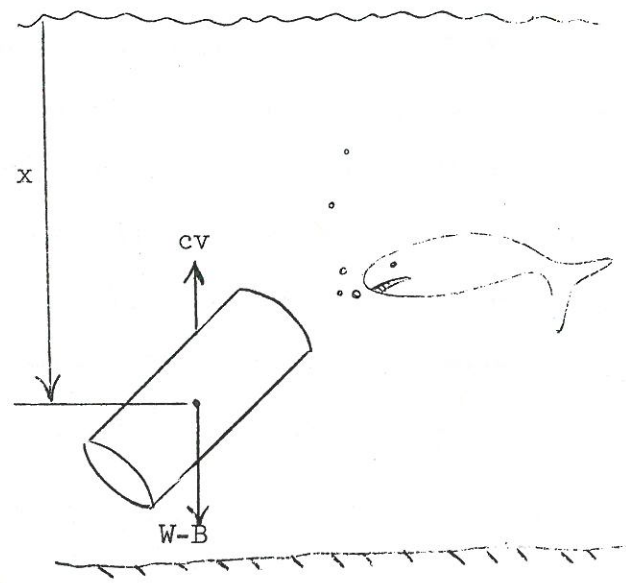


Figure 7.1

$$(7.1)\quad (W - B) - cv = \frac{W}{g}\frac{d^2x}{dt^2} = \frac{W}{g}\frac{dv}{dt}$$

and the variables v and t can be separated so that integration yields

$$\int_0^t -\frac{cg}{W}\,dt = \int_{v_0}^{v} \frac{d[(W-B) - cv]}{[(W-B) - cv]} .$$

Integration produces

$$\log\left|\frac{(W-B) - cv}{(W-B) - cv_0}\right| = -\frac{cg}{W}t ,$$

and with $v_0 < (W-B)/c$ this can be solved explicitly for v:

$$(7.2)\qquad v = \frac{(W-B)}{c} - \left[\frac{(W-B)}{c} - v_0\right] e^{-\frac{cg}{W}t} .$$

For large t the exponential term vanishes and

$$(7.3)\qquad v \longrightarrow \frac{(W-B)}{c} = v_T ;$$

v_T is called a terminal velocity. At the terminal velocity, the downward force (W-B) is opposed by the equal and opposite drag force cv_T. The velocity increases from its initial value v_o to the limiting terminal velocity v_T. Equation (7.2) is inconvenient for our purposes because we wish to find the velocity in terms of depth. We could look at (7.2) as itself a differential equation after recognizing that $v = dx/dt$. Another integration would give $x = x(t)$ and in principle one could find $t = t(x)$ and substitution in (7.2) would give $v = v(x)$. However, this approach is laborious and by using the change of variable of Case III Section 5 we write $\frac{dv}{dt}$ of (7.1) as

$$\text{acceleration} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} v \quad . \tag{7.4}$$

Thus, the original equation (7.1) becomes

$$(W-B) - cv = \frac{W}{g} v \frac{dv}{dx} \quad . \tag{7.5}$$

Before solving this equation it is instructive to obtain an upper bound on the velocity by assuming the drag force to be zero, $c = 0$. The drum is now pulled downward by gravity, less the buoyant force. Equation (7.5) reduces to

$$\int_0^x (W-B)\,dx = \int_0^v \frac{W}{g} v\,dv \tag{7.6}$$

where the initial velocity has been assumed to be zero. When integrated this equation gives

(7.7) $$(W-B)x = \frac{1}{2}\frac{W}{g}v^2 \quad .$$

Assuming the 55 gallon drums (7.35 ft.^{3}) are filled with a material having specific gravity 1.15 and are dropped in 50 fathoms, equation (7.7) gives 50 ft./sec. for the velocity with which they strike bottom. This exceeds the safe value of 40 ft./sec. With 50 ft./sec. as an upper bound on the velocity it is seen that the drag term in (7.5), cv, is less that $8\times10^{-2}(50) = 4$ lb., while the driving force term $(W-B) = (1.15-1)(62.4)(7.35) = 69$ lb. The drag will have little retarding effect. To determine the exact striking velocity integrate (7.5) by separation of variables. We get

$$\int_0^x \frac{g}{W}\,dx = \int_{v_o}^v \frac{v\,dv}{(W-B) - cv} = \int_{v_o}^v \left(-\frac{1}{c}\right)dv + \int_{v_o}^v \frac{(W-B)\,dv}{c[(W-B) - cv]},$$

which with $v_o = 0$ integrates to

(7.8) $$v = \frac{(W-B)}{c}\log\frac{(W-B)}{(W-B) - cv} - \frac{cg}{W}x.$$

We wish to find v for a given value of x, but solution (7.8) does not give an explicit expression $v = v(x)$. Instead, we have a transcendental equation to solve. We can simplify the arithmetic by setting

$$u = \frac{c}{W-B}v \quad , \qquad a = \frac{c^2 g}{W(W-B)}x.$$

Then (7.8) becomes

(7.9) $$u + \log(1-u) + a = 0.$$

With numerical values previously assumed and with $x = 300$ ft. (bottom) we have

$$u = \frac{10^{-2}}{8.6} v \ , \qquad a = .171 \times 10^{-2} \ .$$

Equation (7.9) can be solved by trial and error, Newton's Method (see Thomas, Section 9-3), or any other standard method. A good initial guess for u is .058, obtained by taking for v the upper bound 50 ft./sec. To three significant figures (obtained in one step of Newton's Method) $u = .0573$, giving $v = 49.3$ ft./sec.

Thus the drum strikes the bottom with a velocity not far below the 50 ft./sec. found with no drag, and it's impact velocity appreciably exceeds the experimental safe limit. We are about to sign the irate oceanographers' petition, but hesitate in view of the "absolutely no danger" claim. Can you discover an engineering change which might be used to insure that the drums strike bottom with velocities less than 40 ft./sec.? (Hint: inspect equation (7.3)).

Example 7.2. A Water Pollution Problem

In the treatment of polluted water a situation which frequently arises can be characterized as follows. An activated sludge aeration tank contains a concentration c of a pollutant. The raw sewage influent from a city sewage collection system which contains a higher concentration of the pollutant is added to the tank. After being held in the tank in order to

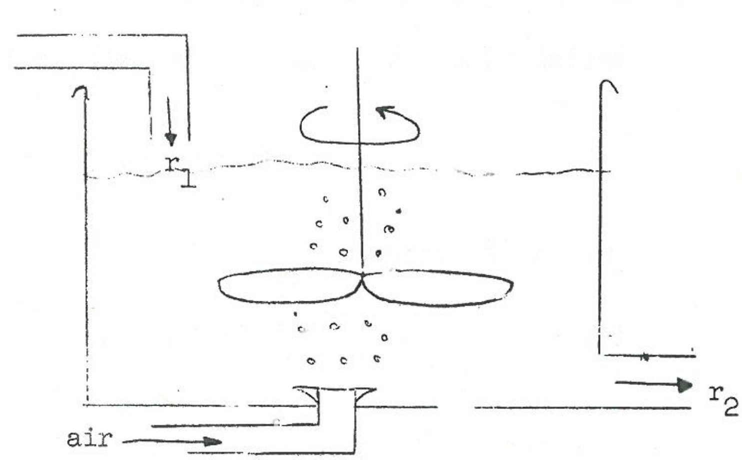


Figure 7.2

permit the bacteria to digest the sewage solids, the mixture is discharged into a river. Figure 7.2 shows the aeration tank into which is steadily fed r_1 gal./min. of influent having a pollutant concentration c_1 lb./gal. The effluent flowing out of the tank at a steady rate r_2 gal./min. is the mixture of bacterial cells and undigested pollutants. The propellor-like stirring device symbolizes an idealization of instantaneous mixing so that the mixture in the tank can be assumed to have perfect uniformity. Initially the tank contains V_0 gal. of wastewater having z_0 lbs. of pollutant. With constant rates of flow in and out, the volume of water in the tank at any time is $V_0 + (r_1-r_2)t$. The problem is to determine how long one can pump from the tank into the river before the concentration in this discharge exceeds a safe level, say $0.30c_1$.

In any period of time the increase of pollutant in the tank must equal the amount of the pollutant which has been added less the amount removed. Let $z(t)$ be the lbs. of pollutant in the tank at any time t. Then in time Δt min. we apply the conservation of mass

$$(7.10) \qquad z(t + \Delta t) - z(t) = r_1 c_1 \Delta t - r_2 \left[\frac{z(\bar{t})}{V_0 + (r_1-r_2)\bar{t}}\right] \Delta t$$

where

$$(7.11) \qquad \frac{z(\bar{t})}{V_0 + (r_1-r_2)\bar{t}}$$

is the concentration at time $\bar{t}$, between t and $t+\Delta t$, chosen so that all the mass leaving in time Δt is accounted for. Divide equation (7.10) by Δt and take the limit as $\Delta t \to 0$:

$$\lim_{\Delta t \to 0} \left[\frac{z(t+\Delta t) - z(t)}{\Delta t} = r_1 c_1 - r_2 \frac{z(\bar{t})}{V_0 + (r_1-r_2)\bar{t}} \right]$$

which gives

$$\frac{dz}{dt} = r_1 c_1 - (r_2) \frac{z}{V_0 + (r_1-r_2)t} \quad . \tag{7.12}$$

We could also have written equation (7.12) directly by thinking of the conservation of pollutant in terms of rates. Everyone should be able to arrive at equation (7.12) both ways. If the differential equation is rearranged to

$$\frac{dz}{dt} + \frac{r_2}{V_0 + (r_1-r_2)t} z = r_1 c_1 , \tag{7.13}$$

we recognize it to be a first-order linear equation. The integrating factor is $[V_0 + (r_1-r_2)t]^{\frac{r_2}{r_1-r_2}}$ and the student should show that integration gives

$$z[V_0 + (r_1-r_2)t]^{\frac{r_2}{r_1-r_2}} = C + c_1 [V_0 + (r_1-r_2)t]^{\frac{r_1}{r_1-r_2}} \quad . \tag{7.14}$$

If at time $t = 0,\ z = z_0$ so that the initial concentration is $c_0 = z_0/V_0$, then equation (7.14) can be solved for the instantaneous concentration $c = z/(V_0 + (r_1-r_2)t)$ to give the relatively simple form

$$c = c_1 + (c_0-c_1) \left[1 + \frac{(r_1-r_2)}{V_0} t\right]^{-\frac{r_1}{r_1-r_2}} \quad . \tag{7.15}$$

If $r_1 > r_2$ then the physically necessary conclusion is reached that if $t \longrightarrow \infty$ then $c \longrightarrow c_1$ (provided the tank is sufficiently large!).

Further, if $r_2 > r_1$ and $c_0 < c_1$ then the concentration rises from c_0 toward c_1. What is the physical significance of the quantity in the square bracket in (7.15) becoming negative? To find the time elapsed until the discharge concentration c is $0.30c_1$, we must solve equation (7.15). Assume c_0 is $0.05c_1$, $r_1 = 500$ gal./min., $r_2 = 250$ gal./min., and $V_0 = 250,000$ gal. After dividing by c_1 in (7.15) and substituting the numerical values we get

$$0.30 = 1 + (0.05 - 1) \left[1 + \frac{250}{250 \times 10^3} t\right]^{-2},$$

which when solved for t gives 160 min. or 2.7 hours.

Thus the solution to (7.15) indicates that the process could be operated for a 2.7-hr. period without violating the requirement that the pollutant concentration of the effluent be less than $0.30c_1$. If however the influent to this sewage treatment process were to continue (as is usually the case) then an additional aeration tank would be required to accept flow r_1 since continued use of the first tank would necessarily violate the safe level for the pollutant. The initial tank would then be continuously aerated until the concentration in the tank was reduced from $0.30c_1$ to c_0.

Example 7.3. An Insulation Problem

Heat flow by conduction is a process in which thermal energy is transmitted as a result of the excitation of matter about a state of equilibrium. The flow of heat in solids takes place exclusively by this process, while in liquids and gases the processes of conduction, convection and radiation can occur simultaneously. The basic law of heat conduction, which can be determined on the basis of either experiment or a theoretical molecular model, has the form

(7.16) $$dq_n = - kdA \frac{\partial T}{\partial n}$$

where dq_n is the quantity of heat flowing per unit time (BTU/min.) through an element of area dA in a direction n along the normal to area dA. See Figure 7.3.

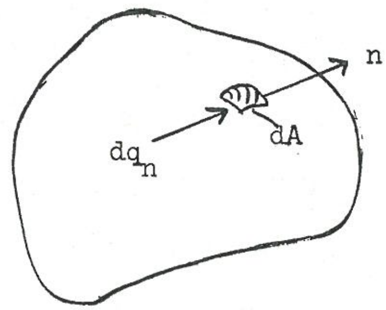


Figure 7.3

The quantity of heat flowing is directly proportional to the temperature gradient $\frac{\partial T}{\partial n}$ along normal n. Constant k (BTU/min. ft. °F) is called the thermal conductivity and is a measure of how well a given material conducts heat. The negative sign in (7.16) indicates that heat flows in the direction of a temperature drop, $-\partial T/\partial n$. The gradient in (7.16) is written as a partial derivative because the temperature could be a function of the three space coordinates as well as time. Equation (7.16) is the fundamental law governing the process of conduction.

Of primary interest in technological applications is the flow of heat between two materials at different temperatures, when the substances are separated by a wall. Consider a plane wall of thickness h and thermal conductivity k, the surfaces of which are maintained at constant temperatures T_1 and T_2 by the substances on either side. Assume steady state, one dimensional heat flow so that $\frac{\partial T}{\partial n} = \frac{dT}{dx}$. The rate of heat flow Q through area A of the wall is given by equation (7.16) as

$$\int_0^Q dq_n = -k \frac{dT}{dx} \int_0^A dA$$

or

$$Q = -kA \frac{dT}{dx} .$$

If this expression is integrated

$$Q \int_0^x dx = -kA \int_{T_1}^{T} dT ,$$

we get

(7.17) $$Qx = -kA(T-T_1)$$

and the temperature within the wall drops linearly with thickness. For the full thickness $x = h$ and (7.17) is

(7.18) $$Q = -\frac{kA}{h}(T_2-T_1) .$$

The heat flow is driven by the temperature difference T_1-T_2.

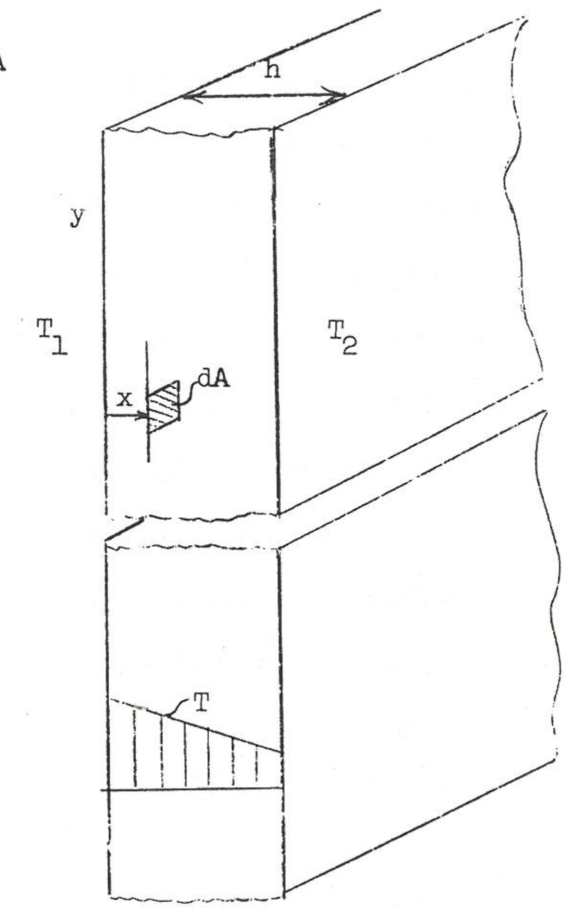


Figure 7.4

In many applications the wall is not plane so that the area through which heat flows is not constant. Consider the storage of gas in the liquid state in a pressurized steel sphere. A cooling system which can remove heat at the rate of 50,000 BTU/hr. is inserted in the liquid in order to keep it at -300°F when the maximum external temperature is 100°F. In order to apply the law of heat conduction, equation (7.16), we must identify either an infinitesimal area dA or a finite area normal to the temperature gradient $\frac{\partial T}{\partial n}$. Because of the spherical symmetry, heat will flow radially inward and $\frac{\partial T}{\partial n}$ in equation (7.16) becomes $\frac{dT}{dr}$ for n is along a radius which is perpendicular to a spherical area $4\pi r^2$. Thus, integrating (7.16) over a spherical surface gives

$$\int_0^Q dq_n = -k\frac{dT}{dr}\int_0^{4\pi r^2} dA$$

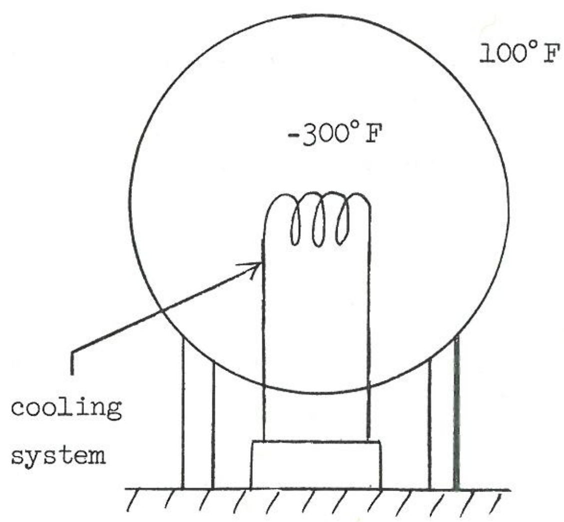


Figure 7.5

so that

$$(7.19) \qquad Q = -4\pi k r^2 \frac{dT}{dr} .$$

We can derive a differential equation which does not involve Q by considering the flow of heat through a spherical shell of thickness Δr. Assuming $\frac{dT}{dr}$ positive, then from (7.19) the inward rate of heat flow across the surface at radius $(r+\Delta r)$ is

$$(7.20) \qquad Q = -4\pi k(r+\Delta r)^2\left(\frac{dT}{dr}\right)_{r+\Delta r} .$$

The inward rate of heat flow across the surface at radius r is

$$Q = -4\pi k r^2 \left(\frac{dT}{dr}\right)_r \quad . \tag{7.21}$$

If steady state conditions prevail, then the rate of heat flow into the spherical shell, equation (7.20), must equal the rate of heat flow out of the shell, equation (7.21). Thus

$$-4\pi k(r+\Delta r)^2\left(\frac{dT}{dr}\right)_{r+\Delta r} = -4\pi k r^2\left(\frac{dT}{dr}\right)_r$$

and this can be written as

$$r^2\left[\left(\frac{dT}{dr}\right)_{r+\Delta r} - \left(\frac{dT}{dr}\right)_r\right] + 2r\Delta r\left(\frac{dT}{dr}\right)_{r+\Delta r} + \Delta r^2\left(\frac{dT}{dr}\right)_{r+\Delta r} = 0.$$

If we now divide by Δr and take the limit as $\Delta r \to 0$ we get

$$\lim_{\Delta r \to 0}\left[\frac{r^2\left[\left(\frac{dT}{dr}\right)_{r+\Delta r} - \left(\frac{dT}{dr}\right)_r\right]}{\Delta r} + 2r\left(\frac{dT}{dr}\right)_{r+\Delta r} + \Delta r\left(\frac{dT}{dr}\right)_{r+\Delta r} = 0\right]$$

In the limit the first term is $r^2\frac{d^2T}{dr^2}$, the second $2r\,\frac{dT}{dr}$, and the third term vanishes. The resulting differential equation for the temperature in a sphere is

$$r^2\frac{d^2T}{dr^2} + 2r\,\frac{dT}{dr} = 0 \quad . \tag{7.22}$$

As described in Case III Section 5 this second-order equation can be reduced to first order by the substitution $\frac{dT}{dr} = u$. Equation (7.22) becomes

$$\frac{du}{dr} + \frac{2}{r}\,u = 0$$

which can be integrated either by separating variables or by the integrating factor method for linear equations. Integration gives

(7.23) $$u = \frac{dT}{dr} = + \frac{C}{r^2}$$

which can be integrated again by separation of variables to yield

(7.24) $$T = - \frac{C}{r} + D$$

in which C and D are constants of integration. If at the outside radius r_1 of the sphere the temperature is T_1 and at the inner radius r_2 the temperature is T_2 then these two conditions when substituted in (7.24) give

$$C = r_1 r_2 \frac{(T_1 - T_2)}{(r_1 - r_2)} \quad , \qquad D = \frac{T_1 r_1 - T_2 r_2}{(r_1 - r_2)} \quad .$$

Equation (7.24) becomes

(7.25) $$T = \left[(T_1 r_1 - T_2 r_2) - r_1 r_2 (T_1 - T_2) \frac{1}{r} \right] \frac{1}{(r_1 - r_2)} \quad .$$

Further, from (7.23) $\frac{dT}{dr} = \frac{C}{r^2}$ so that (7.19) becomes

(7.26) $$Q = - 4\pi k r_1 r_2 \frac{(T_1 - T_2)}{(r_1 - r_2)} \quad .$$

Equation (7.25) gives the temperature distribution as a function of r while (7.26) gives the rate of heat flow in terms of the conditions at each boundary.

Equation (7.26) is convenient because it relates quantities which naturally occur in design. Assume that for reasons of storage capacity r_2 must be 4 ft. and that from considerations of strength the shell thickness $r_1 - r_2 = 1$ inch. Steel has a thermal conductivity $k = 25$ BTU/hr.ft.°F so that with $T_1 - T_2 = 100 - (-300) = 400°F$ equation (7.26) gives

$$Q = -4\pi(25)(4 + \frac{1}{12})(4)(400)/(\frac{1}{12}) = -24.6 \times 10^6 \text{ BTU/hr.}$$

Thus, this combination of parameters would give a rate of heat flow inward to the liquid gas far in excess of the capacity of the cooling unit, 50,000 BTU/hr. The liquid would soon vaporize; the pressure would rise drastically and burst the spherical shell. The difficulty arises from the combination of parameters whereby a temperature drop of 400°F, is taken over a small distance, 1 inch, by a material with a relatively high conductivity, k = 25. This approach is defective because it is a mistaken concept to use the thickness of steel alone as a temperature reducing medium. From (7.19) we see that the rate of heat flow depends directly on the thermal conductivity and the temperature gradient. Thus, for a given rate of flow, if the temperature gradient is relatively high then the conductivity must be relatively low.

The engineering problem would be solved by taking the main temperature drop through an insulating material attached to the outside of the sphere. The shell thickness would still be determined on the basis of strength (1"). The temperature drop in the steel is determined from (7.26) with $Q = -50,000$, $k = 25$, $r_2 = 4$, $r_1 = 4 + \frac{1}{12}$ so that $(T_1 - T_2)$ becomes

$$T_1 - T_2 = 5/6.15 = 0.813$$

and with $T_2 = -300°F$ we get an outer shell temperature of $T_1 = -299.2°F$. Thus, the steel can essentially be neglected as far as impeding the flow of heat is concerned. For a fiberglass wool insulating material $k = 0.024$ BTU/(hr.ft.°F) so that (7.26) gives for the fiberglass wool spherical covering

$$-50,000 = -4\pi \frac{(0.024)(4.083 + t)}{t}(4.083)(100 - (-299.2))$$

where t is the shell thickness, $r_1 = r_2 + t$. This equation gives $t = 0.040$ ft. or about 1/2 inch. The major temperature drop is taken in the 1/2" fiberglass insulation whose thermal conductivity is so small that the rate of heat flow is 50,000 BTU/hr., the capacity of the cooling unit.

Example 7.4. A Low Pass Filter

A simple circuit which passes low frequency signals but inhibits the passage of high frequency signals (filters them out) consists of resistor R connected in series with a capacitor of capacitance C. Connected in parallel with the capacitor is a load resistor R_L; see Figure 7.6. An input voltage $E_1(t)$ is

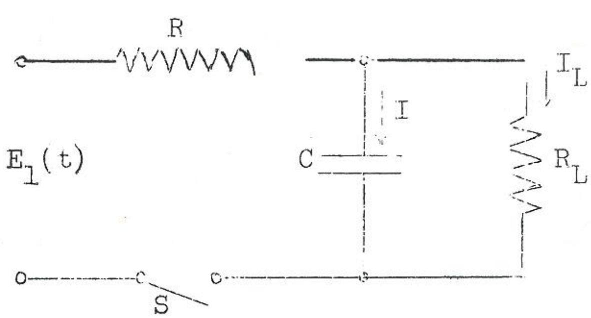


Figure 7.6

applied to the circuit, and with switch S closed the question is to determine the voltage across the load resistor R_L.

With current I in the capacitor branch and I_L in the load resistor branch we can write the voltage equation around the outside circuit as

(7.27) $$R(I + I_L) + E = E_1(t)$$

where E is the voltage drop across both R_L and C so that

(7.28) $$E = R_L I_L = \frac{1}{C} q .$$

Current $I = \frac{dq}{dt} = C \frac{dE}{dt}$ so that (7.27) becomes upon elimination of I_L

(7.29) $$RC \frac{dE}{dt} + (1 + \frac{R}{R_L})E = E_1(t) .$$

First consider the case when a constant voltage $E_1(t) = E_1$ is applied to the circuit and switch S closed. Equation (7.29) becomes

$$RC \frac{dE}{dt} + (1 + \frac{R}{R_L})E = E_1 .$$

From Section 6, the solution can be expressed as the sum of the complementary solution

(7.30) $$Ae^{-\frac{1}{RC}(1 + \frac{R}{R_L})t}$$

and a particular solution, which by inspection is

(7.31) $$E_1/(1 + \frac{R}{R_L}) .$$

Hence, the solution is

(7.32) $$E = E_1/(1 + \frac{R}{R_L}) + Ae^{-\frac{1}{RC}(1 + \frac{R}{R_L})t}$$

If, when the switch is closed at time $t = 0$, there is no charge on the capacitor then $E(0) = 0$ and (7.32) can be solved for **A**. Equation (7.32) gives the load voltage as

(7.33) $$E = \frac{E_1}{(1 + \frac{R}{R_L})}\left[1 - e^{-\frac{1}{RC}(1 + \frac{R}{R_L})t}\right] .$$

The voltage rises from zero to the constant value (7.31) in which E is proportional to E_1. This kind of phenomenon occurs so frequently in physical systems that special names have been given to the complementary and particular solutions. The complementary solution (7.30), which decays exponentially with time, is called a transient and the particular solution (7.31) which is left after the transient period is called the steady state. Thus, we describe the total solution (7.33) as being the sum of a transient and a steady state solution. This type of behavior also occurred in Example 7.1 where the steady state is the constant terminal velocity and the accelerating motion is the transient.

Now consider the case when the input voltage is sinusoidal with amplitude E_1 and frequency f so that $E_1(t) = E_1 \sin 2\pi ft$. Equation (7.29) becomes

(7.34) $$RC\frac{dE}{dt} + (1 + \frac{R}{R_L})E = E_1 \sin 2\pi ft .$$

The complementary solution is again (7.30) and we now seek a particular solution of (7.34). From Example 6.5 we conclude that the particular solution has the form

$$E = A \sin 2\pi ft + B \cos 2\pi ft \ .$$

When substituted in (7.34) and the coefficients of sine and cosine equated, A and B are determined and the particular solution is

$$(7.35) \qquad E = \frac{E_1/RC}{\left(\frac{1 + R/F_L}{RC}\right)^2 + (2\pi f)^2} \left\{ \left[\frac{(1 + R/R_L)}{RC}\right] \sin 2\pi ft - [2\pi f] \cos 2\pi ft \right\} .$$

This expression can be simplified by using the trigonometric identity

$$(7.36) \qquad D \sin (2\pi ft - \phi) = [D \cos \phi] \sin 2\pi ft - [D \sin \phi] \cos 2\pi ft.$$

Equate the coefficients of $\sin 2\pi ft$ and $\cos 2\pi ft$ in the square brackets of (7.35) and (7.36) to obtain expressions for the two constants D and ϕ:

$$(7.37) \qquad D \cos \phi = \frac{1 + R/R_L}{RC} , \qquad D \sin \phi = 2\pi f.$$

The two equations in (7.37) when solved for D and ϕ give

$$(7.38) \qquad D^2(\cos^2\phi + \sin^2\phi) = D^2 = \left(\frac{1 + R/R_L}{RC}\right)^2 + (2\pi f)^2,$$

$$\tan \phi = \frac{2\pi f}{\left[\frac{1 + R/R_L}{RC}\right]} .$$

The result is that the $\sin 2\pi ft$ and $\cos 2\pi ft$ terms in (7.35) can be combined into one term of the form $D \sin (2\pi ft - \phi)$ where D and ϕ are given by (7.38). Particular solution (7.35) becomes

(7.39) $$E = \frac{E_1}{\sqrt{(1 + R/R_L)^2 + (2\pi fRC)^2}} \sin(2\pi ft - \phi) .$$

The total solution is the sum of complementary solution (7.30) and particular solution (7.39). Again the complementary solution is a transient which decays exponentially to zero leaving the steady state solution (7.39). Observe that the steady state output (7.39) has the same sinusoidal form as the input, $E_1 \sin 2\pi ft$, only the amplitude depends on the frequency, f, and the output voltage lags the input by phase angle ϕ. The dependence of the amplitude on f gives the circuit its filter properties. If $f \rightarrow 0$ the amplitude approaches $\frac{E_1}{1 + R/R_L}$, which is the value of E for equation (7.31), the case with constant input voltage. If f is increased, the amplitude of (7.39) decreases and the output voltage can be made very small for high frequency input signals. Such a circuit is said to be a low pass filter - low frequency input signals are transmitted to the output.

Problems

7.1 A basic difficulty arises in temperature measurement because of the temperature lag in the sensing element. Assume a thermocouple probe, initially at temperature T_o, is placed in a combustion chamber where the temperature oscillates sinusoidally with amplitude T_n and frequency f about a value T_E. The probe changes temperature subject to Newton's law of cooling.

(a) Determine the temperature-time relation for the probe. Identify the transient response.

(b) Find the solution for large t, called the steady state response, and discuss the lag and amplitude for different parameter sizes.

Ans. (a). $T = T_E + T_n \cos\varphi \sin(2\pi f t - \varphi) + (T_o - T_E + T_n \sin\varphi \cos\varphi)e^{-kt}$.

7.2 A capacitor having capacitance C is arranged in series with resistor R and voltage V.

(a) If at time $t = 0$ there is no charge on the capacitor and a constant voltage $V = E$ is applied, find the time variation of the charge on the capacitor and the current in the circuit.

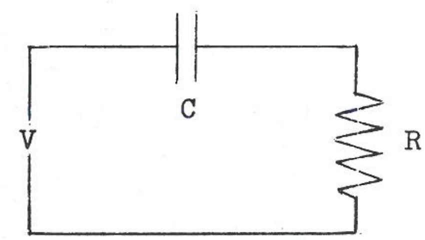


Figure 7.7

(b) If at time $t = 0$ there is no charge on the capacitor and a periodic voltage $V = E \sin\omega t$ is applied, find the current in the circuit and its amplitude and phase relationship with the applied voltage.

7.3 Devise an electric circuit which would be the electrical analog of the system in problem 7.1. Discuss its transient and steady state response.

7.4 Calculate the rate of heat flow Q and temperature distribution T through a unit length of circular cylindrical pipe with inner radius r_i and outer radius r_o and with wall temperatures T_i and T_o if the pipe consists of a metal pipe of thickness h_1 surrounded by an insulating jacket of thickness h_2.

7.5 In the design of a bathyscaph gondola the cooling effect of the ocean must be taken into account. The gondola is essentially a sphere of inner radius r_i and outer radius r_o. It is desired to maintain the inside temperature at T_i for the physical well-being of the occupants. If the deep ocean temperature is T_o determine the variation of temperature and the heat flow through the wall of the shell.

7.6 A tank holding 1000 gallons of salt brine has its bottom covered with a layer of undissolved salt. This dissolves at a rate proportional to the difference between the constant saturation concentration S lb./gal. and the actual concentration of the brine. Fresh water enters the tank at 100 gal./min. and brine leaves the tank at the same rate. Set up the differential equation for the amount, $x(t)$ in lbs., of dissolved salt in the tank. If $x(0) = x_o$ find the amount of salt x at any time t. What is the physical interpretation of the complementary differential equation and its solution? Determine the steady state concentration of salt in the tank.

7.7 In Lanchester's model for combat between riflemen or ships or aircraft (individual targets) the equations

$$\frac{dn_1}{dt} = - k_2 n_2$$

$$\frac{dn_2}{dt} = - k_1 n_1$$

are used where n_1 and n_2 denote the number of troops of Army #1 and Army #2 engaged in the battle. Show the rationale behind the equations and explain the significance of the constants k_1, k_2.

Similarly in area fire (firing into troop concentrations) Lanchester's equations are

$$\frac{dn_1}{dt} = - K_2 n_1 n_2$$

$$\frac{dn_2}{dt} = - K_1 n_1 n_2 \quad .$$

Explain how these are arrived at.

These equations were derived in 1914 and were intended to account for the presence of the machine gun as well as the automatic rifle. Set up your own equations which would take into account high energy nuclear bombing. Reference: Lanchester, Fredrich William, "Mathematics in Warfare" in James R. Newman The World of Mathematics, Simon and Schuster, New York, 1956, p. 2138-2157; from Fredrich William Lanchester, Aircraft in Warfare, Constable and Co. Ltd., 1916.

7.8 In "line of sight" guidance of a missile M, the velocity V of the missile is at any moment directly towards the target T. Let (A,B) be the coordinates of T, (x,y) the coordinates of M, θ the angle the velocity vector makes with the x-axis, and V the magnitude of the velocity. Then

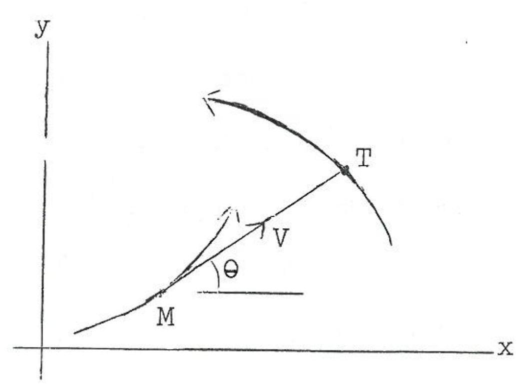


Figure 7.8

$$\frac{dx}{dt} = V \cos \Theta = V \frac{A-x}{\sqrt{(x-A)^2 + (y-B)^2}} ,$$

$$\frac{dy}{dt} = V \sin \Theta = V \frac{B-y}{\sqrt{(x-A)^2 + (y-B)^2}} .$$

Assume that T moves with velocity 2000 ft./sec. on a circle with radius 20 miles and center at the origin, and that the missile has velocity 4000 ft./sec. If the missile is at the origin when the target is on the x-axis when and where will the missile hit the target? [Use Euler's method on the computer, with h = 1 sec. You may want to decrease h when the missile gets close to the target.] Ans. $x \cong 87,800$ ft., $y \cong 58,700$ ft., $t \cong 31.6$ sec.

7.9 A diabetic can take too much insulin and create a dangerous condition known as hypoglycemia - an insufficient supply of glucose. To correct this deficiency as rapidly as possible glucose is fed intravenously until the proper balance is restored. Glucose is added at a steady rate r mg./min. Assume the human system absorbs glucose from the blood at a rate proportional to the amount present in the blood, with a rate of elimination constant k. Determine the differential equation for the concentration of glucose in the blood if V is the volume of blood circulated. Find the solution to the differential equation and interpret it physically.

7.10 In electromagnetic devices a magnetic flux ϕ is established in a magnetic circuit by passing current i through a coil which is coupled with the magnetic circuit. A constant source of voltage E is in series with switch S and the coil, which has resistance R. When the switch is closed, the current rise in the coil circuit is opposed by the induced voltage $\frac{d(N\phi)}{dt}$, where N is the number of coil turns. Kirchhoff's law for the voltage drop around the coil circuit gives

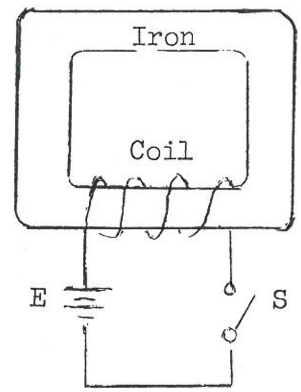


Figure 7.9

$$E = Ri + \frac{d(N\phi)}{dt} .$$

This equation can be solved only after the relation between coil current i and flux ϕ is known.

(a) Assume that i is proportional to ϕ. Determine the current-time relation and sketch the resulting curve.

(b) If the flux-current relation is

$$\phi = \phi_o(1 - e^{-ai})$$

first find the governing differential equation in terms of i and then find the equation in terms of ϕ. Set up a recursion formula to obtain a numerical solution to either equation.

Answer to (a). $i = k\phi$. $\therefore \frac{iR-E}{i_oR-E} = e^{-kRt/N}$.

7.11 Heat energy is radiated according to the Stephan-Boltzmann law: a body at absolute temperature U (°Kelvin) radiates energy of a given frequency at a rate proportional to U^4. If a body is in an environment having constant absolute temperature U_o, then the body receives radiant energy from the environment.

(a) Show that the differential equation describing the rate of temperature change due to radiation is

$$\frac{dU}{dt} = - k (U^4 - U_o^4)$$

and write the recursion formula for an Euler step-by-step solution.

(b) If $k = 0.18 \times 10^{-10}$ $(°K)^{-3}$/min. for a body with initial temperature $U_i = 500°K$ in an environment having $U_o = 600°K$, determine the temperature of the body after 1 hr. Do this calculation on the computer with a CORC program.

(c) Show why the $U = U(t)$ values obtained in part (b) are close to an exponential curve starting at temperature U_i and rising to approach U_o.

7.12 It is desired to decelerate a rocket sled moving on a horizontal track by having a scoop attached to the sled move through a trough of water. This arrangement produces a drag force on the sled proportional (with constant R) to the instantaneous velocity of the sled. The mass of the sled is constant, m, and it has an initial velocity v_o when the scoop is inserted in the water.

(a) Determine the velocity of the sled in terms of the time (from when the scoop is inserted).

(b) Derive a formula for the time it takes the velocity to drop to $\frac{1}{n}$ its initial value v_o.

(c) Determine the velocity of the sled in terms of the distance travelled from the position where the scoop is first inserted.

7.13 A chemical batch blending process has an input to vat I of a solution containing C_1 lb. of acid per gal. entering at a rate of 3 gal./min. The contents of vat I are removed at a rate of 3 gal./min. with 2 gal./min. entering at II and 1 gal./min. bypassing vat II. Additional acid with a concentration $2C_1$ lb./gal. enters vat II at a rate of 2 gal./min. and the contents of vat II are removed at a rate of 4 gal./min.

(a) Set up the differential equations which govern the amount of acid in each vat.

(b) Set up an expression for the lb. of acid per minute leaving the system at P.

(c) Determine the steady state solution to the equations of part (a).

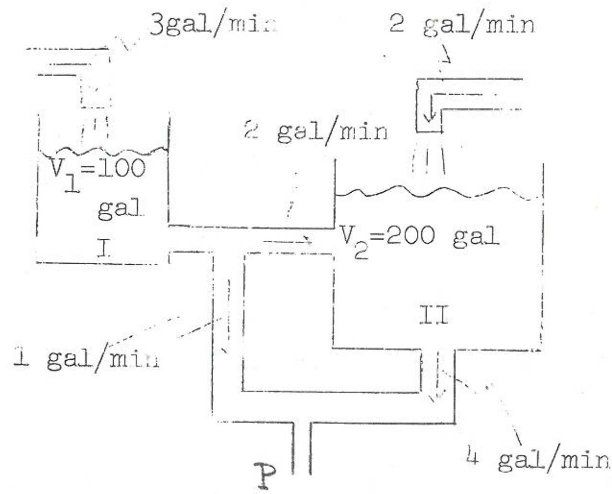


Figure 7.10

7.14 Chemicals A and B must be mixed with caution, as their reaction is highly exothermic. When brought together at an elevated temperature, α lb. of A combines instantaneously with 1 lb. of B to form $(1+\alpha)$

lbs. of a third substance C. Tank 1, containing volume V_1, is heated, so that the reaction occurs in this tank. Tank 2, containg volume V_2, is kept cold, so that no reaction can occur in it. At any time t let $A_1(t)$ and $A_2(t)$ be the number of pounds of chemical A in tanks 1 and 2 respectively, and let $B_2(t)$ be the number of pounds of chemical B in tank 2. The mixing process is accomplished by pumping $r\ \frac{\text{gal.}}{\text{min.}}$ of the mixture from tank 2 into tank 1, while $r\ \frac{\text{gal.}}{\text{min.}}$ of the contents of tank 1 are pumped into tank 2.

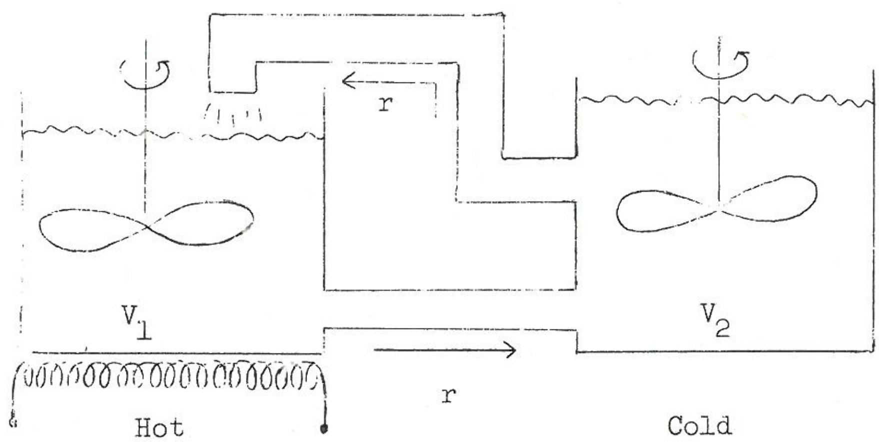


Figure 7.11

(a) Derive the equations for the time rates of change of $A_1(t)$, $A_2(t)$ and $B_2(t)$.

(b) If $C_1(t)$ and $C_2(t)$ are the number of pounds of chemical C in tanks 1 and 2 at time t, derive the equation for the time rate of change of $C_1(t)$.

7.15 A flexible cable is fastened at ends A and B and hangs under the influence of gravity. Take the x-axis horizontal, and let the y-axis pass through point C, the lowest point on the curve. T is the tension in the cable at any point P and H is the constant horizontal tension at C. See Figure 7.12. If W is the total weight downward of any section of cable, then equilibrium of this section requires

$$T \cos \theta = H$$

$$T \sin \theta = W.$$

If these equations are divided we obtain

$$\frac{\sin \theta}{\cos \theta} = \frac{dy}{dx} = \frac{W}{H} .$$

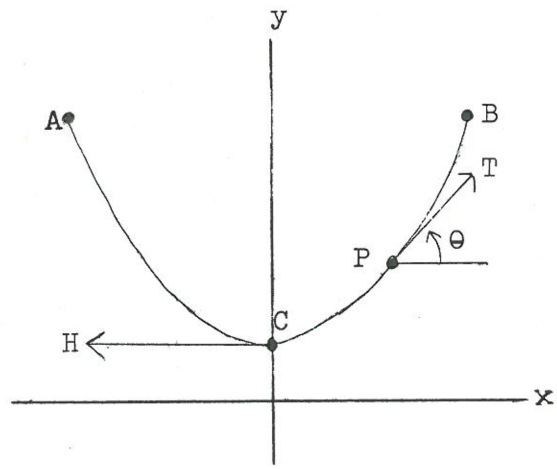


Figure 7.12

The derivative of this equation is

$$\frac{d^2y}{dx^2} = \frac{1}{H}\frac{dW}{dx} .$$

If a uniform cable hangs under its own weight, then the weight of section CP is

$$W = ws$$

where w is the weight per unit length of the cable and s the arc length CP. Using the fact that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

prove that the equation of the cable is

$$y = \frac{H}{2w}\left(e^{\frac{w}{H}x} + e^{-\frac{w}{H}x}\right) + C.$$

7.16 Answer the question at the top of p. 1.87.