CHAPTER 3

Infinite Series

Introduction and Basic Concepts

1. Cutting Up a Cheese Cake.

Suppose I have one cheese cake. Suppose I also have a friend. Following a somewhat unfashionable ethic, I bisect the cake and give one half to my friend and retain one half for myself. A second friend arrives. I again share my portion equally with him, so that he receives one quarter of the original cheese cake and I retain the remaining quarter. (See Fig. 1.1).

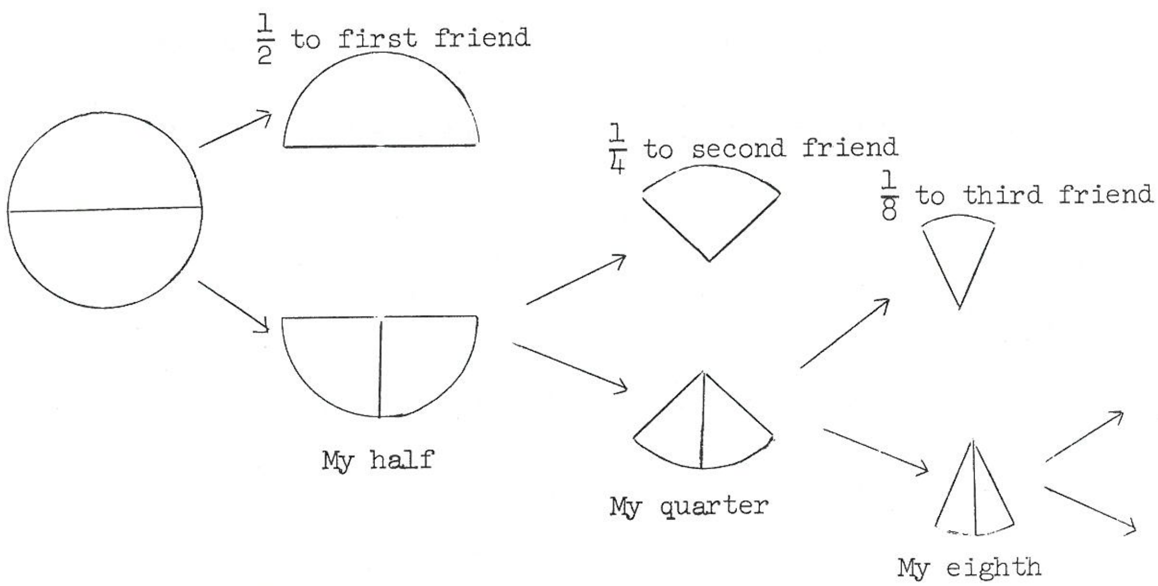


Figure 1.1

Suppose that a third friend arrives. Maintaining my principle of sharing equally, I give him half of my portion, i.e. I give him one eighth of the original cake, and retain the other eighth for myself.

Suppose that friends keep arriving one at a time, and each friend gets half of my current portion. What fraction of the cake do I eventually give away? Let us call this question the "Cheesecake Problem."

Note first that the fraction of the original cake which is given to the first friend is $\frac{1}{2}$; to the second friend $\frac{1}{2}(\frac{1}{2}) = \frac{1}{2^2}$; to the third friend $\frac{1}{2}(\frac{1}{4}) = \frac{1}{2^3}$ and so on. Thus the i^{th} friend gets $\frac{1}{2^i}$ of the original cake. At the time that n friends have taken their share, the fraction of the cake that I have given away is $\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$. At that time my current share is the same as that of the friend who has most recently taken his portion, i.e. $\frac{1}{2^n}$. Since the fraction which I give away plus the fraction which I retain must equal 1 we obtain

$$\left[\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}\right] + \frac{1}{2^n} = 1 ,$$

or, solving for the fraction of cake given away:

$$\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n} . \tag{1.1}$$

Equation (1.1) represents an interesting fact of arithmetic which we shall arrive at by a different route a little later (Section 4).

The term $\frac{1}{2^n}$ on the right side of equation (1.1) represents the fraction of cake which I retain after sharing with n successive friends. As n becomes larger and larger, the quantity $\frac{1}{2^n}$ becomes smaller and smaller. Thus the right side of equation (1.1) becomes closer and closer to 1. Consequently the sum on the left, which represents the total fraction of the cake given away, becomes closer and closer to 1 as n gets larger and larger. We indicate this fact by the notation

$$\lim_{n \to \infty} \left[\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} \right] = 1 ,$$

or by the notation

$$\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \ldots = 1 .$$

Note the additional three dots on the left hand side. [These ideas will be discussed in more precise terms in Section 2.]

Thus we might say that the answer to the question is 1. This answer, however, might be disputed on various grounds, some of which we shall now examine.

The first objection is sociological and is based on the fact that I have only a limited number of friends. Thus the part of my argument in which I let n become larger and larger may be questioned on the grounds that it becomes meaningless as soon as n is larger than the number of friends that I have. Moreover, even if everyone on earth were my friend (and they might well be, to a man who shares his cheese cake) there are still only a finite number of people on the earth; the present world population is surely less than 10^{10}. As soon as n exceeds this number, the computations become questionable.

One answer to this objection is that the statement of the problem did not require that the friends be distinct. That is, any of my friends might present himself again and again to claim his half of my current portion. While this disposes of the objection, it may sound rather legalistic, since if the statement of the problem were altered slightly, so as to require that each recipient be distinct, the objection would reappear. The answer then would reside in the meaning of the phrase: "Suppose that friends keep arriving one at a time ..." just preceding the statement of the question. This phrase means that there is no limit to number of times I share the cake; after each bisection, another will occur. Words mean what people agree that they mean, and all mathematicians would agree to the above meaning. Thus if one objects to the plausibility of the supposition "that distinct friends keep arriving one at a time ..." he is not objecting to our answer to the problem but rather to the plausibility of the assumptions in the statement of the problem itself. This is often a perfectly legitimate, in fact praiseworthy, activity in engineering and scientific practice. Reexamining the basic assumptions in a situation is often extremely fruitful. On the other hand, it is also often fruitful to examine the mathematical and logical consequences of assumptions that may not in fact be true. In mathematics, when the problem says "Suppose that X is true" then everyone must agree for the remainder of that discussion that X is true, whether he believes it or not, in order to arrive at the consequences of that assumption. At the end of the mathematical argument the consequences of the assumption are displayed. The mathematician does not claim that these consequences are true or false; his statements are of the form "If X is true then Y is true." It is the job of the engineer or scientist to decide what degree of credence to give to X; but if he believes that X is true then he must accept the fact that Y is also true. In the present problem then, the phrase "Suppose that friends keep arriving one at a time ..." requires that we suppose this to be true for the remainder of the discussion, regardless of whether it is physically realizable or not.

The second objection is technological and is concerned with the problem of devising an instrument capable of slicing the cake more than a few times (or even once, exactly in half). This is a cogent consideration in engineering practice, but again it is an objection to the statement of the problem and not to the answer. In a mathematical investigation we make the suppositions which are demanded in the statement of the problem. The question of hardware implementation must have its day, but that is another day.

The third objection is physical, namely that after a certain number of slices we would be breaking up molecules of cheese cake into chemical elements which no longer would be identifiable as cheese cake. Further subdivisions would require that we split the atoms and their electron shells into equal parts and that is probably impossible. Again we must dispose of this objection on the grounds that when a mathematical problem asks us to assume something, then we must agree to assume it for the course of that discussion or else leave the room. By now however, the reader is probably convinced that the problem, as stated, is somewhat divorced from the world of reality. This is true. It is also true that any mathematical formulation of a real world situation fails to represent the exact facts of the situation. The use of mathematics in engineering is consequently an art rather than a science; the art of knowing when a mathematical formulation is "good enough" for a particular application and when it is not, so that the formulation must be altered. Proficiency in this comes with experience. In the present course we hope to provide some of this experience but we shall be also concerned with the mathematics itself.

The final objection is a cogent one, based on the following contradiction.

To say that we eventually give away all of the cake implies that we have none to ourselves. But at any instant we always have some cake. Consequently there is no time at which we have none to ourselves.

Note that the argument of the preceding paragraph establishes first that the underlined statement is true; but then establishes that there is no time at which it is true; that is, that it is never true. This contradiction stems from the fact that the "Cheesecake Problem" is indeed stated in imprecise terms. Great quantities of time and mental effort can be (and have been) wasted in discussion which can never reach a conclusion until the question is restated in more precisely defined terms. One of the major contributions of mathematics to science and human affairs has been the idea of precise definitions. In particular, within mathematics itself, precise definitions are absolutely necessary. Thus, to remove the vagueness from the statement of the "Cheesecake Problem," let us reformulate it as follows: "What is the limit, as n goes to infinity, of the fraction of the cake that is donated to n successive recipients?"

Because the word "limit" has a precise technical meaning (see Section 2 below), this question has an unambiguous meaning to anyone trained in mathematics and the answer is unequivocally "1." We shall discuss such terms as "limit" in more detail in Section 2 below. Here we shall only note the fact that if the sum $\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$ is "replaced" by 1, the "error" is precisely $\frac{1}{2^n}$. If n is large enough, then 1 will be a good approximation to the sum.

Problems

1.1 How many friends can we serve before we are required to split atoms? (Consider the best possible case and the worst possible case.)

1.2 Radioactive carbon has a half life of 5720 years; i.e. half of the radioactive particles become non-radioactive in 5720 years. How long would it take for 99% of the radioactivity to disappear? Assuming that the fraction of radioactive carbon in the CO_2 in the atmosphere has been constant for millions of years, describe how the above analysis could be used to estimate the age of fossils. (Essentially this method of dating fossils won a Nobel Prize for its discoverer, N.F. Libby.)

1.3 A capacitor loses one fourth of its charge in 5 seconds. If it has a charge of 10 coulonbs initially, when does the charge fall to that on a single electron? What happens then?

1.4 For a certain mechanical vibrator it is said that "the time to damp to half-amplitude is 10 seconds." If it starts with a vibration of amlitude 1 inch, for how long would the vibrations be detectable, assuming that a vibration of amplitude .0001 inch is detectable?

1.5 In a certain chemical process, one half a solute dissolves in 10 seconds. Approximately how long does it take for 99.9% of the solute to dissolve?

1.6 Suppose that a golf ball dropped on the floor rises to $\frac{1}{2}$ the height from which it was dropped. Approximately how many bounces can be detected, assuming that we can detect a bounce of .001 inch and the ball is dropped from a height of h feet? What is the total distance travelled by the ball? [Ans. $n \leqq (3 + \log_{10} 12h)/\log_{10} 2$; $d = 3h$.]

1.7 Discuss Zeno's paradox about Achilles and the Tortoise (also called the paradox of the Hare and Turtle).

2. Convergence of a Series of Constants.

In the previous section we were concerned with assigning a value to an expression of the form

$$\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} + \dots . \tag{2.1}$$

This is called an infinite series or more briefly merely a series.

More generally if we are given any sequence of numbers $a_1, a_2, \dots a_n, \dots$, the expression

(2.2) $$a_1 + a_2 + \ldots + a_n + \ldots$$

is called an infinite series. The number a_n is called the n^{th} term of the series.

Now, while we can always add together a finite number of terms, it is manifestly impossible even for the fastest computer to add together an infinite number of terms. Consequently there is some question as to what expression (2.2) actually signifies. To assign a numerical value or sum to a series such as (2.2) we first go back to the simpler notion of an infinite sequence, or more briefly a sequence. Examples of sequences are

(2.3) $$\left\{1, 2, 3, 4, \ldots, n, \ldots\right\} ,$$

(2.4) $$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\right\} ,$$

(2.5) $$\left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots, \frac{(-1)^n}{n}, \ldots\right\} ,$$

(2.6) $\left\{0, 1, 0, 1, 0, \ldots\right\}$, where the terms are alternately 1 and 0.

Note the commas between the terms instead of the "+" signs of the series (2.2). The student should be aware of the fact that while the words sequence and series are sometimes used as synonyms in everyday speech, their meanings in mathematical discourse are quite distinct, as indicated in the text. We say that a sequence $\left\{a_1, a_2, \ldots, a_n, \ldots\right\}$ has been defined when a rule has been specified which assigns to each positive integer, n, a corresponding number a_n. Another way of saying this is that a sequence is a function whose domain of definition is the positive integers and whose range of values is the real numbers. In fact the notation $a(n)$ is sometimes used instead of a_n, particularly in computer languages where all symbols must appear on the same horizontal line of type.

The numbers a_n are called the <u>terms</u> of the sequence. The notation for a sequence whose n^{th} term is a_n is $\{a_1, a_2, \ldots, a_n, \ldots\}$ or, if the rule is obvious, $\{a_1, a_2, a_3, \ldots\}$ or, still more briefly, $\{a_n\}$. Sometimes the index n counts, not from the first term, but from some later point in the sequence.

<u>Example 2.1</u>. We frequently encounter sequences defined by recursion formulas, each term being expressed as a function of earlier terms. For example the positive integers are defined by the recursion formulas $a_1 = 1$, $a_{n+1} = a_n + 1$. Another example is the Fibonacci Sequence, defined by

$$a_1 = 1,\ a_2 = 2,\ a_n = a_{n-1} + a_{n-2} \quad \text{for} \quad n > 2.$$

The terms of this sequence, called Fibonacci Numbers, start out with

$$1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ \ldots .$$

(See <u>The Fibonacci Numbers</u>, N.N. Vorobyov, D.C. Heath and Co., Boston, 1963.) Recursion formulas are particularly adaptable to automatic machine computation and consequently are of considerable importance in modern applied mathematics.

Next we need the idea of <u>convergence of a sequence to a limit</u>. Consider for example the sequence $\left\{\frac{3}{1}, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \ldots, \frac{2n+1}{n}, \ldots\right\}$. This can also be written in the form $\left\{3, 2\frac{1}{2}, 2\frac{1}{3}, \ldots, 2 + \frac{1}{n}, \ldots\right\}$. Clearly the successive terms get closer and closer to the number 2. In fact the distance from the n^{th} term to 2 is $\frac{1}{n}$. (See Fig. 2.1.) By taking n sufficiently large we can make this distance as small as we wish. We say that the sequence converges to the limit 2.

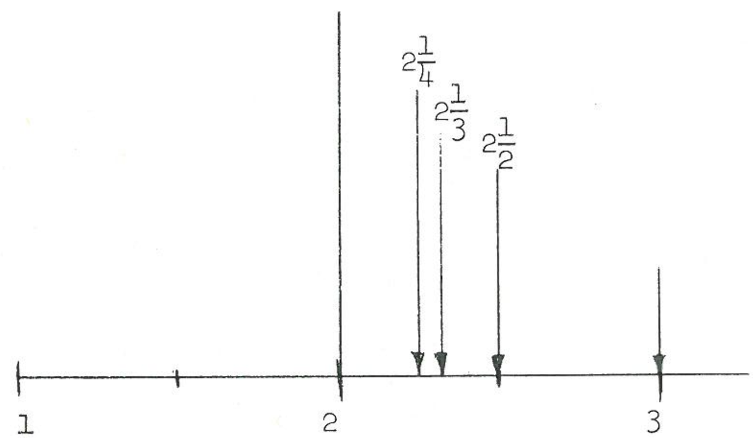


Figure 2.1

The reader will notice the resemblance of the notion of the limit of a <u>sequence</u> to the notion of the limit of a <u>function</u> (as in $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$). In both, one has the intuitive idea of certain real numbers (the terms of a sequence or the values of a function) approaching a limit. But the reader is warned that there are important differences, too, between these two uses of the limit concept and should try to keep them distinct in his mind.

We make the notion of the convergence of a sequence to a limit precise by means of the following definition.

<u>Definition 2.1.</u> We say that a sequence $\{a_n\}$ <u>converges to the limit L</u> if, for any preassigned positive number ϵ, there is an integer N such that, for all n greater than N, the distance from a_n to L is less than ϵ.

This definition will probably be hard for the student to digest at one gulp. He should read it carefully several times. What it says, in words, is this: The number L is a "good" approximation for all the terms in the sequence which are sufficiently late in the sequence. How "good" is the approximation? As good as we wish. How late in the sequence must we go? That depends on how good we wish the approximation to be. The typical situation is that the closer the "tolerance" ϵ, the further in the sequence we must go to insure that all the subsequent elements a_n are within the tolerance ϵ of the limit L.

For example, the sequence $\left\{\frac{3}{1}, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \ldots, \frac{2n+1}{n}, \ldots\right\}$, as was mentioned above, converges to the limit 2. Let us verify that the conditions given in Definition 2.1 are satisfied. The distance from a_n to the limit 2 is $\left|(2 + \frac{1}{n}) - 2\right| = \frac{1}{n}$. Suppose, for example, that a value of $\epsilon = .001$ is assigned. Then the distance from a_n to the limit 2 will be less than ϵ provided that $\frac{1}{n} < .001$, i.e. provided that $n > \frac{1}{.001} = 1000$. Thus when ϵ is chosen to be .001 we may select N (in Definition 2.1) as 1000. (Actually there is nothing to stop us from selecting N to be, say 2000.) The point is, that for each positive number ϵ, there is such an N. In fact in this example we can take N to be any integer larger than $\frac{1}{\epsilon}$.

If there is no number L with the property that $\{a_n\}$ converges to L then we say that the sequence $\{a_n\}$ <u>diverges</u>. Using Definition 2.1 the reader may verify that sequences (2.3) and (2.6) diverge, while (2.4) and (2.5) each converge to the limit zero.

The notation to represent the fact that the sequence $\{a_n\}$ converges to the limit L is

(2.7) $$\lim_{n \to \infty} a_n = L ,$$

or

(2.8) $$a_n \to L, \text{ as } n \to \infty ,$$

or simply $a_n \to L$.

We may also write

(2.9) $$a_n \to \infty$$

and say that a_n <u>diverges to infinity</u> if, however large we take the real number K, there exists N such that $a_n > K$ for $n \geq N$. Thus in (2.3) the sequence diverges to infinity. Sequence (2.6) diverges, but does not diverge to infinity.

Notice the equality sign in (2.7). This represents the fact that the limit <u>is</u> the number L. To say "the limit approaches L" is incorrect; "the limit" is a single real number, and doesn't "approach" anything. The following correct and equivalent statements illustrate proper usage:

The sequence <u>has</u> the limit L;

The terms of the sequence <u>approach</u> L;

The limit of the sequence <u>is</u> L .

It can happen that no term in the sequence is equal to the limit, as in the sequence $\left\{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\right\}$, where no term has the value zero. On the other hand one or more terms in the sequence might have the value of the limit. An extreme example would be the sequence $\{3, 3, 3, \ldots, 3, \ldots\}$, where each term is equal to the limit. The important point is that the conditions of Definition 2.1 are satisfied.

Theorem 2.1. If $\lim_{n \to \infty} a_n = A$ and $\lim_{n \to \infty} b_n = B$, then

(a) $\lim_{n \to \infty} (a_n + b_n) = A + B$,

(b) $\lim_{n \to \infty} (ca_n) = cA$, where c is any constant,

(c) $\lim_{n \to \infty} (a_n b_n) = AB$,

and, if B is not zero, then

(d) $\lim_{n \to \infty} \left(\frac{1}{b_n}\right) = \frac{1}{B}$.

Proof. (a) We must show that for any preassigned positive number ϵ, there exists a number N such that, for all $n > N$, $|(a_n + b_n) - (A + B)| < \epsilon$. Let $\epsilon > 0$ be given. Let $\epsilon_1 = \frac{\epsilon}{2}$ and note that since $\lim_{n \to \infty} a_n = A$, there exists a number N_1 such that, for all $n > N_1$, $|a_n - A| < \epsilon_1$. Let $\epsilon_2 = \frac{\epsilon}{2}$. Since $\lim_{n \to \infty} b_n = B$, there is a number N_2 such that if $n > N_2$ then $|b_n - B| < \epsilon_2$. Now if N is greater than both N_1 and N_2 then, whenever $n > N$,

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)| \leqq |a_n - A| + |b_n - B|$$
$$< \epsilon_1 + \epsilon_2 = \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon .$$

Proofs of the remaining parts are similar and are left as exercises for the student. (See Problem 2.1).

Now that we know what is meant by convergence of a sequence we shall be able to define what we mean by convergence of a series. Recall that in the cake-cutting problem of Section 1 we arrived at the formula for the amount given away by the time n friends had been given their shares, given by equation (1.1),

$$\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n} .$$

We then considered the sequence $\left\{1 - \frac{1}{2^n}\right\}$. This sequence has the limit 1. From this we were led to assign the value 1 to the series $\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} + \dots$.

In general we can proceed in an analogous way. If $\{a_n\}$ is a given sequence, what do we mean by the sum of the series $a_1 + a_2 + \dots + a_n + \dots$? We mean the following. We form the sequence of partial sums $\{S_1, S_2, \dots, S_n, \dots\}$ where

$$S_1 = a_1 ,$$

$$S_2 = a_1 + a_2 ,$$

$$S_3 = a_1 + a_2 + a_3 ,$$

$$\dots .$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n ,$$

$$\dots .$$

Definition 2.2. If the sequence of partial sums $\{S_n\}$, where $S_n = \sum_{i=1}^{n} a_i$, converges to a limit L then we say that the series $a_1 + a_2 + \ldots + a_n + \ldots$ converges to L, or that the sum of the series is L.

The notation to indicate that the series $a_1 + a_2 + \ldots + a_n + \ldots$ converges to L is

$$a_1 + a_2 + \ldots + a_n + \ldots = L \text{ , or } \sum_{n=1}^{\infty} a_n = L \text{ .}$$

If the sequence $\{S_n\}$ does not converge to a limit we say the series $\sum_{n=1}^{\infty} a_n$ does not converge, or that the series diverges.

Example 2.2. (a) In the cake-cutting problem we encountered the series

$$\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \ldots \text{ .}$$

The sequence of partial sums is

$$\left\{ \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots, 1 - \frac{1}{2^n}, \ldots \right\} \text{ .}$$

The sequence converges to 1, hence $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

(b) Consider the series

$$3 + 0 + 0 + 0 + \ldots + 0 + \ldots \text{ .}$$

The sequence of partial sums is

$$\{ 3, 3, 3, \ldots, 3, \ldots \} \text{ ,}$$

and hence the given series converges to the limit 3.

(c) Consider the series

$$1 + 1 + 1 + 1 + \dots + 1 + \dots .$$

The sequence of partial sums is

$$\{1, 2, 3, 4, \dots, n, \dots\} .$$

Since this sequence does not converge, the given series does not converge.

(d) Consider the series

$$1 - 1 + 1 - 1 + 1 \dots .$$

The sequence of partial sums is

$$\{1, 0, 1, 0, 1, \dots\} .$$

Since this sequence does not converge, the given series does not converge.

The difference R_n between the limit S of a convergent series and the sum to n terms S_n is called the <u>remainder after n terms</u>, i.e.

$$R_n = S - S_n \quad \text{or} \quad S = S_n + R_n .$$

If S_n is used as an approximation to L, R_n is called the <u>truncation error</u> of this approximation. In Example 2.2(a) we have $R_n = 1 - (1 - \frac{1}{2^n}) = \frac{1}{2^n}$. In (b) we have for $n > 1$, $R_n = 3 - 3 = 0$. From Definition 2.2 it is clear that if the series converges, then R_n tends to zero as $n \longrightarrow \infty$.

<u>Theorem 2.2</u>. If the series $\sum_{n=1}^{\infty} a_n$ converges, then the sequence $\{a_n\}$ converges to zero.

<u>Proof.</u> Note that $a_n = S_n - S_{n-1}$. It follows from Theorem 2.1 that

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n = 0.$$

The student might find Figure 2.2 helpful in understanding the above proof. Roughly the idea is that the S_n's all approach the same number. Hence successive S_n's must also get close together, thus squeezing the a_n's down to zero. In words, "things close to the same thing are close to each other."

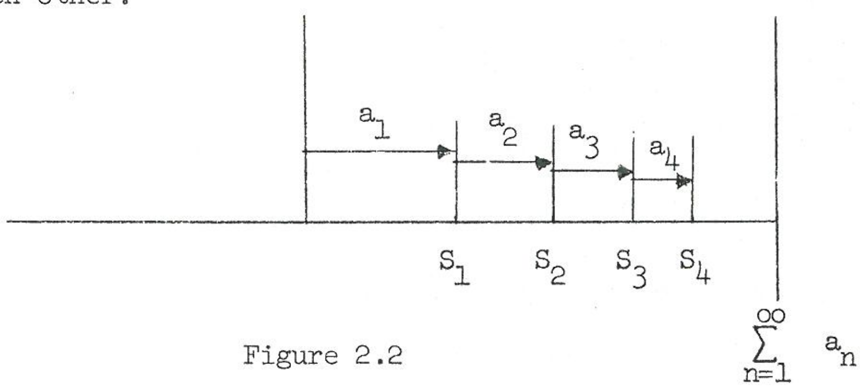


Figure 2.2

This important theorem is often expressed in the abbreviated form: "If a series converges, then the n^{th} term goes to zero."

Theorem 2.2 is illustrated in the Examples 2.2 above. In the series (a) $\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \ldots$, the n^{th} term is $\frac{1}{2^n}$ which tends to zero as n tends to infinity. Since the series converges this is in accordance with Theorem 2.2. In Example 2.2(b) the series $3 + 0 + 0 + \ldots + 0 + \ldots$ has for its n^{th} term 0 (for $n > 1$). The sequence $\{3,0,0,0,\ldots,0,\ldots\}$ converges to zero. This is again in accordance with Theorem 2.2. In Example 2.2(b) the series $1 + 1 + \ldots + 1 + \ldots$ has for its n^{th} term the number 1. The sequence $\{1,1,1,\ldots,1,\ldots\}$ converges to 1, and in accordance with Theorem 2.2 the given series does not converge. In Example 2.2(d) the series $1 - 1 + 1 - 1 + \ldots + (-1)^{n+1} + \ldots$ has for its n^{th} term the number $(-1)^{n+1}$. The sequence $\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ does not converge and, in accordance with the theorem, the given series does not converge.

Notice that Theorem 2.2 can be restated as follows. If the sequence $\{a_n\}$ does <u>not</u> converge to zero then the series $\sum_{n=1}^{\infty} a_n$ does <u>not</u> converge. Note however that the theorem does <u>NOT</u> say that if the sequence $\{a_n\}$ converges to zero then the series $\sum_{n=1}^{\infty} a_n$ converges. In fact, as we shall see shortly, it can indeed happen that the sequence $\{a_n\}$ converges to zero but the series $\sum_{n=1}^{\infty} a_n$ does not converge. This is summed up by saying "the condition $a_n \longrightarrow 0$ is <u>necessary</u> <u>but</u> <u>not</u> <u>sufficient</u> for the convergence of the series $\sum_{n=1}^{\infty} a_n$."

To summarize, Theorem 2.2 can be used in two ways.

(a) If a series is known to converge then we can assert that the terms tend to zero.

(b) If the terms do not tend to zero then we can assert that the series does not converge.

However, if the series does not converge, Theorem 2.2 tells us nothing. If the terms tend to zero, Theorem 2.2 tells us nothing.

The fact that the series $\sum_{n=1}^{\infty} a_n$ may not converge even though the sequence $\{a_n\}$ converges to zero is illustrated by the series

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{\text{two terms}} + \underbrace{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}_{\text{three terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{\text{four terms}} + \ldots + \underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}}_{n \text{ terms}} + \ldots$$

which diverges, although $a_n \longrightarrow 0$. A more famous example is the case of the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots .$$

This series diverges, although the sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ clearly converges to zero. The proof that the series diverges will be found in the following section.

Problems

2.1 Prove the remaining parts of Theorem 2.1.

2.2 Find the limits, if they exist of the following sequences:

(a) $\left\{0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, \ldots\right\}$, the terms being ultimately zero and $\frac{1}{n}$. Ans. 0.

(b) $\left\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\right\}$.

(c) $\left\{\sin 1, \sin 2, \sin 3, \ldots, \sin (n), \ldots\right\}$. Ans. No limit.

(d) $\left\{a_n\right\}$, where $a_n = n + \frac{1}{n}$.

(e) $\left\{x_n\right\}$, where $x_n = 1 + \frac{n}{10^n}$. Ans. 1.

2.3 (a) Show that the set of sequences, using the obvious definitions of addition and multiplication by scalars, forms a vector space V_∞.

(b) Show that the set of all convergent sequences in V_∞ forms a proper vector subspace.

2.4 Evaluate

(a) $\lim_{n \to \infty} \frac{n^2 + 8}{3n^2 + 7n}$

(b) $\lim_{n \to \infty} \frac{\sqrt{n + 2}}{1 + n}$

(c) $\lim_{n \to \infty} \frac{1}{n}$

2.5 Show that the series $\frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n(n+1)} + \ldots$ converges by using the fact that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, so that $S_n = 1 - \frac{1}{n+1}$.

2.6 Prove from Definition 2.2 that if $\sum_{n=1}^{\infty} a_n = L$ and $\sum_{n=1}^{\infty} b_n = M$, then $\sum_{n=1}^{\infty} (a_n + b_n) = L + M$, and, if c is any constant, $\sum_{n=1}^{\infty} (ca_n) = cL$.
[Follow the method of proof of Theorem 2.1.]

2.7 If the series $\sum_{n=1}^{\infty} a_n$ has the sum 2 and the series $\sum_{n=1}^{\infty} b_n$ has the sum 3, what are the sums of the following series?

(a) $\sum_{n=1}^{\infty} (2a_n - 3b_n)$.

(b) $\sum_{n=1}^{\infty} (b_n + \frac{1}{2^n})$.

2.8 Find the remainder after n terms for each of the converging series in Example 2.2.

2.9 State what conclusions can be drawn from the following facts on the basis of Theorem 2.2.

(a) The series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is known to converge. Ans. $\lim_{n \to \infty} \frac{n}{2^n} = 0$.

(b) The sequence $\left\{\frac{1}{n}\right\}$ tends to zero.

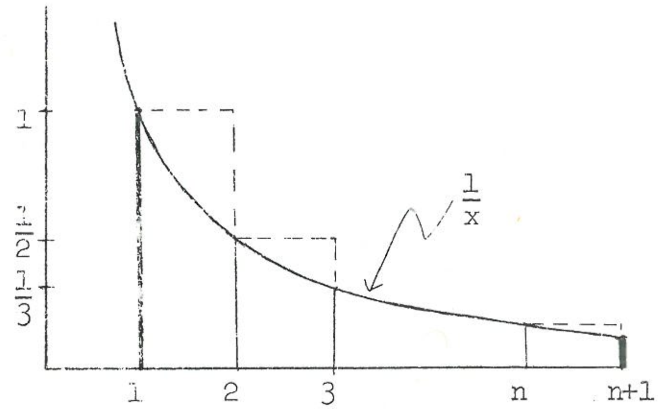
(c) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge. Ans. None.

(d) The sequence $\{\sin(n)\}$ does not tend to zero.

2.10 Show that $\frac{1}{n} + \frac{1}{n+1} - \frac{2}{n+2} = \frac{3n+2}{n(n+1)(n+2)}$. Hence sum the series $\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)(n+2)}$ and calculate the error in replacing the sum by the 100^{th} partial sum.

3. <u>The Generous Donor</u>.

This is a story about the president of a university who besought a rich alumnus for a monetary contribution. "Well," said the alumnus, "I'll tell you what I'll do. This year I'll give you one pound of gold. Next year I'll make it $\frac{1}{2}$ pound; the year after, $\frac{1}{3}$ pound; the following year $\frac{1}{4}$ pound, and so on. In fact I'll write it into my will, so that in the n^{th} year you will receive $\frac{1}{n}$ of a pound." The president, who had forgotten his theory of series and saw only the dwindling income, returned discouraged to the campus. In doleful tones he related the incident to the faculty. An ancient professor of mathematics leaped up and shouted "Ah, but those are the terms of the harmonic series! The sum, if carried far enough, will exceed any preassigned amount. Allow me to elaborate." He hurried to the blackboard and drew the following figure.



"The amount of gold that you will receive in n years," he explained, "is given by the sum______", and he wrote on the blackboard

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} .$$

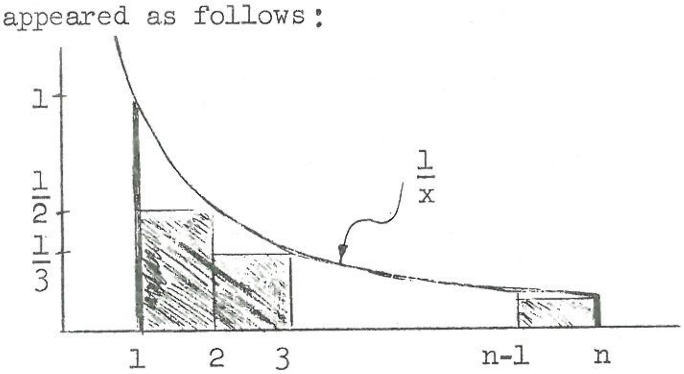
"The solid curve in the figure is the graph of $\frac{1}{x}$. It is easy to see that the terms in S_n are the areas of the successive rectangles shown in the figure. It is clear that the sum of these areas exceeds the area under curve between the heavy vertical lines. Therefore we have the following inequality." He wrote on the board

$$S_n > \int_1^{n+1} \frac{dx}{x} = \log(n+1) .$$

"As n gets larger and larger, the value of log(n+1) gets larger and larger and in fact will eventually exceed any number you wish to name in advance, no matter how large. We shall all be rich!"

The president was delighted with this turn of events. He was preparing to present a motion to change the names on several buildings to that of the generous donor, when a voice was heard from the back of the room.

"Now hold on, just a darn minute!" It was a young instructor of numerical analysis. "What you say is right enough as far as it goes, but remember that the harmonic series diverges <u>very slowly</u>. Let me show you." He made his way to the blackboard under the skeptical gaze of the faculty, then modified the professor's drawing on the blackboard, so that it appeared as follows:



"The areas of the rectangles which I have shaded," he said, "represent the terms in the sum

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} .$$

"Since the sum of the areas of the rectangles is less than the area of the curve between the heavy vertical lines we have

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \int_1^n \frac{dx}{x} = \log n .$$

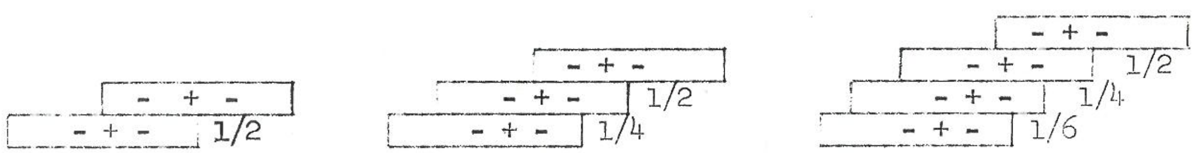
Therefore, again letting S_n denote the amount we will receive in n years, we have

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < 1 + \log n .$$

Consequently, the amount that we will have received after n years will be less than $1 + \log n \cong 1 + 2.3026 \log_{10} n$. For example, after 1 million years we will have collected less than 15 pounds of gold. [$1 + (2.3026)6 < 15$.] Even at $35 an ounce this is less than $8400 in one million years. [$35 \times 16 \times 15 = 8400$.] It is coming in too slowly to do us any good."

Unfortunately the young instructor was correct. The money arrived slowly but surely; but too slowly. The instructor could not be appointed to a tenure position. The ending is not sad, however, for shortly after being terminated at the University, he organized Gizmo-Dynamics, Inc. (cf. Ch. 2, Section 9). After securing a few cost-plus contracts he bought the University, raised everyone's salary and renamed it Herbert H. Gizmo U.

Problem 3.1 In order to stack dominoes with as much overhang as possible one should place them so that the center of gravity of each topmost set falls at the edge of the next lower dominoe. For example the optimal stacking of 2, 3 and 4 dominoes is shown in the figure



2 dominoes overhang $= \frac{1}{2}$

3 dominoes overhang $= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

4 dominoes overhang $= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}$

(a) Show that for any number A, no matter how large, one can make a stack with an overhang of more than A.

(b) Can one make, in this way, a stack with infinite overhang?

(c) For A large, approximately how many dominoes are needed to make a stack with an overhang of A?

(d) With 25 dominoes approximately how large an overhang can be obtained in theory? How much can you actually get by experiment?

Problem 3.2 Fill in the details of the following proof that the harmonic series diverges.

$$\text{Let } t_{2n} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right),$$

$$S_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$$

$$S_{2n} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2n} .$$

Then $t_{2n} = S_{2n} - S_n$. If harmonic series converged to S then $\lim_{n\to\infty} t_{2n} = S-S = 0$. But $t_{2n} \geq 1/2$ for all n. Contradiction.

4. The Geometric Series.

In the cheese cake problem of Section 1 we encountered the series $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} + \ldots$. This is a special case ($r = \frac{1}{2}$) of the series $r + r^2 + \ldots + r^n + \ldots$. If we add an initial 1 to this series we obtain the geometric series in standard form:

$$1 + r + r^2 + \ldots + r^{n-1} + \ldots . \tag{4.1}$$

This is also often written in the form

$$1 + r + r^2 + \ldots + r^n + \ldots$$

in which the "n" counts the terms starting from the second. It should always be clear exactly what the "n" denotes.

The sequence of partial sums can be reduced to convenient form by the following device:

(4.2) $$S_n = 1 + r + r^2 + \dots + r^{n-1} .$$

Multiply by r: $$rS_n = \quad r + r^2 + \dots + r^{n-1} + r^n .$$

Subtract: $$S_n - rS_n = 1 - r^n .$$

Factor out S_n:

(4.3) $$(1-r)S_n = 1 - r^n.$$

Thus we have the identity

(4.4) $$(1-r)(1 + r + r^2 + \dots + r^{n-1}) = 1-r^n ,$$

of which equation (1.1) is a special case. If r is not equal to 1, we can solve equation (4.3) for S_n:

(4.5) $$S_n = \frac{1}{1-r} - \frac{r^n}{1-r} , \ r \neq 1 .$$

We now discuss the convergence of the geometric series for various values of r. If $|r| < 1$ then from (4.5) it is clear that

$$\lim_{n \to \infty} S_n = \frac{1}{1-r} , \text{ since } \left| S_n - \frac{1}{1-r} \right| = \left| \frac{r^n}{1-r} \right| \to 0 \text{ as } n \to \infty$$

If $|r| \geqq 1$ then by Theorem 2.2 $\{S_n\}$ does not converge, since the terms r^n do not converge to zero. We summarize these results:

Theorem 4.1. The geometric series

$$1 + r + \dots + r^n + \dots$$

diverges if $|r| \geq 1$. If $|r| < 1$ the series converges to the limit $\frac{1}{1-r}$, and the remainder after n terms, $\frac{1}{1-r} - S_n$, is equal to $\frac{r^n}{1-r}$.

Suppose two trains are 120 miles apart and headed toward each other on a straight track, each travelling at 60 miles an hour. Suppose a fly starts from the front of the first locomotive and flies at 120 mph toward the second train. On reaching the second locomotive he turns instantly and flies back to the first at 120 mph, then turns and flies toward the second again, and so on. How far does the fly fly?

One solution of this problem is the following. Let d_1 be the distance the fly flies on the first leg, s_1 the distance each train travels in that time, and t_1 the time for the first leg. Then

$$d_1 + s_1 = 120, \quad d_1 = 120t_1, \quad s_1 = 60t_1 .$$

Hence $$120t_1 + 60t_1 = 120 \quad \text{or} \quad t_1 = \frac{2}{3}, \quad d_1 = 80, \quad s_1 = 40 .$$

On the second leg we have a similar computation but with the initial distance reduced to $40 = \frac{1}{3}(120)$. Consequently $d_2 = \frac{1}{3}(80)$, where d_2 is the distance the fly flies on the second leg. Similarly $d_3 = \frac{1}{9}(80)$, and so on. For the total distance flown by the fly we get

$$d_1 + d_2 + d_3 + \dots = 80 + \frac{80}{3} + \frac{80}{9} + \dots = 80(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots)$$

$$= \frac{80}{1 - \frac{1}{3}} = 120 \text{ miles.}$$

A somewhat simpler solution is the following. Since the trains are travelling at 60 mph and are initially 120 miles apart, it is clear that they will crash in 1 hour. Since the fly is continually flying at 120 mph the total distance he must cover in that hour is 120 miles.

There is a story that when the great mathematician John von Neumann was asked this problem he instantly answered "120 miles." "How disappointing," said the questionner, "You solved it like a physicist, using the elapsed time until the crash times the speed of the fly. I thought that as a mathematician you would compute the infinite series."

"But I did compute it by the series," von Neumann replied.

The student should compute $t_1, t_2, \ldots, t_n, \ldots,$ and $s_1, s_2, \ldots, s_n, \ldots$ and verify that $\sum_{n=1}^{\infty} t_n = 1$, $\sum_{n=1}^{\infty} s_n = 60$.

We now proceed to interpret the results of Theorem 4.1 in terms of properties of functions. From this theorem we can write

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots \tag{4.6}$$

for any value of x in the range $-1 < x < 1$. The infinite series on the right hand side of equation (4.6) is called a power series , of which more will be said in subsequent sections of this chapter.

When dealing with power series we will often use "n^{th} term" to mean the term containing x^n, so that in particular the constant term is the "zero-th term." Similarly the "n^{th} term partial sum" S_n may mean the sum up to and including x^n, the "remainder after the n^{th} term" will be the limit minus S_n, and so on. In any particular situation the meaning of the "n" will be made clear in the context.

An important conclusion to be drawn from (4.6) is that the function $1/(1-x)$ is expressible as the limit of a sequence of polynomials

$$S_0 = 1, \quad S_1 = 1+x, \quad S_2 = 1+x+x^2, \ldots .$$

(Here we use the notation indicated in the previous paragraph.) Note that the function $\frac{1}{1-x}$ is defined for all x except $x = 1$ (see Fig. 4.1). The series on the right side of equation (4.6) converges, however, only for x in the interval $-1 < x < 1$.

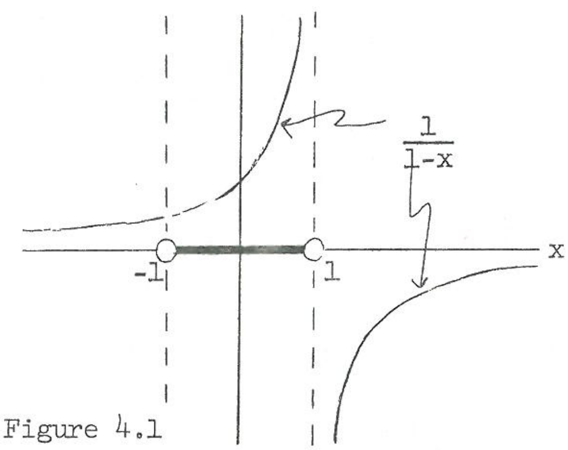


Figure 4.1

We can perhaps provide additional insight by writing equation (4.6) in the form

$$(4.7) \qquad \frac{1}{1-x} = \lim_{n \to \infty} S_n(x), \qquad -1 < x < 1 ,$$

where $S_n(x) = 1 + x + \ldots + x^n$, and graphing the successive terms $S_0(x)$, $S_1(x)$, $S_2(x)$ in Fig. 4.2.

Notice how the successive curves $S_n(x)$ are beginning to approach the function $\frac{1}{1-x}$ for x in the interval $-1 < x < 1$, but not for x outside that interval. One might also note that the approximation is

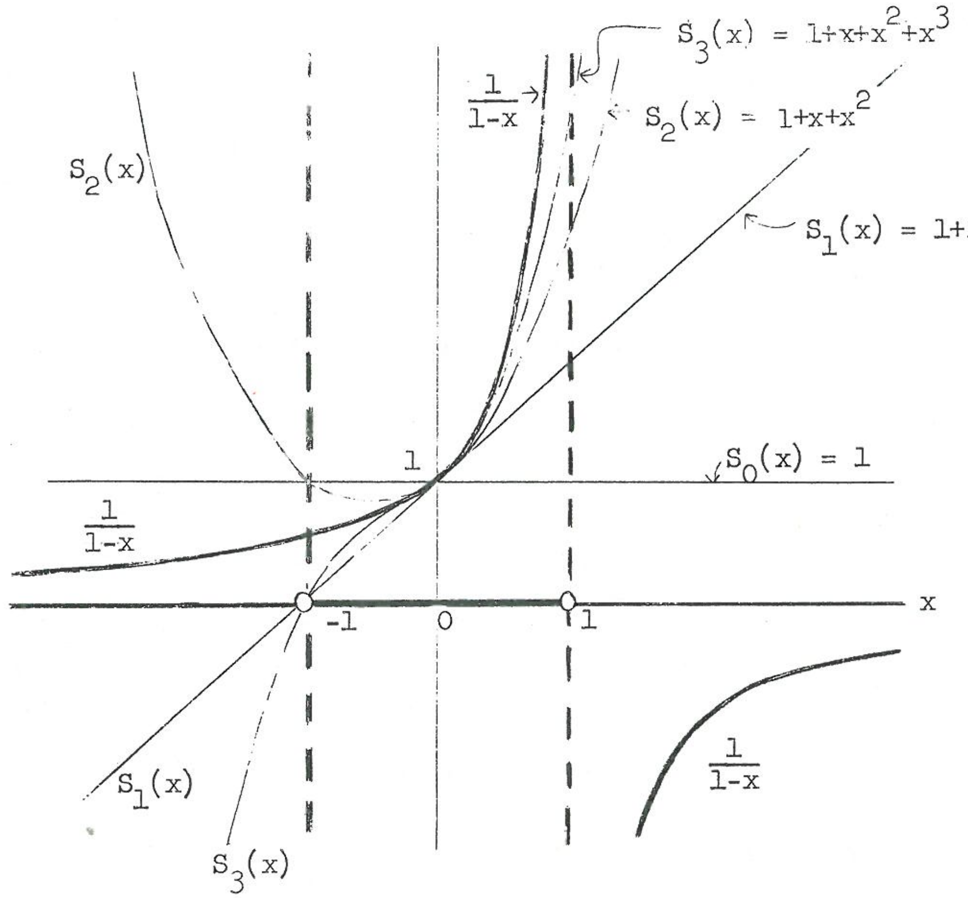


Figure 4.2

best in the neighborhood of $x = o$ and becomes poorer as we approach the values $x = \pm 1$. For any fixed value of x in the interval $-1 < x < 1$, however, the value of $S_n(x)$ will converge to $\frac{1}{1-x}$ as $n \longrightarrow \infty$, in accordance with Theorem 4.1.

We can use the error estimate of Theorem 4.1 to sharpen the statement of equation (4.6):

$$(4.8) \qquad \frac{1}{1-x} = S_n(x) + R_n(x) , \qquad -1 < x < 1 ,$$

where

(4.9) $$S_n(x) = 1 + x + x^2 + \dots + x^n ,$$

and

(4.10) $$R_n(x) = \frac{x^{n+1}}{1-x} .$$

Equations (4.8) to (4.10) can be used for numerical approximations. Suppose, for example, we consider any particular value of x in the interval $-\rho \leqq x \leqq \rho$, where $0 < \rho < 1$. Then $|R_n(x)| \leqq \frac{\rho^{n+1}}{1-\rho}$. This implies that the polynomial $S_n(x)$ can be used as an approximation to the function $\frac{1}{1-x}$ with an error of at most $\frac{\rho^{n+1}}{1-\rho}$ for any x in the interval $-\rho < x < \rho$. To be specific suppose we take $\rho = \frac{1}{2}$ and wish to have two decimal place accuracy. Thus we require

$$|R_n(x)| \leqq .005 .$$

This will be true if $\frac{\rho^{n+1}}{1-\rho} = \frac{1}{2^n} \leqq .005$, that is, if

$$2^n \geqq 200 ,$$

which holds for $n \geqq 8$.

It follows that the polynomial $1 + x + x^2 + \dots + x^8$ may be used as an approximation to the function $\frac{1}{1-x}$ with at least two decimal place accuracy for all x in the interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$.

Equations (4.8) to (4.10) can be used for other numerical approximations. Thus for example, suppose we wish to evaluate numerically the value of log(1-x) for some fixed x in the interval $-1 < x < 1$. Let $|x| \leq \rho < 1$. Then

$$\log(1-x) = -\int_0^x \frac{dt}{1-t} = -\int_0^x [R_{n-1}(t) + S_{n-1}(t)]dt$$

$$= -\int_0^x [1 + t + t^2 + \dots + t^{n-1}]dt - \int_0^x \frac{t^n}{1-t}\,dt$$

(4.11) $$= -[x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n}] - \int_0^x \frac{t^n}{1-t}\,dt .$$

[The first equality is a formula from Calculus, the second is (4.8), the third is (4.9), (4.10) and the last is an elementary integration.] Therefore if we use the quantity $-[x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n}]$ as an approximation for the value of log(1-x), then the absolute error is $\epsilon_n = \left|\int_0^x \frac{t^n}{1-t}\,dt\right|$. This quantity can be estimated in the following way. Since $|t| < \rho$ it follows that $-t > -\rho$, hence that $1-t > 1-\rho$ and $\frac{1}{1-t} < \frac{1}{1-\rho}$. Consequently

$$\left|\int_0^x \frac{t^n}{1-t}\,dt\right| < \left|\int_0^x \frac{t^n}{1-\rho}\,dt\right| .$$

[The reason for this inequality becomes clear if one thinks of the integrals as the limits of sums. The terms $\frac{t_i^n}{1-t_i}$ occurring in the sum for the left side are all less in absolute value than the terms $\frac{t_i^n}{1-\rho}$ on the right.] Therefore $\epsilon_n < \frac{1}{1-\rho}\left|\int_0^x t^n\,dt\right| = \left|\frac{x^{n+1}}{(1-\rho)(n+1)}\right|$ and finally

$$\epsilon_n < \frac{\rho^{n+1}}{(1-\rho)(n+1)} . \tag{4.12}$$

This error can be made as small as one wishes by choosing n sufficiently large. For instance if ρ is $\frac{1}{2}$ and we wish 3 decimal place accuracy, (4.12) would require that we take n so that $\frac{(\frac{1}{2})^{n+1}}{\frac{1}{2}(n+1)} < .0005$.

That is $\frac{1}{2^n(n+1)} < \frac{1}{2000}$, or $2^n(n+1) > 2000$. By trying successive values of n we find that this inequality will hold if $n \geqq 8$. Consequently the polynomial $-[x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^8}{8}]$ can be used as an approximation for the function $\log(1-x)$ with at least 3 decimal place accuracy for all x in the interval $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$.

Equations (4.8) to (4.10) can also be used to obtain other series representations by making a change of variable. For example, by writing $\frac{1}{1+x^2}$ as $\frac{1}{1-(-x^2)}$, and using $-x^2$ in place of x, we obtain

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \dots , \quad \text{for } -1 < x < 1. \tag{4.13}$$

The student might consider it to be peculiar that the function on the left, $\frac{1}{1+x^2}$, is continuous for all x (see Fig. 4.3) while the series on the right diverges for $|x| \geqq 1$. The explanation of this phenomen will be given when we study complex variables in Chapter 4.

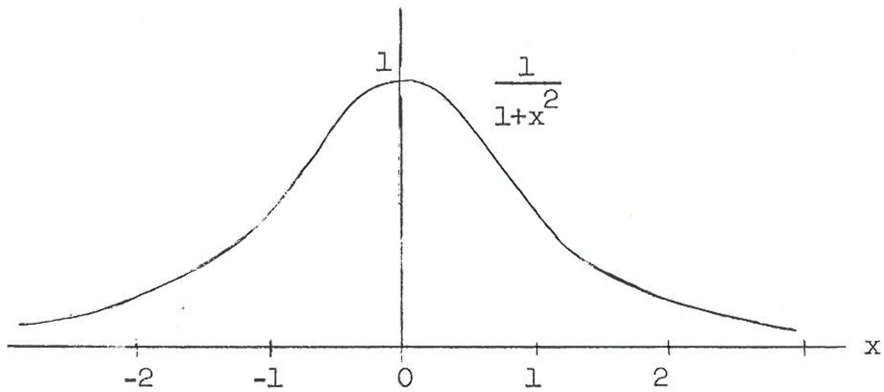


Figure 4.3

Problems

4.1 (a) In approximating the function $\frac{1}{1-x}$ by a polynomial, as described following (4.10), what degree polynomial would be required if we wished to insure 10 decimal place accuracy for x in the interval $-.2 \leqq x \leqq .2$? Ans. 14.

(b) Generalize (a) to find a formula for the degree of the polynomial required, using the method of text, to insure M decimal places of accuracy for x in the interval $-\rho \leqq x \leqq \rho$, where $\rho < 1$.

Ans. $n > \dfrac{M+1 - \log_{10}\left[5(1-\rho)\right]}{-\log_{10}\rho} - 1$.

4.2 Evaluate $\log \frac{1}{2}$ to three decimal places, using the results following (4.11). Ans. -0.693.

4.3 Write a program to evaluate log x by series for $x = .2, .4, .6, .8, 1.0, 1.2, 1.4, 1.6, 1.8$, accurate to three decimal places.

4.4 Find a polynomial which approximates log x for x in the interval $.5 < x < 1.5$ to within two decimal places. Ans. $-\left[(1-x) + \frac{(1-x)^2}{2} + \ldots + \frac{(1-x)^6}{6} \right]$

4.5 Use the method of the penultimate paragraph of the text to find a series (not necessarily a power series) representation of each of the following functions and state the range of values of x for which the series converge.

(a) $\frac{1}{1+2x^2}$. (b) $\frac{1}{1+e^x}$.

(c) $\frac{4}{2+\sin x}$. (d) $\frac{a}{b+cx}$.

Answers.

(a) $1 - 2x^2 + 4x^4 - 8x^6 + \ldots + (-1)^{n-1}2^{n-1}x^{2(n-1)} + \ldots$, for $|x| < \frac{1}{\sqrt{2}}$.

(b) $1 - e^x + e^{2x} - e^{3x} + \ldots + (-1)^{n-1}e^{(n-1)x} + \ldots$, for $x < 0$.

(c) $2 - \sin x + \frac{\sin^2 x}{2} + \ldots + (-1)^{n-1}\frac{(\sin x)^{n-1}}{2^{n-2}} + \ldots$, for all x.

(d) $\frac{a}{b} - \frac{acx}{b^2} + \frac{ac^2x^2}{b^3} - \ldots + \left[(-1)^{n-1}\right]\frac{ac^{n-1}x^{n-1}}{b^n} + \ldots$, for $|x| < \left|\frac{b}{c}\right|$.

4.6 Using the series (4.13),

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots , \quad -1 < x < 1 ,$$

the integration formula

$$\arctan x = \int_0^x \frac{dt}{1+t^2},$$

and a method such as was used to find (4.12), find a polynomial approximation for arctan x, which will be accurate to at least eight decimal places for $|x| < \frac{1}{10}$. Ans. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$.

4.7 How long does the process in Problem 1.6 take? Ans. $(4+3\sqrt{2})\sqrt{\frac{h}{g}} \cong 1.46\sqrt{h}$ seconds.

4.8 If I take out a mortgage for $15,000 to be repaid in equal monthly installments over a twenty year period and such that the amount of interest paid each month should be at the rate of 6% per annum on the unpaid balance of the mortgage, how much is each monthly installment?

4.9 If I leave $1000 in the bank at the rate of 6% per annum how much is the principal at the end of 20 years if the interest is compounded

(a) annually,

(b) quarterly,

(c) monthly,

(d) n times per year,

(e) "continually."

4.10 Derive the formula

$$A_n = \frac{[(1+r)^n - 1]P}{r}$$

for the amount in a sinking fund after n equal deposits of P dollars each, made at the end of each period, with interest compounded at the end of each period at rate r per period.

4.11 At one stage in the making of paper a mixture of wood pulp, soda, and water, called soda pulp, is washed with fresh water in successive steps to remove the soda. Let the soda pulp contain V gallons of water. At each step kV gallons of fresh water are added, the mixture agitated, and kV gallons of soda solution removed. Show that after n such rinses the fraction of soda removed is $1 - (\frac{1}{1+k})^n$. [Hint. If X_n is the concentration of soda after the n-th rinse, then

$$VX_n = VX_{n+1} + kVX_{n+1}.]$$

4.12 In the paradox of Achilles and the Tortoise, show that in overtaking the Tortoise, although Achilles must travel infinitely many successive partial distances, he takes only a finite time to do this. (Cf. Problem 1.7.)

4.13 If a certain amoeba reproduces (by splitting in half) once each minute, how many amoebae would be present at the end of a day?

4.14 If, in a certain chain reaction, each neutron releases (on the average) k other neutrons each nanosecond, how many neutrons would be present one microsecond after a single neutron is released. (Assume there are enough neutrons present to keep the process going for a microsecond.)

4.15 (a) Show that the repeating decimal .135135135... is equal to 5/37.

(b) Use (a) to show that .27135135135... ,(in which the digits 1 3 5 keep repeating) is equal to 251/925.

(c) Prove that every repeating decimal $.a_1 \ldots a_n b_1 \ldots b_m b_1 \ldots b_m \ldots$, (in which $b_1 \ldots b_m$ keeps repeating) represents a rational number.

(d) Express the rational number 22/7 as a repeating decimal.

(e) Prove that any rational number can be expressed as a repeating decimal.

5. Power Series.

In the preceding section we encountered the geometric series

$$1 + x + x^2 + \dots + x^n + \dots .$$

This is a special case (with all $a_n = 1$) of the general power series

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots , \tag{5.1}$$

where $a_0, a_1, \dots, a_n, \dots$ are constants. This is also written $\sum_{n=0}^{\infty} a_n x^n$.

The series (5.1) might converge when certain values are substituted for x and might diverge when other values are substituted for x. It is obvious from (5.1) however that the series surely converges when $x = 0$. For other values of x the situation turns out to be fairly simple. It will be proved in Section 14, below, that one of the following three alternatives must hold.

Theorem 5.1. Given a power series (5.1), either

(a) the series does not converge for any value of x other than $x = 0$, or

(b) there is a positive number R such that the series converges for all x in the interval $|x| < R$ and diverges for all x such that $|x| > R$, or

(c) the series converges for all values of x.

The case (a) is rather unusual and we shall probably not run into it again in this course. The case (c) is sometimes referred to as a special case of (b) with an "infinite radius of convergence." In case (b), at the point $x = R$ the series might or might not converge, and at the point $x = -R$ the series might or might not converge (independently of what happens at $x = R$.) (See Figure 5.1, below). The number R is called the radius of convergence and the interval $-R < x < R$ is called the interval of convergence (regardless of whether or not the series converges at the endpoints).

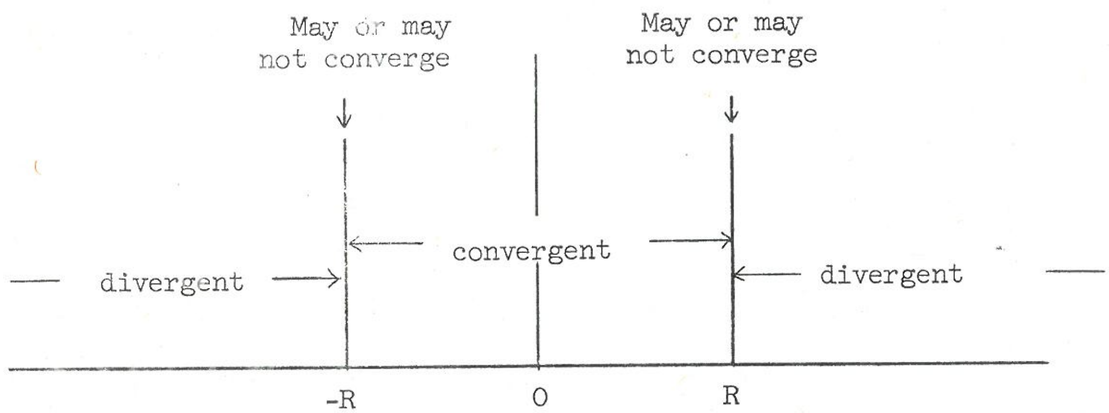


Figure 5.1

Example 5.1. The geometric series converges for $|x| < 1$ and diverges when $|x| > 1$. The radius of convergence is 1. The series happens to diverge at both endpoints, $x = \pm 1$.

The radius of convergence of a power series is often easy to find by means of the following theorem.

Theorem 5.2. If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ then the series $\sum_{n=0}^{\infty} a_n x^n$ converges for all values of x. If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$ then the radius of convergence is $\frac{1}{L}$.

Although we shall not prove Theorem 5.2 until Section 14, we shall find it useful now for determining the interval of convergence of power series.

A convenient way of stating the conclusion of Theorem 5.2 is the following: If $\left|\frac{a_n}{a_{n+1}}\right| \to R$ then R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$. In this formulation the case $R = 0$ is just case (a) of Theorem 5.1, while the case $R = \infty$ is case (c). For it is easy to see that $\left|\frac{a_n}{a_{n+1}}\right| \to \infty$ (see equation (2.9)) if and only if $\left|\frac{a_{n+1}}{a_n}\right| \to 0$.

Example 5.2. For the geometric series $1 + x + x^2 + \ldots + x^{n-1} + \ldots$ we have $a_{n+1} = a_n = 1$. Consequently $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 = L$. Therefore the radius of convergence is $R = \frac{1}{L} = 1$.

Example 5.3. Find the interval of convergence of the power series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots .$$

Since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1}$, we have $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$. Therefore by Theorem 5.2 the series converges for all values of x.

Example 5.4. Find the radius of convergence of the power series

$$1 + 2x + 4x^2 + \ldots + 2^n x^n + \ldots .$$

Since $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \frac{2^{n+1}}{2^n} = \lim_{n \to \infty} 2 = 2$, we see that the radius of convergence is $\frac{1}{2}$.

Example 5.5. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} nx^n$. Since $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)}{n} \right| = \lim_{n \to \infty} \frac{n+1}{n} = 1$, we see that the interval of convergence is $|x| < 1$.

In many ways power series can be manipulated as if they were polynomials. If we had two polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n ,$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n ,$$

then we would have the formulas

(1) $f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n ,$

(2) $cf(x) = ca_0 + ca_1 x + \ldots + ca_n x^n ,$

(3) $f(x)g(x) = (a_0 b_0) + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \ldots$

$$+ (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0)x^n + \ldots + a_n b_n x^{2n} ,$$

(4) $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + na_n x^{n-1} ,$

(5) $\int_0^x f(t)dt = a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \ldots + \frac{a_n x^{n+1}}{n+1} .$

Suppose now we have two power series, having radii of convergence R_1, R_2 respectively:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots , \text{ for } |x| < R_1 ,$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots , \text{ for } |x| < R_2 .$$

Then, in analogy to formulas (1) to (5), the following equations hold, at least for the indicated values of x:

$$(1') \quad f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots + (a_n + b_n)x^n + \dots,$$
$$\text{for } |x| < \min(R_1, R_2),$$

$$(2') \quad cf(x) = ca_0 + ca_1 x + \dots + ca_n x^n + \dots, \text{ for } |x| < R_1,$$

$$(3') \quad f(x)g(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \dots$$
$$+ (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0)x^n + \dots, \text{ for } |x| < \min(R_1, R_2),$$

$$(4') \quad f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + na_n x^{n-1} + \dots, \text{ for } |x| < R_1,$$

$$(5') \quad \int_0^x f(t)\,dt = a_0 x + a_1 \frac{x^2}{2} + \dots + a_n \frac{x^{n+1}}{n+1} + \dots, \text{ for } |x| < R_1.$$

In equations (4') and (5') the radius of convergence is in fact precisely R_1. Although we shall not prove these facts here,* we shall nevertheless find them to be very useful. Equation (4') is particularly important. It is sometimes stated in the form: "Power series may be differentiated termwise, in the interval of convergence. The radius of convergence of the resulting series is the same as that of the original series." It follows that the differentiated series can in turn be differentiated, and so on. Therefore a power series can be differentiated any number of times, in the interval of convergence, without altering the radius of

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*Proofs may be found, for example, in A. E. Taylor, Advanced Calculus, Ginn and Co., 1955, or R. Creighton Buck, Advanced Calculus, McGraw Hill Book Co., 1956. Also see Problem 17.2 below.

convergence. We express this by saying that the <u>power series is infinitely differentiable in the interval of convergence</u>.

Example 5.6. In equation (4.13) we found that

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ , for } |x| < 1.$$

By differentiating each side we see that

$$\frac{-2x}{(1+x^2)^2} = \sum_{n=0}^{\infty} 2n(-1)^n x^{2n-1} = \sum_{n=1}^{\infty} (2n)(-1)^n x^{2n-1}$$

$$= \sum_{m=0}^{\infty} (2m+2)(-1)^{m+1} x^{2m+1} = -2x \sum_{m=0}^{\infty} (m+1)(-1)^m x^{2m} \text{ ,}$$

that is,

$$\frac{1}{(1+x^2)^2} = \sum_{m=0}^{\infty} (m+1)(-1)^m x^{2m} \text{ , } |x| < 1 \text{ .}$$

Similarly by integrating each side of (4.13) from 0 to x, we find

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ , } |x| < 1.$$

The series for $\frac{1}{(1+x^2)^2}$ can also be obtained by multiplying the series for $\frac{1}{1+x^2}$ by itself, thus

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$$

$$\begin{array}{rrrrr}
\hline
1 & -\,x^2 & +\,x^4 & -\,x^6 & + \ldots \\
 & -\,x^2 & +\,x^4 & -\,x^6 & + \ldots \\
 & & x^4 & -\,x^6 & + \ldots \\
 & & & -\,x^6 & + \ldots \\
 & & & & + \ldots \\
\hline
\end{array}$$

$$\frac{1}{(1+x^2)^2} = 1 - 2x^2 + 3x^4 - 4x^6 + \ldots .$$

In this case the general formula for the n-th term is apparent, but one cannot generally expect this to happen when two series are multiplied.

Example 5.7. A more elaborate application of formula (3'), which will be useful later, is obtained by multiplying the series for e^x and e^y, to prove the important identity $e^x \cdot e^y = e^{x+y}$. In Section 7 we shall find that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \quad \text{for all } x.$$

We multiply as in the preceding example:

$$\begin{array}{llllllll}
1 + \frac{x}{1!} & + \frac{x^2}{2!} & + \frac{x^3}{3!} & + \ldots & + & \frac{x^n}{n!} & + \ldots & \\
1 + \frac{y}{1!} & + \frac{y^2}{2!} & + \frac{y^3}{3!} & + \ldots & + & \frac{y^n}{n!} & + \ldots & \\
\hline
1 + \frac{x}{1!} & + \frac{x^2}{2!} & + \frac{x^3}{3!} & + \ldots & + & \frac{x^n}{n!} & + \ldots & \\
\quad \frac{y}{1!} & + \frac{xy}{1!1!} & + \frac{x^2y}{2!1!} & + \ldots & + & \frac{x^{n-1}y}{(n-1)!1!} & + \ldots & \\
 & + \frac{y^2}{2!} & + \frac{xy^2}{1!2!} & + \ldots & + & \frac{x^{n-2}y^2}{(n-2)!2!} & + \ldots & \\
 & & + \frac{y^3}{3!} & + \ldots & + & \frac{x^{n-3}y^3}{(n-3)!3!} & + \ldots & \\
 & & & + \ldots & & & & \\
 & & & & + & \frac{y^n}{n!} & + \ldots & \\
 & & & & & & + \ldots . &
\end{array}$$

Hence, adding by columns,

$$e^x e^y = 1 + \frac{(x + y)}{1!} + \frac{1}{2!}(x^2 + \frac{2!}{1!1!} xy + y^2) + \frac{1}{3!}(x^3 + \frac{3!}{2!1!} x^2 y + \frac{3!}{1!2!} xy^2 + y^3)$$

$$+ \dots + \frac{1}{n!}(x^n + \frac{n!}{(n-1)!1!} x^{n-1} y + \frac{n!}{(n-2)!2!} x^{n-2} y^2 + \dots + y^n) + \dots$$

$$= 1 + \frac{(x + y)}{1!} + \frac{(x + y)^2}{2!} + \frac{(x + y)^3}{3!} + \dots + \frac{(x + y)^n}{n!} + \dots$$

$$= e^{x+y} .$$

This verifies a well-known identity for the exponential function of a real variable. In Chapter 4 we shall use this manipulation of series to prove the same identity when x and y are complex numbers.

Problems

5.1 Find the radius of convergence of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

(b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

(c) $\sum_{n=0}^{\infty} n^2 x^n$.

(d) $\sum_{n=0}^{\infty} \sqrt{n}\, x^n$.

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$.

(f) $\sum_{n=0}^{\infty} 10^{-n} x^n$.

(g) $\sum_{n=1}^{\infty} \frac{(10)^{-n} x^n}{(n+2)}$.

(h) $\sum_{n=1}^{\infty} (10)^n x^n$.

(i) $\sum_{n=1}^{\infty} n! x^n$.

(j) $\sum_{n=1}^{\infty} n(n+1)(\log n) x^n$.

[Answers. 1, 1, 1, 1, ∞, 10, 10, $\frac{1}{10}$, 0, 1.]

5.2 Starting with the geometric series $\sum_{n=o}^{\infty} x^n = \frac{1}{1-x}$, $|x| < 1$, obtain, by differentiation and integration, formulas for $\sum_{n=1}^{\infty} nx^n$, $\sum_{n=1}^{\infty} n^2x^n$ and $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Answer. $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$, $\sum_{n=1}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}$, $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$

5.3 In Section 7 we shall find that the series

(i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \ldots ,$

(ii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + \ldots ,$

(iii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots ,$

converge for all values of x. Use this to find the terms through x^4 for the power series of the functions in (a) and (b):

(a) $2 \sin x \cos x$, by multiplying (i) and (ii),

(b) $\sin 2x$ (replace x by $2x$ in (i)).

(c) Compare the results of (a) and (b), thus verifying (3').

(d) Similarly compute the terms through x^4 for the power series of $\cos^2 x - \sin^2 x$ and compare with the power series of $\cos 2x$.

(e) Similarly compute the terms through x^4 for the power series of $\cos^2 x + \sin^2 x$.

(f) Compute the terms through x^4 for the series of e^{-x}, by replacing x by $-x$ in equation (iii).

(g) Use the product formula (3') in the text, equation (iii) above and part (f) to find the terms through x^4 for the power series of the function $e^x e^{-x}$.

(h) Find the first five non-zero terms of the series for $\frac{e^x+e^{-x}}{2}$. This function is called the hyperbolic cosine of x and is usually written cosh x.

(i) Find the first five non-zero terms of the series for $\frac{e^x-e^{-x}}{2}$. This function is called the hyperbolic sine of x and is usually written sinh x.

5.4 Show that the set of all power series with a given interval of convergence forms a vector space. Is the space finite-dimensional? If so, what is its dimension?

5.5 With any sequence $\{a_n\} = \{a_0, a_1, a_2, \ldots, a_n, \ldots\}$ can be associated the power series $a_0 + a_1t + a_2t^2 + \ldots + a_nt^n + \ldots$. If this series converges in any interval $-R < t < R$ it defines a function $A(t)$ in this interval. $A(t)$ is called the <u>generating function</u> of the sequence $\{a_n\}$. Of the many ways in which generating functions are useful in investigating properties of sequences we give one example.

The Fibonacci sequence (cf. Example 2.1) is defined by $f_0 = f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$ if $n > 1$.

(a) If $F(t)$ is the generating function of the Fibonacci sequence, show that $(1 - t - t^2)F(t) = 1$, and hence that $F(t) = \frac{1}{1 - t - t^2}$.

(b) Prove the identity

$$\frac{1}{1 - t - t^2} = \frac{1}{\sqrt{5}}\left(\frac{a}{1-at} - \frac{b}{1-bt}\right), \tag{5.2}$$

where

$$a = \frac{1 + \sqrt{5}}{2} = 1.61803 \ , \quad b = \frac{1 - \sqrt{5}}{2} = -0.61803 \ .$$

(c) By expanding the right hand side of (5.2) in power series and equating coefficients of like powers of t show that

(5.3) $$f_n = \frac{1}{\sqrt{5}} (a^{n+1} - b^{n+1}) \ .$$

(d) Check (5.3) for $n = 0$ and $n = 1$.

(e) Use (5.3) and a table of logarithms to estimate the size of f_{100}.
Ans. 5.74×10^{20}.

6. Representation of Functions by Power Series.

In the previous section we stated that a convergent power series represents an infinitely differentiable function which can be differentiated termwise inside its interval of convergence. Before going on, let us be certain that we know the precise meaning of that statement. Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a power series having an interval of convergence I. For any fixed value x in the interval I the series converges to a certain sum, which we shall call $f(x)$. At each point x in the interval I this function has a derivative $f'(x)$. At each point x in the interval I the series $\sum_{n=0}^{\infty} na_n x^{n-1}$ converges; the sum of this series is the number $f'(x)$. Similarly at each point x of the interval I the function $f(x)$ has a second derivative $f''(x)$; the series

$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$ converges to $f''(x)$, and so on. This is the meaning of the statement that a convergent power series represents an infinitely differentiable function which can be differentiated termwise inside its interval of convergence.

In the previous paragraph we started with a power series and arrived at an infinitely differentiable function. Now we want to turn the procedure around. Suppose we have a function $f(x)$ which is infinitely differentiable, i.e. having derivatives of every order in some interval $-R < x < R$. Is there a power series $\sum_{n=0}^{\infty} a_n x^n$ such that $f(x)$ is the sum of the series for all x in the interval $-R < x < R$? If we can find such a series we say that <u>the series represents $f(x)$ in the interval $-R < x < R$</u>.

<u>Example 6.1</u>. The function $\frac{1}{1-x}$ is infinitely differentiable in the interval $-1 < x < 1$. It is represented by the geometric series $1 + x + x^2 + \ldots + x^{n-1} + \ldots$ for all x in that interval. (See equation (4.6)).

<u>Example 6.2</u>. The function $\log(1-x)$ is infinitely differentiable in the interval $-1 < x < 1$. It is represented by the series $- x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^n}{n} - \ldots$ in that interval. (See Problem 5.2 and equation (4.11).)

These two examples show functions that are infinitely differentiable in an interval $-R < x < R$ and that have power series representations in the same interval. That we can have the first condition without the second is shown in the next example.

<u>Example 6.3</u>. The function $\frac{1}{1 + x^2}$ is infinitely differentiable in the interval $-2 < x < 2$ - in fact, in $-R < x < R$ for any value of R. It

has a power series representation $1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \dots$, (see equation (4.13)) but this series converges only in the interval $-1 < x < 1$.

Given a function $f(x)$, infinitely differentiable in an interval $|x| < R$, the problem of determining a power series $\sum_{n=0}^{\infty} a_n x^n$ to represent it in that interval requires the determination of the numbers a_n. <u>If the representation is possible</u>, it is easy to show that the numbers a_n are given by the formula

(6.1) $$a_n = \frac{f^{(n)}(0)}{n!} \quad n = 0, 1, 2, \dots .$$

(Here we use the notation $f^{(n)}(0)$ for the value of the n^{th} derivative of $f(x)$ evaluated at $x = 0$, and the conventions $0! = 1$ and $f^{(0)} = f$.)

To see this, suppose the representation is possible, i.e. suppose that there is some power series $\sum_{n=0}^{\infty} a_n x^n$ such that

(6.2) $$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots ,$$

for $|x| < R$. Setting $x = 0$ on each side of equation (6.2) we obtain

$$f(0) = a_0 .$$

Since a power series may be differentiated termwise inside its domain of convergence, we differentiate each side of (6.2) to obtain

(6.3) $$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + na_n x^{n-1} + \dots ,$$

for $|x| < R$. Setting $x = 0$ on each side of (6.3) we obtain

$$f'(0) = a_1 .$$

Differentiating (6.3) we obtain

$$f''(x) = 2a_2 + 3\cdot 2a_3 x + \dots + n(n-1)a_n x^{n-2} + \dots ,$$

and setting $x = 0$ we find

$$f''(0) = 2a_2 .$$

Continuing in this way we verify equation (6.1).

Thus: <u>If a function $f(x)$ can be represented by a power series in the interval $|x| < R$ then the representation is given by</u>

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots , \tag{6.4}$$

or more briefly

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} . \tag{6.5}$$

Regardless of whether or not the above steps can be justified, the right hand side of equation (6.4) or (6.5) is called the <u>Maclaurin series for $f(x)$</u>. Actually it is clear that the Maclaurin series of a function can be formed whenever $f(x)$ has a derivative of each order at $x = 0$, regardless of whether or not $f(x)$ can be represented by a power series. What we have shown so far is that <u>if</u> $f(x)$ can be represented by a power series in an interval $|x| < R$, <u>then</u> that power series is the Maclaurin series for $f(x)$. Before pursuing the question of whether $f(x)$ can in fact be represented by a power series, i.e. whether its Maclaurin series converges to $f(x)$ we should get some facility in computing Maclaurin series.

Example 6.4. Find the Maclaurin series for sin x.

$f(x) = \sin x,$ $f(0) = \sin 0 = 0,$

$f'(x) = \cos x,$ $f'(0) = \cos 0 = 1,$

$f''(x) = - \sin x,$ $f''(0) = - \sin 0 = 0,$

$f'''(x) = - \cos x,$ $f'''(0) = - \cos 0 = -1,$

$f^{(4)}(x) = \sin x,$ $f^{(4)}(0) = \sin 0 = 0,$

$f^{(5)}(x) = \cos x,$ $f^{(5)}(0) = \cos 0 = 1,$ etc.

Thus the Maclaurin series for sin x is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^m x^{2m+1}}{(2m+1)!} + \dots .$$

Problems

6.1 Compute the Maclaurin series for each of the functions:

(a) $\cos x$. (d) $3x^3 + 7x - 1$. (g) $\log (1-x)$. (j) $\sin x^2$.

(b) e^x. (e) $\sqrt{1 + x}$. (h) $(1+x)^{3/2}$. (k) $\frac{1}{1+x^2}$.

(c) $\frac{1}{1+x}$. (f) $\tan x$. (i) $\arcsin x$. (l) $\sin 2x$.

Answers (a) - (e).

(a) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{(-1)^m x^{2m}}{(2m)!} + \dots$

(b) $1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$

(c) $1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots$

(d) $-1 + 7x + 3x^3$

(e) $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots$

6.2 Using the Maclaurin series for e^x and $\cos x$, write the Maclaurin series, up to and including the terms in x^3, for

(a) $e^x \cos x$,

(b) e^{x+1},

(c) $e^{x^2} \cos x$,

(d) $e^{x+2} \cos x^2$.

Hint for (b): Note that $1 + (x+1) + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \dots$ is NOT the answer!

6.3 Compare the first three terms of the Maclaurin series for $\tan x$ with what you get by dividing the series for $\sin x$ by the series for $\cos x$.

7. Taylor's Formula for the Remainder.

The question which arose at the end of the previous section, as to whether the Maclaurin series of a function $f(x)$ represents the function $f(x)$ on some interval $|x| < R$, involves two parts. (a) Does the Maclaurin series converge for each x_0 in the interval $|x| < R$? (b) If the Maclaurin series for $f(x)$ converges at a point x_0 in the interval $|x| < R$ does the sum of the series equal $f(x_0)$? To investigate these questions we introduce the quantity $R_n(x)$, the diference between $f(x)$ and the n-th partial sum of its Maclaurin series,

$$R_n(x) = f(x) - \left[f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)x^n}{n!} \right] . \tag{7.1}$$

One calls $R_n(x)$ the error or the remainder after the n-th term. [Note that this terminology does not exactly agree with our earlier usage of the "remainder" as the difference between the n-th partial sum of a series and its limit. Here the "remainder" $R_n(x)$ is given by equation (7.1) regardless of whether or not the Maclaurin series converges and regardless of whether or not its sum, if it does converge, is $f(x)$. If the series does converge to $f(x)$, then these reduce to the same thing. This is the usual state of affairs.] From equation (7.1) it is obvious that the Maclaurin series will converge to $f(x)$ if and only if the remainder $R_n(x)$ converges to zero as $n \longrightarrow \infty$. Moreover $R_n(x)$ shows us exactly the error committed by replacing $f(x)$ by its Maclaurin series. Fortunately we can find a formula for the remainder $R_n(x)$. It is called Taylor's Formula.

Suppose the function $f(x)$ has $n+1$ continuous derivatives for x in the interval $|x| < R$. Then for any point a in this interval we have

$$f(a) - f(0) = \int_0^a f'(t)dt .$$

The integral on the right will now be integrated by parts according to the formula

$$\int_a^b udv = uv \Big]_a^b - \int_a^b vdu, \text{ with } u = f'(t),\ dv = dt,\ du = f''(t)dt,\ v = t - a.$$

Hence

$$f(a) - f(0) = f'(t)(t-a) \Big]_0^a - \int_0^a (t-a)f''(t)dt ,$$

or

$$f(a) = f(0) + f'(0)a - \int_0^a (t-a)f''(t)dt .$$

Integrating by parts again, this time with $u = f''(t)$, $dv = (t-a)dt$, $du = f'''(t)dt$, $v = \frac{(t-a)^2}{2}$, we get

$$f(a) = f(0) + f'(0)a - f''(t)\frac{(t-a)^2}{2} \Big]_0^a + \int_0^a \frac{(t-a)^2}{2} f'''(t)dt,$$

or

$$f(a) = f(0) + f'(0)a + \frac{f''(0)}{2!} a^2 + \int_0^a \frac{(t-a)^2 f'''(t)}{2} dt .$$

Continuing in this way we get

$$f(a) = f(0) + f'(0)a + \frac{f''(0)a^2}{2!} + \dots + \frac{f^{(n)}(0)a^n}{n!}$$

$$+ (-1)^n \int_0^a \frac{(t-a)^n}{n!} f^{(n+1)}(t)dt .$$

Replacing a by x we get <u>Taylor's Formula with a Remainder in Integral Form:</u>

(7.2) $$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n$$

$$+ \int_0^x \frac{(x-t)^n f^{(n+1)}(t)}{n!} dt.$$

Consequently the remainder $R_n(x)$ is given by the Formula

$$R_n(x) = \int_0^x \frac{(x-t)^n f^{(n+1)}(t)}{n!}\,dt\,. \tag{7.3}$$

This can be put into more convenient form as follows. The mean-value theorem for integrals (cf. R.C. Buck, op. cit.) states that if $f(x)$ is a continuous function on the interval (a,b), and $g(x) \geqq 0$ on (a,b), then

$$\int_a^b f(x)\,g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$

for some suitable value of ξ on (a,b). Thus from equation (7.3) we obtain, taking $g(x) = (x-t)^n$,

$$R_n(x) = \int_0^x \frac{(x-t)^n f^{(n+1)}(t)\,dt}{n!} = \frac{f^{(n+1)}(\xi)}{n!}\int_0^x (x-t)^n dt \tag{7.4}$$

$$= -\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-t)^{n+1}\Big]_0^x = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1},$$

where ξ is a certain number between zero and x.

Thus Taylor's Formula with a Remainder in Derivative form is:

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}. \tag{7.5}$$

This is also known as the "extended mean-value theorem," (see Thomas, Sec. 3-9) since the special case $n = 0$ gives the mean-value theorem $f(x) = f(0) + f'(\xi)x$. Although equation (7.4) gives us the exact value of the remainder, our lack of information regarding the value of ξ prevents us from computing the remainder directly. Nevertheless, as we shall see, (7.4) can often be used to derive an upper bound for the remainder which is often sufficient to demonstrate the convergence of the Maclaurin series to the given function. It is also very useful in numerical work in providing a bound on the error committed by replacing a function by its polynomial approximation.

For example, in the Maclaurin series for sin x we find that each derivative $f^{(k)}(x)$ is either $\pm$ sin x or $\pm$ cos x. Therefore $|f^{(n+1)}(x)| \leqq 1$. Therefore, by (7.4), $|R_n(x)| \leqq \frac{|x|^{n+1}}{(n+1)!}$. For any fixed x this converges to zero. Consequently the Maclaurin series for sin x converges to sin x for all x. Thus we can write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \frac{(-1)^m x^{2m+1}}{(2m+1)!} + \dots .$$

Furthermore we can find a bound for the error after a finite number of terms. For instance taking n = 2 in equation (7.5) we find

$$(7.6) \qquad |\sin x - x| = |(\cos \xi) x^3/6| \leqq |x^3/6| .$$

Thus x is a very good approximation for sin x when x is small. For $|x| < .1$, (= 5.7°), for instance, the error is less than .00017.

For a closer approximation we take n = 4 in equation (7.5) to obtain

$$\sin x = x - \frac{x^3}{3!} + \left[\frac{d^5(\sin x)}{dx^5}\right]_{x=\xi} \frac{x^5}{5!} .$$

Hence

$$\left|\sin x - \left(x - \frac{x^3}{3!}\right)\right| \leqq \left|\frac{x^5}{120}\right| .$$

The following table gives the Maclaurin series for certain functions and the intervals in which they converge.

$$\text{(7.7)} \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^m x^{2m+1}}{(2m+1)!} + \ldots , \text{ all } x,$$

$$\text{(7.8)} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^m x^{2m}}{(2m)!} + \ldots , \text{ all } x,$$

$$\text{(7.9)} \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots , \text{ all } x,$$

$$\text{(7.10)} \qquad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n+1}x^n}{n} + \ldots , \quad |x| < 1,$$

$$\text{(7.11)} \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots , \quad |x| < 1,$$

$$\text{(7.12)} \qquad (1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots + \frac{m(m-1)\ldots(m-(n-1))}{n!}x^n \ldots,$$

$|x| < 1$, (all x if m is a non-negative integer).

From the above series one can, by changing the variable, find other series. For example

$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \ldots + \frac{(-1)^m x^{4m+2}}{(2m+1)!} + \ldots , \text{ all } x,$$

$$\cos \sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \ldots + \frac{(-1)^n x^n}{(2n)!} + \ldots , \quad x \geq 0,$$

and so on.

<u>Example 7.1</u> To illustrate a typical application of these methods to the evaluation of integrals consider the evaluation of the integral

$$\int_0^x \sin t^2 \, dt, \text{ for values of } x \text{ such that } 0 \leq x < 0.1 .$$

(The integral cannot be expressed in terms of elementary functions.) Since, from (7.6), $|\sin x - x| \leqq |\frac{x^3}{6}|$, it follows that

$$|\sin x^2 - x^2| \leqq \frac{x^6}{6} .$$

Hence

$$\int_0^x \sin t^2 \, dt = \int_0^x (\sin t^2 - t^2 + t^2) dt = \int_0^x t^2 dt + \int_0^x (\sin t^2 - t^2) \, dt$$

$$= \frac{x^3}{3} + \int_0^x (\sin t^2 - t^2) \, dt .$$

Equation (7.6) shows that t^2 can be taken as an approximation to $\sin t^2$, and so we consider $\int_0^x t^2 \, dt = \frac{x^3}{3}$ as an approximation to $\int_0^x \sin t^2 \, dt$.

We have

$$\int_0^x \sin t^2 \, dt - \int_0^x t^2 \, dt = \int_0^x (\sin t^2 - t^2) \, dt.$$

Hence

$$\left| \int_0^x \sin t^2 \, dt - \frac{x^3}{3} \right| = \left| \int_0^x (\sin t^2 - t^2) dt \right| \leqq$$

$$\int_0^x \left| \sin t^2 - t^2 \right| dt.$$

This last inequality is a case of the general relation

$$\left| \int_a^b f(x) \, dx \right| \leqq \int_a^b | f(x) | \, dx \text{ , if } a < b \text{ ,}$$

which can be seen to follow from the definition of a definite integral as the limit of a sum. (See Buck, loc. cit., p. 56.)

Now from (7.6) we have $\left| \sin t^2 - t^2 \right| \leqq \frac{t^6}{6}$, and so

$$\left| \int_0^x \sin t^2 \, dt - \frac{x^3}{3} \right| \leqq \int_0^x \frac{t^6}{6} \, dt = \frac{x^7}{42} < \frac{10^{-7}}{42} .$$

Hence $\int_0^x \sin t^2 \, dt$ may be approximated by $\frac{x^3}{3}$ with an accuracy of at least 8 decimal places if $0 \leqq x < 0.1$.

Problems

7.1 Show that the Maclaurin series for cos x converges to cos x for all x and find a bound for the error, after n terms.

7.2 Show that the Maclaurin series for e^x converges to e^x for all x and find a bound for the error, after n terms.

7.3 Evaluate the integrals below to two decimal places.

(a) $\int_0^{0.2} e^{-x^2} \, dx$.

(b) $\int_0^{.5} \frac{\sin x}{x} \, dx$.

(c) $\int_0^{1} e^x \cos \sqrt{x} \, dx$.

7.4 Write a program to evaluate, to 6 decimal places, each of the integrals in Problem 7.3.

7.5 Show that $e^x \geqq 1 + x$ for any value of x, and that, for $|x| < .1$, e^x may be replaced by $1 + x$ with an error less than 6% of x.

7.6 Verify formulas (7.7) to (7.12).

7.7 Evaluate $\int_0^{1/2} \frac{\sin x}{x} dx$ to three decimal places. Ans. 0.493. Show that the approximation $\frac{71}{144}$ is in error by less than 5.3×10^{-5}.

7.8 Evaluate $\int_0^{1/2} e^{-x^2} dx$ to three decimal places. Ans. 0.461.

7.9 We have seen that

$$\sin x = \lim_{m \to \infty} S_m(x),$$

where

$$S_m(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{m+1} \frac{x^{2m-1}}{(2m-1)!} \quad .$$

(a) Program and run the computation of $S_m(x)$ for $m = 1, 4, 7$ for each values of x from 0 to 6.0 at intervals of 0.5.

Also print sin x for each of these values of x, for comparison.

(b) Using the above results make careful graphs of $S_1(x)$, $S_4(x)$, $S_7(x)$ and sin (x) for x in the interval $0 \leqq x \leqq 6$ (all graphs, carefully labeled, on the same sheet of 8 1/2 x 11 graph paper). From the graphs of $S_4(x)$ and sin x estimate the maximum error of $S_4(x)$ in the given interval and compare with the bound obtained from Taylor's Formula (7.4).

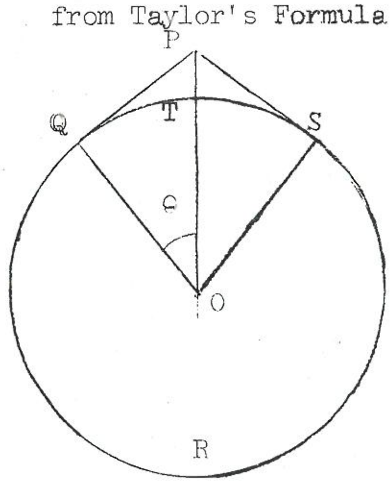
7.10 

Figure 7.1

(a) Around a circle of radius r a loop of cord PQRSP is wrapped. The cord is slightly longer than the circumference of the circle, its length being $2r(\pi + \epsilon)$, so when it is pulled tight at P it looks like Figure 7.1. Assuming that ϵ is very small, show that to a first approximation $\theta = \sqrt[3]{3\epsilon}$.

[Hint. The length of the cord is $2r \tan\theta + 2r(\pi - \theta) = 2r(\pi + \epsilon)$, so $\tan\theta - \theta = \epsilon$.]

(b) If $r = 4000$ miles and $2r\epsilon = 1$ mile find an approximate value of the height TP.

7.11 Sketch the curve $x(t) = e^{-at}(A \sin bt + B \cos bt)$, where a, b, A, B are all positive but A is very small compared to B. It is desired to estimate T, the value of t at which the maximum of $x(t)$ nearest to $t = 0$ occurs. By setting $\dot{x}(T) = 0$ and making suitable approximations show that

$$bT \cong \frac{bA - aB}{aA + bB}.$$

7.12 In Example 7.1 of Chapter 1 we encountered the equation (7.9):

$$u + \log(1-u) + a = 0, \qquad (7.13)$$

a being a small constant (specifically, $a = .00171$). To solve this for u, using tables to find log(1-u), requires very accurate log tables, since the terms u and log (1-u) nearly cancel each other. To avoid this we expand log(1-u) in a Maclaurin series,

$$\log(1-u) = -u - \frac{1}{2}u^2 - \frac{1}{3}u^3 - \dots ,$$

and then (7.13) becomes

$$\frac{1}{2}u^2 + \frac{1}{3}u^3 + \frac{1}{4}u^4 + \dots = a . \tag{7.14}$$

As a first approximation we can take

$$\frac{1}{2}u^2 = a, \qquad u = \sqrt{2a} .$$

For further approximations write (7.14) in the form

$$\frac{1}{2}u^2 = a - \frac{1}{3}u^3 - \frac{1}{4}u^4 - \dots ,$$

and use the <u>iteration</u> <u>method</u>. That is, given an approximation u_n we get a better approximation u_{n+1} from

$$\frac{1}{2}u_{n+1}^2 = a - \frac{1}{3}u_n^3 - \frac{1}{4}u_n^4 - \dots .$$

Use this method to solve (7.13), with $a = .00171$, to 5 decimal places.

7.13 If the integral $\int_0^{1/2} xe^x dx$ is evaluated by expanding the integrand in a power series find the two decimal place value of the integral and an upper bound for the remainder if three terms plus remainder are used in the expression. **Ans.** 0.17; 0.002.

7.14 Six terms of the power series for e^x are used to compute e. How many places of accuracy are obtained? Ans. 2.

7.15 Evaluate $\int_0^1 e^{-x^2}\,dx$ to 3 decimal places.

8. Taylor Series.

Taylor's Formulas (equations (7.2) and (7.5)) can be generalized so as to accommocate the case in which the interval is not necessarily centered at the origin, but at an arbitrary point a. Suppose that we are given a function $f(x)$ which has $n+1$ continuous derivatives for x in the interval $|x - a| < R$. The same argument as in the previous section shows that

$$(8.1)\qquad f(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n(x),$$

$$\text{for } |x-a| < R,$$

where

$$(8.2)\qquad R_n(x) = \int_a^x \frac{(x-u)^n f^{(n+1)}(u)}{n!}\,du, \text{ or}$$

$$(8.3)\qquad R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

for some ξ between a and x.

The series

$$(8.4)\qquad f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \dots$$

is called the <u>Taylor series for $f(x)$ about the point $x = a$</u>. [Note that the Maclaurin series is the special case of the Taylor series in which $a = 0$.] If it converges to $f(x)$, i.e. if $\lim_{n\to\infty} R_n(x) = 0$, then we say that $f(x)$ can be expanded in a Taylor series about the point $x = a$.

<u>Example 8.1</u>. The function log x cannot be expanded in a Maclaurin series since it is not defined at $x = 0$. We can however expand it in a Taylor series about, say, the point $x = 1$. We compute

$$f(x) = \log x, \qquad f(1) = 0,$$

$$f'(x) = \frac{1}{x}, \qquad f'(1) = 1,$$

$$f''(x) = -\frac{1}{x^2}, \qquad f''(1) = -1,$$

$$f'''(x) = \frac{2}{x^3}, \qquad f'''(1) = 2,$$

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}, n \geqq 2, \qquad f^{(n)}(1) = (-1)^{n-1}(n-1)!, n \geqq 2.$$

Hence we have the Taylor series

$$\log x = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \ldots + \frac{(-1)^{n-1}}{n}(x-1)^n + \ldots . \tag{8.5}$$

Note: This could also be obtained by replacing x by $1-x$ in the Maclaurin series for $\log(1-x)$. The series converges for $|x - 1| < 1$; [actually for $0 < x \leqq 2$ (see Example 16.1)].

Problems

8.1 Expand the given function in a Taylor Series about the indicated point.

(a) $\sin x;\ x = \pi/3.$ (b) $\cos x;\ x = \pi/4.$

(c) $\tan x;\ x = 5\pi/4.$ (d) $e^x;\ x = 1.$

(e) $\frac{1}{1+x};\ x = 1.$ (f) $x^5;\ x = 2.$

(g) $\arcsin x;\ x = 1/2.$ (h) $\arctan x;\ x = 1.$

(i) $(x+1)^3;\ x = 1.$ (j) $\sqrt{x};\ x = 1.$

(k) $x^{7/2};\ x = 1.$ (l) $(1+x)^{2/3};\ x = 1/2.$

8.2 (a) Find the first four terms of the Taylor Series about $x = \pi/3$ for the function $f(x) = \cos x$. [Note: "Four terms" means including the term in $(x-\pi/3)^3$ but not $(x-\pi/3)^4$.]

(b) Use Taylor's Formula with the remainder in derivative form to find a bound on the error committed by approximating $\cos x$ by $\left[\frac{1}{2} - \frac{\sqrt{3}}{2}(x-\pi/3) - \frac{1}{4}(x-\pi/3)^2\right]$ for x in the interval $|x - \pi/3| < 1/2$.

Answer. (a) $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}(x-\pi/3) - \frac{1}{4}(x-\pi/3)^2 + \frac{\sqrt{3}}{12}(x-\pi/3)^3 + \dots .$

(b) $|R| = \left|\frac{\sin \xi\,(x-\pi/3)^3}{6}\right| \leqq \frac{1}{6\cdot 2^3} = \frac{1}{48}.$

8.3 In order to evaluate $\sqrt[3]{10}$ an engineer uses as an approximation the first three terms of a Taylor Series for $x^{1/3}$, thus:

$$x^{1/3} \cong 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2;$$

$$\sqrt[3]{10} \cong 2 + \frac{1}{6} - \frac{1}{72} \cong 2.15.$$

Find a bound for the error in this approximation.

8.4 (a) Find the first three terms in the Taylor series expansion of the function $f(x) = \sin x$ about the point $x = \frac{\pi}{2}$.

(b) Sketch the first two terms and their sum in the neighborhood of $x = \frac{\pi}{2}$ on a neat, legible diagram.

(c) For what values of x does the expansion of part (a) converge?

Answer. (a) $\sin x = 1 - \frac{1}{2!}(x - \frac{\pi}{2})^2 + \frac{1}{4!}(x - \frac{\pi}{2})^4 + \ldots$, (c) all x.

8.5 Newton's law of gravitation gives the force of attraction between two bodies as

$$F = \frac{\gamma m M}{r^2}. \tag{8.6}$$

In engineering we often take the gravitational force (weight) of a body as constant near the surface of the earth. Taking the radius of the earth as $a = 4000$ miles, expand the right side of (8.6) in a Taylor series about $r = a$, and show that the error in the engineering approximation at a height of 40 miles above the earth does not exceed 2% of the sea-level weight. More generally show that the error, in percentage of sea-level weight, is less than $\frac{h}{20}$ where h is the height in miles.

8.6 (a) Evaluate $\sin 31°$ to three decimal places by expanding $\sin x$ in a Taylor series about the point $x = \pi/6$.

(b) Write a program for the digital computer to compute $\sin x$, for x in the range $25° \leq x \leq 35°$ accurate to 6 decimal places.

Answer. (a) 0.515.

8.7 Use a Taylor series for e^{x^2} about the point $x = 0.8$ to evaluate the integral $\int_{.6}^{1} e^{x^2} dx$ accurate to two decimal places. Answer. 0.78.

8.8 Find an approximate **formula** for the length of the arc of a circle of radius R in terms of the length of the subtending chord.

9. Taylor Series Solutions of Differential Equations.

Taylor series are useful in the solution of differential equations. To illustrate the technique we consider first the simple equation

(9.1) $$y' = y + 1 ,$$

with the initial condition

(9.2) $$y(1) = 3 .$$

We proceed by tentatively assuming that the solution $y(x)$ can be represented by its Taylor series expansion about the initial point $x = 1$:

(9.3) $$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \ldots + \frac{y^{(n)}(1)}{n!}(x-1)^n + \ldots$$

To evaluate the quantities $y(1)$, $y'(1)$, ..., $y^{(n)}(1)$, ... appearing on the right side of (9.3) we form Table 9.1. The left hand column of Table 9.1 is obtained by successively differentiating each side of (9.1). The right hand column is formed by starting with (9.2) and successively using the corresponding equation in the left hand column, with the previously determined value of the $y^{(n)}(1)$.

Derivative	Value at $x = 1$
	$y(1) = 3$
$y' = y + 1$	$y'(1) = 4$
$y'' = y'$	$y''(1) = 4$
$y''' = y''$	$y'''(1) = 4$
$y^{(4)} = y'''$	$y^{(4)}(1) = 4$

Table 9.1

Therefore, if there is a power series solution it is of the form

$$(9.4) \qquad y(x) = 3 + 4(x-1) + \frac{4}{2!}(x-1)^2 + \dots + \frac{4}{n!}(x-1)^n + \dots .$$

Since $\lim_{n \to \infty} \frac{4/(n+1)!}{4/n!} = \lim_{n \to \infty} \frac{1}{n+1} = 0$ we see that the series in (9.4) converges for all values of x. Therefore we can differentiate it termwise to obtain

$$y'(x) = 4 + 4(x-1) + \frac{4}{2!}(x-1)^2 + \dots + \frac{4}{(n-1)!}(x-1)^{n-1} + \dots ,$$

$$y' = 1 + y .$$

Consequently $y(x)$ as given by (9.4) indeed satisfies the differential equation (9.1). Also, it has been constructed to satisfy the initial condition $y(1) = 3$. Therefore equation (9.4) represents the solution to the given problem. The student might note that the right side of (9.4) is precisely $4e^{x-1} - 1$, which is of course the solution to equation (9.1) with the initial value $y(1) = 3$ as would be obtained by the methods of Section 5 or 6, Chapter 1.

In this particular problem we were able to obtain a formula for the general term in the Taylor series and thus establish the convergence of that series. From this we could verify that the series was indeed a solution to the given problem. Although one can always compute as many terms of the series as desired, it often turns out that one cannot find a simple formula for the general term, or for the ratio of $\frac{a_{n+1}}{a_n}$. In such a case it is difficult to prove directly the convergence of the resulting series in order to establish that it is a solution. Fortunately this may not be necessary because there are fairly general theorems assuring us that a Taylor series solution exists, giving an interval in which it converges and providing estimates of the error after n terms. (See for example W. Kaplan, Ordinary Differential Equations, Addison-Wesley Publishing Company, Reading, Mass., 1958.) In this case one does not have to study the series itself to establish its convergence or bound the error.

Sometimes one forms a Taylor series solution such as (9.4) without any investigation of its convergence. In this case it is called a formal Taylor series solution to the problem (it has the form of a Taylor series and is the solution to the problem if the problem has a Taylor series solution).

Analogous methods can be used to solve systems of differential equations or differential equations of higher order. To illustrate we find the first few terms of the formal Taylor series solution of the simultaneous system

$$\frac{dx}{dt} = x^2 + y^2 + t^2 ,$$

$$\frac{dy}{dt} = xyt + e^t ,$$

with the initial values $x(0) = 1$, $y(0) = -1$. The result is

$$x(t) = 1 + 2t + t^2 + \frac{8}{3} t^3 + \ldots ,$$

$$y(t) = -1 + t - \frac{1}{6} t^3 + \ldots ,$$

with the computation shown in Table 9.2.

DERIVATIVES

x	y
$\dot{x} = x^2 + y^2 + t^2$	$\dot{y} = xyt + e^t$
$\ddot{x} = 2(x\dot{x} + y\dot{y} + t)$	$\ddot{y} = xy + x\dot{y}t + \dot{x}yt + e^t$
$\dddot{x} = 2(x\ddot{x} + (\dot{x})^2 + y\ddot{y} + (\dot{y})^2 + 1)$	$\dddot{y} = 2(x\dot{y} + \dot{x}y + \dot{x}\dot{y}t) + (x\ddot{y} + \ddot{x}y)t + e^t$

VALUES AT $t = 0$

x	y
$x(0) = 1$	$y(0) = -1$
$\dot{x}(0) = 2$	$\dot{y}(0) = 1$
$\ddot{x}(0) = 2(2-1) = 2$	$\ddot{y}(0) = -1 + 1 = 0$
$\dddot{x}(0) = 2(2 + 4 + 1 + 1) = 16$	$\dddot{y}(0) = 2(1 - 2) + 1 = -1$

Table 9.2

Problems

9.1 Find a Taylor series solution for the differential equation

$$y' + y = e^{x}$$

with the initial value $y(0) = 0$. Verify the series solution by comparing it with the closed form solution of the differential equation.

Answer. $y = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \; (= \sinh x = \frac{e^{x} - e^{-x}}{2})$.

9.2 Find the first four terms of the Taylor series solution of the differential equation $y' = e^{xy}$ with the initial condition $y(1) = 3$.

Answer. $y = 3 + 20.086(x-1) + 231.85(x-1)^2 + 3470.88(x-1)^3 + \dots$.

9.3 Find a Taylor series solution for the differential equation

$$y'' + x^2 y' + y = 0$$

with the initial condition $y(0) = 1$, $y'(0) = -1$.

Answer. $y = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{3x^4}{4!} + \frac{5x^5}{5!} - \frac{15x^6}{6!} + \dots$.

9.4 Find the first three terms of the Taylor series solution $x(t)$, $y(t)$, of the system of differential equations

$$\dot{x} + xy = t^3 - 1 , \qquad \ddot{y} + x^2 + y^2 = 0 ,$$

with the initial conditions

$$x(0) = 0 , \qquad y(0) = 0, \qquad y(0) = 1 .$$

Answer. $x = -t + \frac{t^3}{3} + \frac{t^4}{4} + \dots , \; y = t - \frac{t^4}{6} + \frac{t^6}{45} + \dots$.

9.5 The differential equation

$$2xy' = 3y$$

has the solution $y = cx^{3/2}$. This function does not have a Maclaurin series expansion. In what way does the method described in the text above break down for this problem?

9.6 Find the first four terms of the Maclaurin Series of the solution to the differential equation

$$\frac{dy}{dx} = 1 + x + y^2$$

satisfying the initial condition $y(0) = 1$.

Answer. $y(x) = 1 + 2x + \frac{5x^2}{2} + 3x^3 + \dots$.

9.7 (a) Find the first four terms of the Taylor series solution about $x = 1$ of the differential problem:

$$\frac{dy}{dx} = \frac{2x^2 + y}{x}, \qquad y(1) = 1 .$$

(b) Noting that $y(x) = 2x^2 - x$ is also a solution to the problem prove that for the solution in (a) $\frac{d^n y}{dx^n} = 0$ at $x = 1$, for $n \geq 3$.

9.8 Find the first four terms of the Taylor Series about $x = 2$ of the solution of the differential equation

$$\frac{dy}{dx} = x + y^2$$

subject to the condition

$$y(2) = -1.$$

Answer. $y(x) = -1 + 3(x-2) - \frac{5}{2}(x-2)^2 + \frac{14}{3}(x-2)^3 + \dots$.

10. Indeterminate Forms.

We can use Taylor's theorem to evaluate certain limits (of functions). Thus suppose that $f(x)$ and $g(x)$ are functions represented by their Taylor series at $x = a$ and that $f(a) = g(a) = 0$. Then the ratio $\frac{f(x)}{g(x)}$ is indeterminate at $x = a$, so that we need some trick to evaluate

$$\lim_{x \to a} \frac{f(x)}{g(x)} .$$

To be quite precise, let us suppose that $f(a) = f'(a) = \dots = f^{(m-1)}(a) = 0$, while $f^{(m)}(a) \neq 0$; $g(a) = g'(a) = \dots = g^{(n-1)}(a) = 0$, while $g^{(n)}(a) \neq 0$, where m and n are positive integers. Then we have the following theorem.

<u>Theorem 10.1</u>. If $m > n$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$;

if $m = n$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$;

if $m < n$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist.

<u>Proof</u>. Since $f(a) = f'(a) = \dots = f^{(m-1)}(a) = 0$, Taylor's theorem gives us $f(x) = \frac{f^{(m)}(a)(x-a)^m}{m!} + \frac{f^{(m+1)}(a)(x-a)^{m+1}}{(m+1)!} + \dots$;

similarly $g(x) = \frac{g^{(n)}(a)(x-a)^n}{n!} + \frac{g^{(n+1)}(a)(x-a)^{n+1}}{(n+1)!} + \dots$.

$$\text{Thus } \frac{f(x)}{g(x)} = \frac{\frac{(x-a)^m}{m!}\left[f^{(m)}(a) + (x-a)\left(\frac{f^{(m+1)}(a)}{m+1} + \dots\right)\right]}{\frac{(x-a)^n}{n!}\left[g^{(n)}(a) + (x-a)\left(\frac{g^{(n+1)}(a)}{n+1} + \dots\right)\right]} .$$

This means that, when x is close to a, $\frac{f(x)}{g(x)}$ is very much like

$\frac{n!}{m!} \cdot \frac{f^{(m)}(a)}{g^{(n)}(a)} \cdot (x-a)^{m-n}$. The conclusion of the theorem now follows

since $(x-a)^{m-n} \to 0$ as $x \to a$ if $m > n$ and has no limit if

$m < n$, while if $m = n$, $\frac{f(x)}{g(x)}$ plainly converges to $\frac{f^{(n)}(a)}{g^{(n)}(a)}$.

<u>Example 10.1.</u> $\frac{e^x - (1+x)}{\sin x} = \frac{\frac{x^2}{2!} + \cdots}{x - \cdots} = \frac{\frac{x}{2!} + \cdots}{1 - \cdots}$. Hence $\lim_{x \to 0} \frac{e^x - (1+x)}{\sin x} = 0$.

In this case $m > n$.

<u>Example 10.2.</u> $\frac{t \sin t}{1 - \cos t} = \frac{t(t - \frac{t^3}{3!} + \cdots)}{1 - (1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots)}$

$$= \frac{t^2(1 - \frac{t^2}{3!} + \cdots)}{t^2(\frac{1}{2!} - \frac{t^2}{4!} + \cdots)} = \frac{(1 - \frac{t^2}{6} + \cdots)}{(\frac{1}{2} - \frac{t^2}{24} + \cdots)} .$$

Hence

$$\lim_{t \to 0} \frac{t \sin t}{1-\cos t} = 2.$$

In this case $m = n$.

<u>Example 10.3.</u> $\frac{e^x - 1}{x \log(1-x)} = \frac{x + \frac{x^2}{2!} + \cdots}{-x(x + \frac{x^2}{2} + \cdots)} = \frac{1 + \frac{x}{2!} + \cdots}{-x(1 + \frac{x}{2} + \cdots)}$,

which becomes infinite as x tends to 0. In this case $m < n$.

A more general result of this nature is L'Hospital's Rule which we now state without proof.

<u>L'Hospital's Rule:</u>* If $f(x)$ and $g(x)$ each have a derivative in an interval $|x-a| < R$ and $f(a) = g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{10.1}$$

provided the limit on the right side of equation (10.1) exists. If the functions have Taylor series then l'Hospital's rule can be deduced from Theorem 10.1.

Even if $f(x)$ and $g(x)$ do not have Taylor series, l'Hospital's rule may still be used. For example

$$\lim_{x \to 0} \frac{x^{4/3}}{\sin x} = \lim_{x \to 0} \frac{\frac{4}{3}x^{1/3}}{\cos x} = 0 .$$

The $\lim_{x \to a} \frac{f(x)}{g(x)}$ is often called "the indeterminate form $\frac{0}{0}$", but this is only a form of speaking. The symbol $\frac{0}{0}$ by itself has no meaning.

If the resulting form is also indeterminate, i.e. $f'(a) = g'(a) = 0$, then if $f'(x)$ and $g'(x)$ have derivatives, we can repeat the procedure.

To illustrate the application of l'Hospital's rule, the limits in Examples 10.1, 10.2, 10.3, become

$$\lim_{x \to 0} \frac{e^x-1-x}{\sin x} = \lim_{x \to 0} \frac{e^x-1}{\cos x} = 0 ,$$

$$\lim_{t \to 0} \frac{t \sin t}{1-\cos t} = \lim_{t \to 0} \frac{t \cos t + \sin t}{\sin t} .$$

This limit can be evaluated as

$$\lim_{t \to 0} (\cos t \frac{t}{\sin t} + 1) = 1 + 1 = 2,$$

or alternatively

$$\lim_{t \to 0} \frac{-t \sin t + 2 \cos t}{\cos t} = 2,$$

and

$$\lim_{x \to 0} \frac{e^x-1}{x \log (1-x)} = \lim_{x \to 0} \frac{e^x}{\frac{-x}{1-x} + \log(1-x)} ,$$ which becomes infinite as x tends to zero.

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*For a proof see Thomas, Section 16-7, or R.P. Agnew, <u>Calculus</u>, McGraw-Hill Book Co., New York, 1962, p. 330.

Warning: L'Hospital's rule is based on the assumption that $f(a) = g(a) = 0$. If this condition is not satisfied, application of the rule will generally lead to wrong results. For example:

$$\lim_{x \to 0} \frac{7x + 3}{2x + 5} = \frac{3}{5}, \qquad \text{not } \frac{7}{2}.$$

L'Hospital's rule is applicable when x approaches a from only one side, as in the example

$$\lim_{x \to 0} \frac{e^x - 1}{\sqrt{x}} = \lim_{x \to 0} \frac{e^x}{\frac{1}{2}x^{-1/2}} = \lim_{x \to 0} 2x^{1/2}e^x = 0.$$

A more radical extension of the rule is to the case where the condition $f(a) = 0, g(a) = 0$ is replaced by $f(x) \to \infty$, $g(x) \to \infty$ as $x \to a$. We shall not prove this here.* This is called the "indeterminate form $\frac{\infty}{\infty}$."

<u>Example 10.4.</u> $\lim_{x \to 0} x \log x = \lim_{x \to 0} \frac{\log x}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} (-x) = 0.$

This rule is also applicable in case $x \to \infty$. To see this let $s = \frac{1}{x}$, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{s \to 0} \frac{f(\frac{1}{s})}{g(\frac{1}{s})} = \lim_{s \to 0} \frac{-\frac{1}{s^2}f'(\frac{1}{s})}{-\frac{1}{s^2}g'(\frac{1}{s})} = \lim_{s \to 0} \frac{f'(\frac{1}{s})}{g'(\frac{1}{s})} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}.$$

<u>Example 10.5.</u> $\lim_{n \to \infty} \frac{n^3}{e^n} = \lim_{n \to \infty} \frac{3n^2}{e^n} = \lim_{n \to \infty} \frac{6n}{e^n} = \lim_{n \to \infty} \frac{6}{e^n} = 0.$

<u>Example 10.6.</u> $\lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0.$

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* For a proof see Agnew, loc. cit., p. 331 or "L'Hospital's Rule," A.E. Taylor, American Mathematical Monthly 59 (1952) 20-24.

Other applications of l'Hospital's rule, or of power series, can be made to the indeterminate form 1^∞, 0^0, $\infty - \infty$ etc. by the techniques illustrated in the following examples.

Example 10.7. To find $\lim_{x \to 0} (1 + x)^{\frac{1}{x}}$, let $A = \lim_{x \to 0} (1 + x)^{\frac{1}{x}}$. Then since the logarithm is a continuous function,

$$\log A = \log \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = \lim_{x \to 0} \log(1 + x)^{\frac{1}{x}} = \lim_{x \to 0} \frac{\log(1 + x)}{x}$$

$$= \lim_{x \to 0} \frac{1}{1 + x} = 1.$$

Hence $A = \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$.

Example 10.8. To find $\lim_{x \to 0} x^x$, let $A = \lim_{x \to 0} x^x$. Then

$$\log A = \log \lim_{x \to 0} x^x = \lim_{x \to 0} \log x^x = \lim_{x \to 0} x \log x = \lim_{x \to 0} \frac{\log x}{\frac{1}{x}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} (-x) = 0.$$

Hence $A = \lim_{x \to 0} x^x = 1$.

Example 10.9. $$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x \to 0} \frac{\sin x - x}{x \sin x} = \lim \frac{x - \frac{x^3}{3!} + \ldots - x}{x(x - \frac{x^3}{3!} + \ldots)}$$

$$= \lim \frac{-\frac{x}{3!} + \ldots}{1 - \frac{x^2}{3!} + \ldots} = 0.$$

Problems

10.1 Evaluate the following indeterminate forms.

(a) $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$.

(b) $\lim_{x \to 1} \frac{\log x - 1 + x}{1-x}$

(c) $\lim_{x \to \infty} \frac{\sin x}{x}$.

(d) $\lim_{x \to 0} \left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right)$.

(e) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sqrt{x}}\right)$.

(f) $\lim_{x \to \infty} x^{\frac{1}{x}}$.

(g) $\lim_{x \to \infty} (1+x)^{\frac{1}{x}}$.

(h) $\lim_{x \to 0} (1 - \sin x)^{\frac{1}{x}}$.

(i) $\lim_{x \to \infty} \frac{\log x}{x^{\alpha}}$, $\alpha > 0$.

(j) $\lim_{x \to \infty} x^n e^{-x}$.

(k) $\lim_{x \to 0} \frac{\sin 2x - 2x \cos x}{x^3}$.

Answers. $\frac{1}{2}$, -2, 0, $\frac{1}{3}$, ∞, 1, 1, $\frac{1}{e}$, 0, 0, $-\frac{1}{3}$.

10.2 Discuss the following derivation of l'Hospital's Rule.

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f(x)/x}{g(x)/x} = \frac{f'(0)}{g'(0)} .$$

10.3 Evaluate

(a) $\lim_{x \to 0} \frac{e^x - 1}{\sin 3x}$. (b) $\lim_{x \to 0} \frac{e^{3x} - e^{2x} - x}{1 - \cos x}$. (c) $\lim_{x \to 1} \frac{\sqrt{2-x} - 1}{1-x}$.

10.4 Evaluate (a) $\lim_{x \to 0} x^{x^x}$, (b) $\lim_{x \to 0} x^{x^2}$

CONVERGENCE OF SERIES

11. Basic Facts about Series and Sequences.

Thus far in this chapter we have defined the notion of convergence of an infinite series and have made certain statements, notably in connection with power series, about the convergence or divergence of particular series and the sum of a series if it does converge. The time has now come to examine the question of convergence more carefully and to justify our earlier statements.

We shall develop some techniques for:

(a) determining whether a given series converges or diverges;

(b) establishing bounds on the remainder after n terms of a convergent series.

Before proceeding we recall some results from Section 2 and some other basic facts about sequences and series.

(A) If $\sum_{n=1}^{\infty} a_n$ converges then $a_n \longrightarrow 0$.

(B) The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. One has the estimates:

$$\log(N + 1) \leqq \sum_{n=1}^{N} \frac{1}{n} \leqq 1 + \log N.$$

(C) The geometric series $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$, diverges if $|r| \geqq 1$. The error after n terms is $R_n = \frac{1}{1-r} - S_n = \frac{1}{1-r} - \frac{1-r^n}{1-r} = \frac{r^n}{1-r}$.

(D) If $\sum_{n=1}^{\infty} a_n = L$ and $\sum_{n=1}^{\infty} b_n = M$ then $\sum_{n=1}^{\infty} (a_n + b_n) = L + M$,

$\sum_{n=1}^{\infty} (ca_n) = cL$, where c is any constant.

(E) From the definition of convergence it follows that convergence and divergence are not affected by the alteration of any finite number of terms of the series. Of course, the sum of the series will be affected by such a change, but when we are discussing merely whether a given series converges or diverges we are free to alter any finite number of terms.

(F) One property of real numbers that we shall have occasion to use is so basic that it is often taken as an axiom, known as the Axiom of Continuity*:

If a sequence $\{a_n\}$ has the property

(i) $a_{n+1} \geqq a_n$, $n = 1, 2, \ldots$,

and if there is a number M such that

(ii) $a_n \leqq M$, $n = 1, 2, \ldots$,

then $\{a_n\}$ converges.

We express condition (i) by saying that $\{a_n\}$ is monotonic increasing, and (ii) by saying that $\{a_n\}$ is bounded above. Thus the axiom of continuity can be stated: If a monotonic increasing sequence is bounded above, it converges.

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*See, for example C.B. Morrey, University Calculus, Addison-Wesley Publishing Co., Reading, Mass., 1962.

Example 11.1. The sequence

$$\{1,\ 1.1,\ 1.11,\ 1.111,\ \ldots\}$$

is monotonic increasing and bounded above by 2. It converges to the limit

$$1 + \frac{1}{10} + \frac{1}{100} + \ldots + \frac{1}{10^n} + \ldots = \frac{1}{1 - \frac{1}{10}} = \frac{10}{9}.$$

Problems

11.1 **Show that if there were no irrational numbers, the Axiom of Continuity would not be true, i.e. find a bounded monotonic increasing sequence of rational numbers that does not have a rational number as limit.**

11.2 If $\sum_{n=1}^{\infty} a_n = 2$ and $\sum_{n=1}^{\infty} b_n = 3$ evaluate $\sum_{n=1}^{\infty} (2a_n - 3b_n + \frac{1}{2^n})$. Ans. -4.

12. Absolute and Conditional Convergence.

If a series $\sum_{n=1}^{\infty} a_n$ has the property that the corresponding series formed by using the absolute values of the terms converges, then the given series is said to be absolutely convergent. That is, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty} |a_n|$ converges. The choice of words is justified by the fact that an absolutely convergent series does, in fact, converge, as we shall prove in the following section (Corollary 13.1). Of course the limits of the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ will be different unless $a_n \geq 0$ for $n = 1, 2, \ldots$; but if

$\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. On the other hand, it can happen that $\sum_{n=1}^{\infty} a_n$ converges, while $\sum_{n=1}^{\infty} |a_n|$ diverges; then the series $\sum_{n=1}^{\infty} a_n$ is said to be <u>conditionally convergent</u>. That is, a series is conditionally convergent if it is convergent but is not absolutely convergent.

13. <u>Comparison Test</u>.

One way of telling whether a series converges or diverges is to compare it, term by term with another series whose behavior we already know. We first consider series with no negative terms.

<u>Theorem 13.1</u>. (<u>Comparison Test</u>.) Let $a_n \geqq 0$ and $b_n \geqq 0$. Then

(a) If $\sum_{n=1}^{\infty} b_n$ converges and if $a_n \leqq b_n$ for $n = 1, 2, \ldots$, then $\sum_{n=1}^{\infty} a_n$ converges;

(b) If $\sum_{n=1}^{\infty} b_n$ diverges and if $a_n \geqq b_n$ for $n = 1, 2, \ldots$, then $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Proof</u>. (a) The sequence $S_n = \sum_{m=1}^{n} a_m$ is monotonic increasing. It is is also bounded, because $S_n \leqq \sum_{m=1}^{n} b_m \leqq \sum_{m=1}^{\infty} b_m$. Hence, by the Axiom of Continuity of Section 11 the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) We prove this by contradiction. Suppose $\sum_{n=1}^{\infty} a_n$ were convergent. Then we could apply part (a) of the theorem to b_n, since $b_n \leqq a_n$ and conclude that $\sum_{n=1}^{\infty} b_n$ converged, contrary to the hypothesis. Therefore $\sum_{n=1}^{\infty} a_n$ diverges.

Corollary 13.1. If a series $\sum a_n$ is absolutely convergent then it is convergent; i.e., if $\sum |a_n|$ converges then $\sum a_n$ converges.

Proof. Let $b_n = \frac{|a_n| + a_n}{2}$. Then $b_n \geqq 0$ and $b_n \leqq |a_n|$. Since $\sum_{n=1}^{\infty} |a_n|$ converges, by hypothesis, it follows from Theorem 13.1 that $\sum_{n=1}^{\infty} b_n$ converges.

Similarly let $B_n = \frac{|a_n| - a_n}{2}$; then $\sum_{n=1}^{\infty} B_n$ converges. Consequently, by property (D), Section 11, the series $\sum_{n=1}^{\infty} (b_n - B_n) = \sum_{n=1}^{\infty} a_n$ converges.

Note that what this proof really does is to show that the series consisting of only the positive terms of $\sum a_n$, and the series consisting of only the negative of the negative terms, each converge by the comparison test. Since they converge their difference converges, and their difference is exactly the given series.

Corollary 13.2. If $|a_n| \leqq b_n$, and if $\sum b_n$ converges then $\sum a_n$ converges absolutely, and the error after N terms is bounded by $\sum_{n=N+1}^{\infty} b_n$.

Proof. From Theorem 13.1 and Corollary 13.1 we conclude that $\sum a_n$ converges. The error after N terms then satisfies

$$R_N = \left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \right| = \left| \sum_{n=N+1}^{\infty} a_n \right| \leq \sum_{n=N+1}^{\infty} |a_n| \leq \sum_{n=N+1}^{\infty} b_n .$$

Example 13.1. The series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{(-1)^{n-1}}{n10^{n-1}}$ converges absolutely, since $|a_n| = \frac{1}{n10^{n-1}} \leq \frac{1}{10^{n-1}} \equiv b_n$ and $\sum_{n=1}^{\infty} b_n$ is the geometric series with $r = \frac{1}{10}$. For the error after N terms we have

$$\left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \right| \leq \sum_{n=N+1}^{\infty} b_n = \frac{1}{10^N} + \frac{1}{10^{N+1}} + \ldots = \frac{1}{10^N(1 - \frac{1}{10})} = \frac{1}{9\cdot 10^{N-1}} .$$

Thus, for example, after 3 terms we have

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{20} + \frac{1}{300} + R_3 = \frac{143}{150} + R_3 ,$$

(13.1) where $|R_3| \leq \frac{1}{900} = .0011\ldots$.

In Section 16 below we shall find a technique for obtaining a sharper estimate for the error after N terms for series of this particular form. In this case we shall see that $-\frac{1}{4000} < R_3 < 0$ (see Example 16.2).

As a consequence of Property (E) of Section 11, the conditions $a_n \leq b_n$ and $a_n \geq b_n$ in Theorem 13.1 (a) and (b) need hold only for n larger than some chosen value N. Using this observation we can state a form of the Comparison Test that is often more convenient than the original one.

Theorem 13.2. Let $a_n > 0$, $b_n > 0$ for sufficiently large n. If $\lim_{n \to \infty} \frac{a_n}{b_n} = L$, where L is neither 0 nor ∞, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Proof. Apply Definition 2.1 to the sequence $\left\{\frac{a_n}{b_n}\right\}$, taking $\epsilon = \frac{L}{2}$. Then for $n > N$ we have

$$\left| \frac{a_n}{b_n} - L \right| < \frac{L}{2} .$$

This inequality is equivalent to

$$\frac{1}{2} L < \frac{a_n}{b_n} < \frac{3}{2} L ,$$

or, since we can assume N large enough that a_n and b_n are both positive,

$$\frac{1}{2} Lb_n < a_n < \frac{3}{2} Lb_n .$$

Now if $\sum b_n$ converges so does $\sum \frac{3}{2} Lb_n$, by Property (D) of Section 11. Hence from $a_n < \frac{3}{2} Lb_n$ we conclude from Theorem 13.1 that $\sum a_n$ converges. If $\sum b_n$ diverges we use the inequality $a_n > \frac{1}{2} Lb_n$ to prove that $\sum a_n$ diverges.

Example 13.2. Test the convergence of the series $\sum_{10}^{\infty} a_n$, where

$$a_n = \frac{5n - 27}{n^2 + 15\log n} .$$

First we try to get some estimate of the behavior of a_n for large n. If n is large the term -27 is small compared with $5n$, and $15\log n$ is small compared with n^2. Hence a_n behaves like $\frac{5n}{n^2} = \frac{5}{n}$. We therefore take as a comparison series the harmonic series $\sum b_n = \sum \frac{1}{n}$. Using l'Hospital's Rule and a little algebra we get

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{5n^2 - 27n}{n^2 + 15\log n}$$

$$= \lim_{n \to \infty} \frac{10n - 27}{2n + 15/n}$$

$$= \lim_{n \to \infty} \frac{10}{2 - \frac{15}{n^2}} = 5.$$

Since $\sum b_n$ diverges so does $\sum a_n$.

Problems

13.1 By comparing with the series $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots$ show that the series for the decimal expansion of any number between zero and one, namely $\frac{a_1}{10} + \frac{a_2}{100} + \frac{a_3}{1000} + \ldots$ is absolutely convergent and find a bound on the error after N terms.

13.2 By comparing with the geometric series $1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots$ show that the series $1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 2^2} + \ldots + \frac{1}{n2^{n-1}} + \ldots$ converges and find a bound on the error after N terms. Compare with the result of integrating the Maclaurin series for $\frac{1}{1-x}$. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

Answer. $\epsilon_N \leqq \frac{1}{2^{N-1}}$, in fact $\epsilon_N \leqq \frac{1}{(N+1)2^{N-1}}$. $\sum_{n=1}^{\infty} \frac{1}{n2^n} = \log 2$.

13.3 Show that the series $\frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots$ diverges.

Answer. Compare with harmonic series.

13.4 Using the fact that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} + \ldots$ converges to 1 (see Problem 2.5) show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and find a bound on the error after N terms. Answer. $\epsilon_N \leqq \frac{1}{N}$.

13.5 Prove that the harmonic series is divergent on the basis of the fact that its terms are greater than or equal to those of the divergent series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \ldots .$$

14. Ratio Test.

Theorem 14.1. (Ratio Test) If $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. If $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent. If $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, or does not exist, then the series might or might not converge and further investigation is required.

Proof. If $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$ choose a number ρ such that $r < \rho < 1$.

Then there is an integer N such that, for all $n \geqq N$, $\left|\frac{a_{n+1}}{a_n}\right| < \rho$.

Hence

$$|a_{N+1}| < |a_N| \rho,$$

$$|a_{N+2}| < |a_{N+1}| \rho < |a_N| \rho^2,$$

$$|a_{N+3}| < |a_{N+2}| \rho < |a_N| \rho^3, \text{ etc.}$$

The terms on the right are the terms of the geometric series times the constant $|a_N|\rho$; i.e. $|a_N|\rho(1 + \rho + \rho^2 + \ldots)$. Since $\rho < 1$ this series converges, and by the Comparison Test the series $\sum a_n$ is absolutely convergent. If $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ then the condition $\lim_{n \to \infty} a_n = 0$ does not hold and by property (A) the series must diverge.

To clarify and illustrate the last part of the theorem consider the four series

(a) $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$,

(b) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} + \ldots$,

(c) $1 + \frac{1}{2} + \frac{1}{3^3} + \frac{1}{4} + \frac{1}{5^3} + \ldots + \frac{1}{n^{2+(-1)^{n+1}}} + \ldots$,

(d) $1 + \frac{1}{2^4} + \frac{1}{3^2} + \frac{1}{4^4} + \frac{1}{5^2} + \ldots + \frac{1}{n^{3+(-1)^n}} + \ldots$.

For series (a) $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \frac{n}{n+1} = 1$. The series is the harmonic series, which diverges. For series (b) $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n \to \infty} \frac{n^2}{(n+1)^2} = 1$. It will be shown below (Corollary 15.1) that this is a convergent series. (Or see Problem 13.4 above.) For series (c) the ratios $\left|\frac{a_{n+1}}{a_n}\right|$ are alternatively of the form $\frac{n}{(n+1)^3}$ and $\frac{n^3}{n+1}$ and so the limit $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist. In fact, the series diverges, as may be seen from the fact that the even-numbered terms are $(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots) = \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \ldots)$

and the last factor is the divergent harmonic series, while the remaining terms are all positive. For series (d), $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ does not exist. This series converges, because its terms are less than or equal to those in (b).

In carrying out the ratio test we may well find ourselves trying to evaluate limits,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| ,$$

where $a_n \to 0$ or $|a_n| \to \infty$ as $n \to \infty$. We may then avail ourselves, if necessary, of l'Hospital's rule, provided that there is a continuous function $f(x)$ of the real variable x such that

$$f(n) = |a_n| ;$$

for, in that case,

$$\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| . \tag{14.1}$$

Equation (14.1) asserts that to evaluate $\lim_{x \to \infty} \frac{f(x+1)}{f(x)}$ (if it exists) we may allow x to tend to infinity through integer values.

Consider for example the series $\sum \frac{n^n b^n}{n!}$, where b is a positive constant. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n!(n+1)^{n+1} b^{n+1}}{(n+1)!n^n b^n} = (1 + \frac{1}{n})^n b .$$

Now $\lim_{n \to \infty} (1 + \frac{1}{n})^n = \lim_{x \to \infty} (1 + \frac{1}{x})^x = \lim_{y \to 0} (1 + y)^{\frac{1}{y}}$, (setting $y = \frac{1}{x}$)

$= e$ (Example 10.7) .

Thus $\left|\frac{a_{n+1}}{a_n}\right| \to eb$ and so the series converges if $b < \frac{1}{e}$ and diverges if $b > \frac{1}{e}$.

The following theorem provides the proof of the facts about power series stated in Section 5.

Theorem 14.2. The power series $\sum_{n=1}^{\infty} a_n x^n$ either

(a) converges for all values of x, or

(b) converges for no values of x other than $x = 0$, or

(c) there exists a number R (the radius of convergence) such that the series converges absolutely for $|x| < R$ and diverges for $|x| > R$.

Proof. That (a) is possible can be seen by considering the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ which, as we saw in Problem 7.2, converges to e^x for all values of x.

That (b) is possible can be seen by considering the series $\sum_{n=1}^{\infty} n!x^n$. For any specified value of x other than zero the terms $n!x^n$ do not tend to zero and so the series does not converge.

If neither (a) nor (b) holds let $b \neq 0$ be a value of x for which the series converges and c a value of x for which the series diverges. Let x be a number such that $|x| < |b|$. Since the series $\sum_{n=1}^{\infty} a_n b^n$ converges $a_n b^n \to 0$ and so there is an N such that for all $n > N$,

$|a_n b^n| < 1$, i.e. $|a_n| < \frac{1}{|b^n|}$. Hence $|a_n x^n| < \left|\frac{x^n}{b^n}\right| = r^n$ where $r < 1$. By comparison with the geometric series it follows that $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely. Hence $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely for all x such that $|x| < |b|$.

Next let x be any number such that $|x| > |c|$. If $\sum_{n=1}^{\infty} a_n x^n$ were convergent it would follow by the previous argument that $\sum_{n=1}^{\infty} a_n c^n$ would converge, contrary to assumption. Hence $\sum_{n=1}^{\infty} a_n x^n$ diverges for all x such that $|x| > |c|$.

Let us draw a figure to illustrate the situation, taking b, c to be positive for convenience.

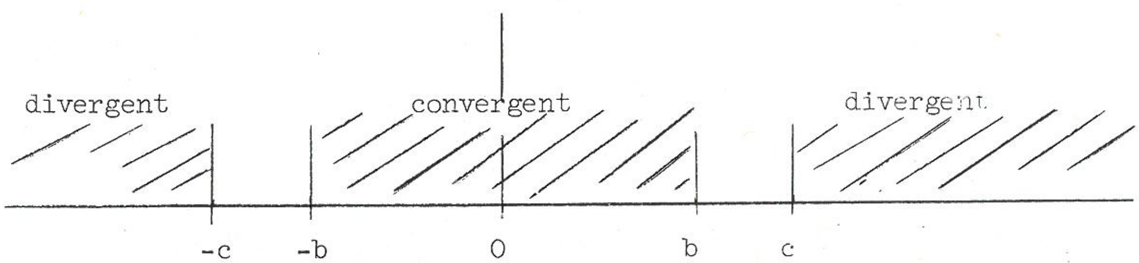

Figure 14.1

Summing up so far, we have proved that if the series converges at any point it also converges at every point nearer to the origin and that if it diverges at any point it also diverges at every point further from the origin. Moreover there is a point (b, say) not at the origin at which the series converges and a point (c, say) at which the series diverges and, necessarily, $|b| < |c|$.

It is now possible to complete the argument perfectly rigorously, using the Axiom of Comtinuity, to show that the series $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence.* We will not give the formal details; but, intuitively, all we do is to 'stretch' the interval $-b < x < b$ uniformly about 0 so that it always continues to consist only of points at which the series converges. This process must stop by the time we reach the interval $-c < x < c$, since any larger interval contains a point (namely c) at which the series diverges. Wherever we have to stop stretching, say when the interval has been expanded to $-R < x < R$, we have reached the interval of convergence.

Theorem 14.3. If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ then the power series $\sum_{n=1}^{\infty} a_n x^n$ converges for all values of x. If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$ then the radius of convergence is $\frac{1}{L}$.

To prove this theorem we apply the ratio test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = |x| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| .$$ If

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0$ then $\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = 0 < 1$ for all x. If

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \neq 0$ then $\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| L$. This quantity is less than 1 if $|x| < \frac{1}{L}$ and greater than 1 if $|x| > \frac{1}{L}$. Hence $L = \frac{1}{R}$.

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*See, for instance, A.E. Taylor, Advanced Calculus, Ginn and Co. New York, 1955, p. 606.

Problems

14.1 Use the ratio test to verify the radius of convergence of the power series (7.7) to (7.12).

14.2 Use the ratio test to investigate the convergence or divergence of the series $\sum_{n=1}^{\infty} a_n$, where a_n has the given value.

(a) $\frac{10^n}{n!}$. (b) $\frac{n^3}{2^n}$.

(c) $\frac{n^{10}}{10!}$. (d) $\frac{(3n)!}{(n!)^2}$.

(e) $\frac{\sin(n)}{n^3}$. (f) $\frac{\log n}{3^n}$.

(g) $\frac{n!}{n^n}$.

Answers. C, C, *, D, *, C, C. *ratio test gives no information.

14.3 Where possible in Problem 14.2 find a bound for the error after n terms.

14.4 Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{n^2 x^n}{2^n}$. Answer. 2.

14.5 Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{e^n}{n}(x-3)^n$ and determine whether the series converges at the end points.

Answer. $|x-3| < \frac{1}{e}$; (C,D).

15. Integral Test.

This test has already been illustrated for the harmonic series in the story of the Generous Donor, (Section 3). More generally suppose the series $\sum_{n=1}^{\infty} a_n$ consists of positive terms of the form $a_n = f(n)$ where $f(x) \geqq 0$ and $f(x)$ is a continuous monotonic decreasing function for $x \geqq 1$; i.e. $f(x) \leqq f(y)$ whenever $x \geqq y$. Then, as indicated in Figure 15.1 we have the inequalities

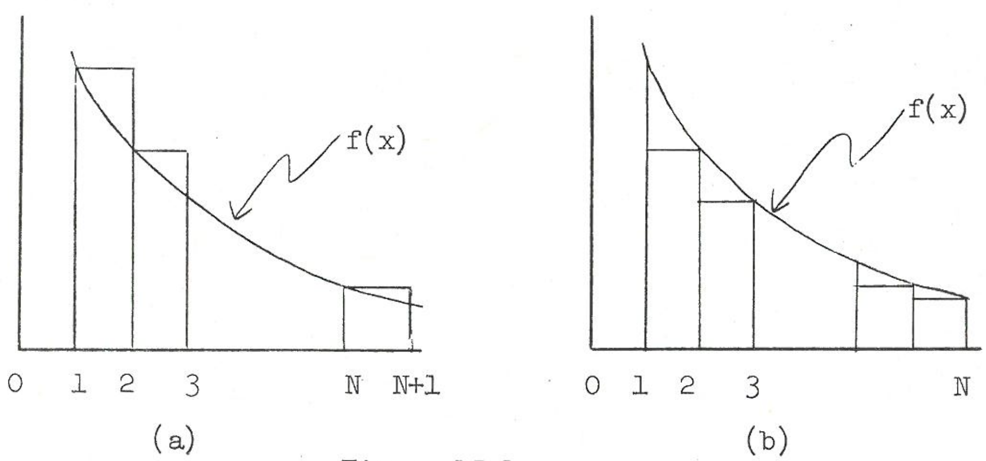

Figure 15.1

$$\int_1^{N+1} f(x)dx \leqq \sum_{n=1}^{N} a_n \leqq a_1 + \int_1^N f(x)dx \ . \tag{15.1}$$

From this it follows that the series $\sum_{n=1}^{\infty} a_n$ converges or diverges according as $\int_1^{\infty} f(x)dx$ converges or diverges. For testing whether the series converges the lower limit 1 in the integral can be replaced, if desired, by any positive constant.

If the series converges, an estimate of the error after N terms is given by

$$(15.2) \qquad \int_{N+1}^{\infty} f(x)\,dx \leqq \left| \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \right| = \sum_{N+1}^{\infty} a_n \leqq \int_{N}^{\infty} f(x)\,dx,$$

as may be seen from a similar sketch.

An important application of this result is to the so-called "p-series."

Corollary 15.1. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leqq 1$.

This follows from the fact that for $p > 1$,

$$\int_{1}^{\infty} \frac{dx}{x^p} = \frac{-1}{(p-1)x^{p-1}} \Big]_1^{\infty} = \frac{1}{p-1} .$$

For $p = 1$ the p-series is the divergent harmonic series and for $p < 1$ the integral

$$\int_{1}^{\infty} \frac{dx}{x^p} = \frac{x^{1-p}}{1-p} \Big]_1^{\infty}$$

diverges. (The case $p < 1$ can also be settled by comparison with the harmonic series.) The special case $p = 2$ was shown in the discussion following Theorem 14.1.

Example 15.1. To illustrate the error estimate, take the case $p = 2$.

$$(15.3) \qquad \int_{N+1}^{\infty} \frac{dx}{x^2} \leqq \left| \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{N} \frac{1}{n^2} \right| = \epsilon_N \leqq \int_{N}^{\infty} \frac{dx}{x^2} .$$

Therefore

$$\frac{1}{N+1} \leq \epsilon_N \leq \frac{1}{N} .$$

From this it can be seen that the series converges fairly slowly. There are various devices to sharpen this estimate, but we shall not be going into them here. (See Thomas, p.780.)

Problems

15.1 Give an example of (a) a convergent series, (b) a divergent series for which the ratio test fails to give a decision but the integral test succeeds.

15.2 Give an example of (a) a convergent and (b) a divergent series where the integral test is not applicable but the ratio test succeeds.

15.3 Show by the integral test that the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges. Find a lower and an upper bound for $\sum_{n=2}^{100} \frac{1}{n \log n}$, i.e. numbers L and U such that $L \leq \sum_{n=2}^{100} \frac{1}{n \log n} \leq U$. [Note that $\int \frac{dx}{x \log x} = \log \log x + C$.]

Answer. L = 1.896, U = 2.616.

15.4 Show that $.78 < \sum_{n=1}^{\infty} \frac{1}{1+n^2} < 1.3$ by using equation (15.1).

15.5 Use (15.1) to estimate the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and compare the bound on the error with that found in Problem 13.4 by a comparison test.

15.6 To sharpen the estimate in equation (15.3) the following method is sometimes useful. Suppose, for example, that $N = 10$, so that equation (15.3) gives the estimate

$$\frac{1}{11} \leqq \left| \sum_{n=1}^{\infty} \frac{1}{n^2} - \left(1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{10^2}\right) \right| \leqq \frac{1}{10} .$$

To obtain a slightly sharper estimate consider the series $\frac{1}{11^2} + \frac{1}{12^2} + \cdots + \frac{1}{(n+10)^2} + \cdots$, and apply equation (15.1) (with $N = \infty$) to the function $\frac{1}{(x+10)^2}$ to obtain $\frac{1}{11} \leqq |R_{10}| \leqq \frac{1}{11} + \frac{1}{121}$. This reduces the length of the uncertainty interval" from $\frac{1}{110}$ to $\frac{1}{121}$. Notice how much better both estimates are than the direct estimate

$$\epsilon_N = \frac{1}{(N+1)^2} + \frac{1}{(N+2)^2} + \cdots \leqq \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N} ,$$

which in this case would give $\epsilon_{10} = \frac{1}{10}$, which is too conservative by a factor of $\frac{1}{N+1}$ in the present case.

15.7 Determine whether or not the series $\frac{2}{3} + \frac{3}{8} + \frac{4}{15} + \cdots + \frac{n+1}{n(n+2)} + \cdots$ converges. Find an estimate for the sum of 100 terms.

16. Alternating Series with Decreasing Terms.

The comparison test, ratio test and integral test establish <u>absolute</u> convergence. They are essentially concerned with series whose terms are positive. Now the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, whereas, as we shall see below, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges. Thus there is need for a test which detects those series which are convergent but not absolutely convergent. We now consider one such test which can be used for series whose terms are alternately positive and negative.

<u>Definition 16.1</u>. A series whose terms alternate in sign is called an <u>alternating series</u>. That is, an alternating series is of the form $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ where each $a_n > 0$.

A sequence whose terms do not increase is called a <u>monotonic decreasing sequence</u>. That is $\{a_n\}$ is monotone decreasing if for each $n \geqq 1$, $a_{n+1} \leqq a_n$.

<u>Theorem 16.1</u>. If the sequence $\{a_n\}$ is monotonic decreasing and converges to zero then the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges; and the remainder R_N after N terms lies between 0 and $(-1)^N a_{N+1}$, i.e. between 0 and the first term omitted.

<u>Proof</u>. The sequence of partial sums of an even number of terms is given by $S_2 = a_1 - a_2$,

$$S_4 = a_1 - (a_2 - a_3) - a_4,$$

$$S_6 = a_1 - (a_2 - a_3) - (a_4 - a_5) - a_6,$$

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2n-2} - a_{2n-1}) - a_{2n}.$$

Since all of the terms in parentheses are non-negative we see that the sequence $\{S_2, S_4, S_6, \ldots\}$ is bounded from above by a_1: it is monotonic increasing sequence, since $S_{2n+2} - S_{2n} = (a_{2n+1} - a_{2n+2}) \geqq 0$. Hence by the axiom of continuity, it converges; say $S_{2n} \longrightarrow S$. Since the sequence $S_1, S_3, S_5, \ldots$ is given by $S_{2n+1} = S_{2n} + a_{2n+1}$ and $a_{2n+1} \longrightarrow 0$, we have also $S_{2n+1} \longrightarrow S$, so that $S_n \longrightarrow S$. Hence the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

For the remainder after N terms we have

$$R_N = \sum_{n=1}^{\infty} (-1)^{n-1} a_n - \sum_{n=1}^{N} (-1)^{n-1} a_n = \sum_{n=N+1}^{\infty} (-1)^{n-1} a_n.$$

An argument similar to the one given above shows that R_N lies between 0 and $(-1)^N a_{N+1}$.

The reader will find it instructive to follow through the proof of the theorem while interpreting the partial sums as points on the real number line. (See Figure 16.1.) He will notice how the partial sums $S_1, S_3, S_5, \ldots$ all exceed S and decrease to S, while the partial sums $S_2, S_4, S_6, \ldots$ all fall short of S and increase to S.

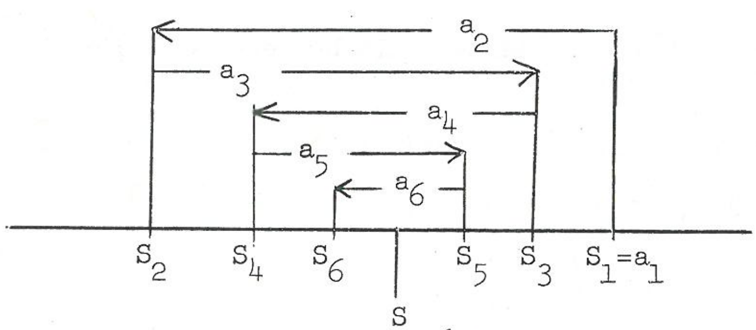

Figure 16.1

Example 16.1. The harmonic series with alternating signs

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots$$

converges. The error after two terms, for example, is between 0 and $+\frac{1}{3}$. The error after three terms is between 0 and $-\frac{1}{4}$. [Compare the Note following equation (8.5).]

Example 16.2. In Example 13.1 we mentioned that a sharper estimate of the error could be made. By virtue of our present knowledge we can say that R_3 is between 0 and $-\frac{1}{4000}$.

Problems

16.1 Let $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$.

(a) Show that S lies between 197/216 and 1549/1728.

(b) Write an appropriate program and use it to compute S to 6 decimal places.

16.2 Compare the error estimate obtained from Theorem 16.1 for the remainder after N terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ with the estimate $\epsilon_N \leqq \sum_{n=N+1}^{\infty} \frac{1}{n^3} \leqq \int_N^{\infty} \frac{dx}{x^3} = \frac{1}{2N^2}$. How many terms would be required to obtain 6 decimal place accuracy using each estimate?

Answer. $\frac{1}{(N+1)^3} < \frac{1}{2N^2}$; 125; 1000.

16.3 Investigate the convergence of the given power series at the end points of the interval of convergence.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

(b) $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

(c) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

(d) $\sum_{n=1}^{\infty} nx^n$.

(e) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$.

(f) $\sum_{n=1}^{\infty} \frac{(-1)^n(x+2)^n}{\sqrt{n+1}}$.

Answers. (a) (C,D); (b) (C,D); (c) (C,C); (d) (D,D); (e) (C,D); (f) (D,C).

17. Uniform Convergence.

In treating the convergence of power series we are concerned with the limits of sequences $\{S_0(x), S_1(x), \ldots, S_n(x), \ldots\}$ where $S_n(x)$ is the n-th partial sum $S_n(x) = \sum_{m=1}^{n} a_m x^m$. More generally, we can consider the sequence $\{f_0(x), f_1(x), \ldots, f_n(x), \ldots\}$ where each term is a function of x. For each fixed value of x this sequence either converges or it does not. For those values of x for which the sequence converges we can define a function $f(x)$ to be the limit of the sequence:

$$f(x) = \lim_{n \to \infty} f_n(x) .$$

In considering the convergence of the sequence $\{f_n(x)\}$ we first fix a value x. Then $\epsilon > 0$ is given, and we must find an N such that

$$(17.1) \qquad | f_n(x) - f(x) | < \epsilon$$

for all $n \geqq N$. Note that the value of N may depend on the value of x selected as well as on ϵ. The idea of uniform convergence is that N can be chosen, dependent on ϵ, but independent of x for all x in some set S. The set S will always in practice be some interval $a \leqq x \leqq b$ or $a < x < b$. More precisely:

<u>Definition 17.1.</u> The sequence of functions $\{f_n(x)\}$ <u>converges uniformly</u> to $f(x)$ for x in the interval $a \leqq x \leqq b$ if, for each $\epsilon > 0$ there exists an N such that for all $n \geqq N$

$$| f_n(x) - f(x) | < \epsilon$$

for all x in the interval $a \leqq x \leqq b$.

A geometric interpretation of uniform convergence is the following. Suppose S is the interval $a \leqq x \leqq b$. (See Figure 17.1, in which we also sketch the sequence of functions $f_n(x)$, and the limit function $f(x)$.) To say that the sequence $\{f_n(x)\}$ converges uniformly for x in the interval $a \leqq x \leqq b$ is to assert that for any $\epsilon > 0$, there is an N large enough such that, for all $n \geqq N$, all of the curves $f_n(x)$ are contained in a strip of 2ϵ centered around $f(x)$ and running from $x = a$ to $x = b$. That is, for all n sufficiently large, all of the curves $f_n(x)$ are contained in the strip.

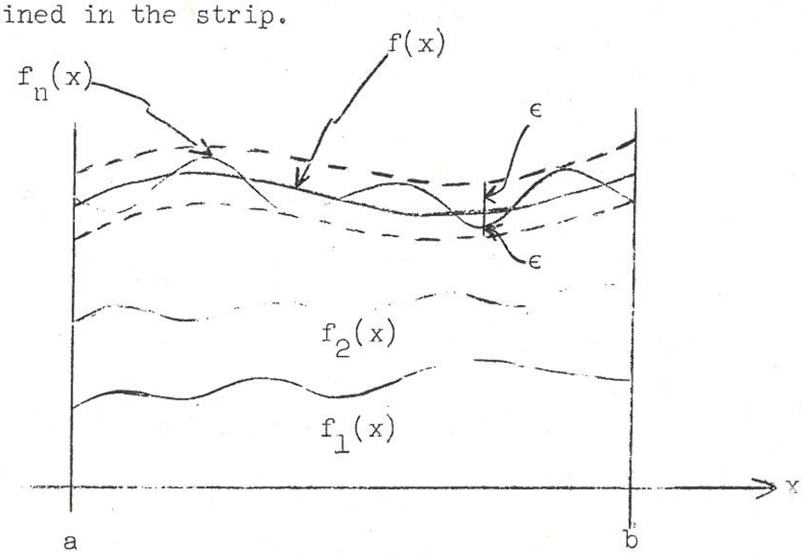

Figure 17.1

As an example of a non-uniformly convergent sequence of functions consider the sequence $f_n(x)$, $n = 1, 2, 3, \ldots$, where

$$
\begin{aligned}
f_n(x) &= nx && \text{for } 0 \leqq x \leqq \frac{1}{n}, \\
&= n\left(\frac{2}{n} - x\right) && \text{for } \frac{1}{n} \leqq x \leqq \frac{2}{n}, \\
&= 0 \quad \text{otherwise,}
\end{aligned}
$$

as illustrated in Figure 17.2.

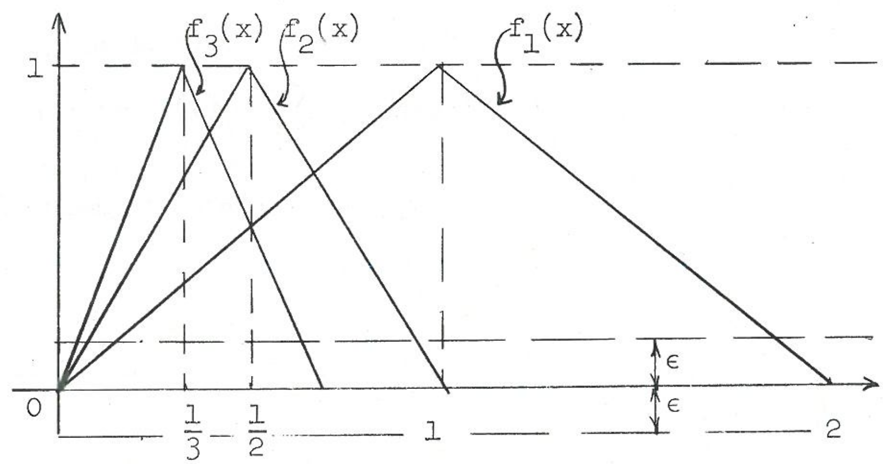

Figure 17.2

Here we have

$$\lim_{n \to \infty} f_n(x) = 0$$

for all x, but for any specified n one can find an x such that $f_n(x) = 1 > \epsilon$. The sequence does not converge uniformly on any interval containing the origin.

If $S_n(x)$ is the n-th partial sum of an infinite series $\sum_{m=1}^{\infty} u_m(x)$, i.e. $S_n(x) = \sum_{m=1}^{n} u_m(x)$, we say the series converges uniformly in a set

S whenever the sequence $\{S_n(x)\}$ converges uniformly in S. The importance of the concept of uniform convergence stems from the following three theorems:

Theorem 17.1. Let I be an interval $|x - a| < r$ on which all of the functions $u_n(x)$ are continuous. If the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly for x in I then the sum $f(x) = \sum_{n=1}^{\infty} u_n(x)$ is a continuous function for all x in I.

Theorem 17.2. Let I be an interval $|x - a| < r$ on which all of the functions $u(x)$ are continuous. If the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly for x in I then the series may be integrated termwise in I; i.e.

$$\int_a^x f(t)dt = \sum_{n=1}^{\infty} \int_a^x u_n(t)dt, \text{ for } |x - a| < r.$$

Theorem 17.3. Let I be an interval $|x - a| < r$ on which each of the functions $u_n(x)$ has a continuous derivative. If the differentiated series $\sum_{n=1}^{\infty} u'_n(x)$ converges uniformly in I then the series $\sum_{n=1}^{\infty} u_n(x)$ converges in I to a differentiable function $f(x)$, and

$$f'(x) = \sum_{n=1}^{\infty} u'_n(x) \text{ for } |x - a| < r.$$

The following test for uniform convergence is called the Weierstrass M-test.

Theorem 17.4. If there exists a convervent series of constants $\sum_{n=1}^{\infty} M_n$

such that, for all x in an interval I,

$$|u_n(x)| \leqq M_n ,$$

then the series $\sum_{n=1}^{\infty} u_n(x)$ converges absolutely and uniformly for x in I.

We shall not prove these theorems here. (Reference, A.E. Taylor, Advanced Calculus and R.C. Buck, Advanced Calculus.)

Problems

17.1 Show that the geometric series $1 + x + x^2 + \ldots + x^n + \ldots$ converges uniformly for x in the interval $|x| \leqq \frac{1}{2}$.

17.2 Show by use of the Weierstrass M-test that if R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and $0 < \rho < R$ then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely and uniformly for $|x| \leqq \rho$. Show also that the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges uniformly for $|x| \leqq \rho$. Apply Theorems 17.2 and 17.3 to show that a power series can be integrated or differentiated termwise inside its interval of convergence as in equations (4') and (5') of Section 5.

17.3 Show that the series $(1-x)x + (1-x)x^2 + \ldots + (1-x)x^n + \ldots$ has for its sequence of partial sums $S_n(x) = x - x^n$. Hence show that the series $\sum_{n=1}^{\infty} (x-1)x^n$ converges for all x such that $-1 < x \leqq 1$. Let $f(x) =$

$\sum_{n=1}^{\infty} (x-1)x^n$ for $-1 < x \leqq 1$. Graph $f(x)$. Show that the series does not converge uniformly in the interval $0 \leqq x \leqq 1$. Graph $S_1(x)$, $S_2(x)$, $S_3(x)$ and compare the behavior with the strip $f(x) \pm \epsilon$ described in the text.