CHAPTER 4

Complex Numbers

1. Introduction. The complex numbers are ordinarily introduced into mathematics as a means of solving certain equations that have no real roots; for example, the equation $x^2 + 1 = 0$ or, more generally, the equation $ax^2 + bx + c = 0$ where $b^2 < 4ac$. In many applications of complex numbers, such as the analysis of oscillatory phenomena, certain geometrical properties are more significant than the the fact that they are solutions of algebraic equations. With such applications in mind we will emphasize the geometrical features of complex numbers.

The relationship between complex numbers and real numbers is much like that between fractions (rational numbers) and integers. A fraction is specified by a pair of integers, written in the form $\frac{m}{n}$ or m/n, and the basic arithmetic operations on fractions are defined in terms of the specifying integers, thus:

$$(m/n) + (p/q) = (mq + np)/(nq), \quad (m/n)(p/q) = (mp)/(nq).$$

Similarly, a complex number is specified by a pair of real numbers, generally written $a + bi$ or some obvious modification of this, and the arithmetic operations are defined by

$$(a + bi) + (c + di) = (a + c) + (b + d)i, \quad (a + bi)(c + di) = (ac - bd) + (ad + bc)i .$$

As far as these definitions are concerned, the $+$ and the i in $a + bi$ have no more significance than the $/$ in m/n. However, we shall soon see that + and i can be interpreted in a way to give a convenient mechanism for the algebraic manipulation of complex numbers; and this, of course, is the reason we use them.

In one important respect the above analogy breaks down. Different pairs of integers can determine the same fraction, for example $1/2 = 2/4 = 10/20$, but a given complex number uniquely determines the two real numbers that specify it.

2. Basic Properties of Complex Numbers.

Definition 2.1. A complex number is an expression of the form $a + bi$, where a and b are real numbers and i is an arbitrary but agreed-upon symbol.*
Equality, addition, and multiplication of complex numbers are defined as follows:

(1) $a + bi = c + di$ if and only if $a = c$ and $b = d$;

(2) $(a + bi) + (c + di) = (a + c) + (b + d)i$;

(3) $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

From these definitions the following basic properties of complex numbers, u, v, w, are easily derived.

(4) Closure: $u + v$ and uv are complex numbers.

(5) Commutative laws: $u + v = v + u$, $uv = vu$.

(6) Associative laws: $(u + v) + w = u + (v + w)$, $(uv)w = u(vw)$.

(7) Distributive law: $u(v + w) = uv + uw$.

(8) Zero element, $0 + 0i$, such that $(0 + 0i) + u = u$.

(9) Negative: if $u = a + bi$ then for $(-u) = (-a) + (-b)i$ we have $u + (-u) = 0 + 0i$.

(10) Unity element, $1 + 0i$, such that $(1 + 0i)u = u$.

(11) Inverse: if $u = a + bi \neq 0$ then for $u^{-1} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i$ we have $uu^{-1} = 1 + 0i$.

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*Electrical engineers, who often use i for an electric current, usually prefer to write $a + bj$ for a complex number. Mathematicians define a complex number simply as an ordered pair of real numbers (a,b) but quickly introduce the $a + bi$ notation (cf. Thomas, Section 17-1).

An algebraic system with properties (4) to (11) is called a field. Other examples of fields are the real numbers and the rational numbers. Since the complex numbers form a field, the complex field, all the techniques of the algebra of real numbers that are based on the "rational operations" of addition, subtraction, multiplication, and division can be applied to complex numbers. In particular we can define $u - v$ as $u + (-v)$ and u/v as uv^{-1}.

Consider the set of complex numbers of the form $x + 0i$. There is an obvious one-to-one correspondence between these complex numbers and the real numbers, namely

$$x + 0i \longleftrightarrow x. \tag{2.1}$$

This correspondence is preserved under the basic operations of addition and multiplication, for if we also have

$$y + 0i \longleftrightarrow y$$

then from (2) and (3) we find

$$(x + 0i) + (y + 0i) = (x + y) + 0i \longleftrightarrow x + y$$

and

$$(x + 0i)(y + 0i) = (xy) + 0i \longleftrightarrow xy.$$

Hence the algebraic properties of the real numbers are exactly the same as those of these special complex numbers.* There is therefore no rea-

*Just as in Section 8 of Chapter 2, where an isomorphism between two vector spaces was defined as a one-to-one correspondence preserved under the basic operations of vectors, so (2.1) defines an isomorphism between the set of real numbers $\{x\}$ and the set of complex numbers $\{x + 0i\}$.

son for distinguishing between them and we can regard the real number x as simply a shorthand form of the complex number $x + 0i$. In particular, the zero and the unity elements of Properties (8) and (10) are the familiar 0 and 1.

For additional shorthand we write bi for $0 + bi$, and i for $1i$. Then $a + bi$ may be considered to be the indicated algebraic combination of a, b, and i, thus justifying our original notation for a complex number. Any rational operation with complex numbers can be carried out by the ordinary processes of algebra, with the additional simplifying identity $i^2 = -1$.

<u>Example 2.1.</u> (a) Instead of trying to memorize Definition 2.1(3) we simply use

$$(a + bi)(c + di) = ac + adi + bci + bdi^2$$
$$= (ac - bd) + (ad + bc)i.$$

(b) $$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bci - adi - bdi^2}{c^2 - d^2i^2}$$
$$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2},$$

provided, of course, that c and d are not both 0.

By Definition 2.1(1) the relation

$$(2.2) \qquad a + bi \longleftrightarrow (a, b)$$

is a one-to-one correspondence between the set of complex numbers and the vector space V_2. Definition 2.1(2) tells us that this correspondence is preserved under addition; and since, for any real number c, we have

$$c(a + bi) = (ca) + (cb)i,$$

it is also preserved under multiplication by real numbers (scalars). The correspondence is therefore an isomorphism, and so the complex numbers can be regarded as elements of a 2-dimensional vector space, and all the vector space concepts and techniques can be applied to them. In particular, the complex numbers 1 and i can be taken as a basis $\{\vec{e}_1, \vec{e}_2\}$ of this vector space. We then have $x + yi = x\vec{e}_1 + y\vec{e}_2$, and so x and y are the components of the vector $x + iy$ with respect to this basis. We can represent the vector space geometrically, as in Figure 2.1, and in the next Section we shall exploit this aspect of complex numbers.

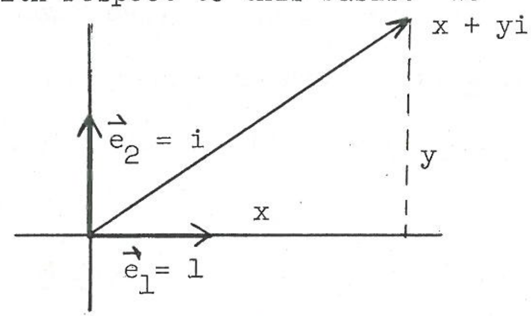

Figure 2.1

The complex field can now be regarded as a 2-dimensional vector space in which a multiplication has been defined that satisfies the field properties (4) to (11). It is natural to wonder if a similar multiplication can be introduced into an n-dimensional vector space with $n \neq 2$. For $n = 1$ this is easy, for the real numbers, with their ordinary multiplication, form a 1-dimensional vector field. On the other hand, it has been proved that no n-dimensional vector field exists for $n > 2$. For example, for $n = 3$, the dot and the cross products that are used in mechanics do not meet the requirements. The dot product gives a scalar, not a vector, and so does not satisfy (4). The cross product is not commutative and there is no unity element. The closest one can get to a field is in the case $n = 4$; there is a system known as "quaternions." which satisfies all the field properties except the commutative law of multiplication. Quaternions were very popular at one time, but their use has declined in recent years and we shall say no more about them.

Some special terminology used in connection with complex numbers must be noted. In the complex number $z = x + yi$, where x and y are real, x is called the <u>real part</u> of z and y the <u>imaginary part</u> of z. We write $x = \text{Re}(z)$ or $R(z)$, and $y = \text{Im}(z)$ or $I(z)$. The length $\sqrt{x^2 + y^2}$ of the vector z (see Figure 2.1) is called the <u>absolute value</u> or <u>modulus</u> of z and written $|z|$. The <u>conjugate</u> of z is $\bar{z} = x - iy$.

<u>Example 2.2</u>. $R(2 - 3i) = 2$, $I(2 - 3i) = -3$, $|2 - 3i| = \sqrt{13}$, $\overline{2 - 3i} = 2 + 3i$.

<u>Problems</u>

2.1 Reduce each of the following to the form $a + bi$.

(a) $(2 - 3i)(1 + 2i) - (3 + 2i)i$.

(b) $(3 + 4i)(3 - 4i)$.

(c) $(1 + i)(2 + i)(3 + i)(4 + i)$.

(d) $\sum_{n=0}^{3} (1 + ni)^2$.

(e) $\sum_{n=1}^{3} (1 + ni)^n$.

(f) $\frac{3 - 2i}{3 + 4i}$.

(g) $\frac{3 - 2i}{3 + 4i} + \frac{3 + i}{3 - i}$.

(h) $\frac{2 - i}{3 + i} \cdot \frac{1 + 2i}{1 + 3i} + (7 + 5i)^2$.

(i) $3z^2 + 7z - (3 + i)$ when $z = 1 - 2i$.

2.2 Use the binomial theorem to express $(1 - i)^4$ in the form $a + bi$.

2.3 Show that $\frac{1}{2} + \frac{\sqrt{3}}{2} i$ is a root of $x^2 - x + 1 = 0$.

2.4 Show that $1 - 2i$ is a root of $x^2 - 2x + 5 = 0$ and of $x^2 + x + 2(2 + i) = 0$

2.5 Solve the following equations for real numbers x and y:

$a(x + iy)^2 + b(x + iy) + c = 0$, (a) for a, b, c real, (b) for a, b, c complex.

(c) $\frac{1 + i}{1 - i} + (2 + 5i)^2(x - 2iy) = 2x - iy$.

(d) $\frac{1 - i}{x + 2iy} = (\frac{5 + i}{5 - i})^2 + 2i$.

2.6 (a) Show that the matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

where a and b are real numbers, are, under the correspondence

$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \longleftrightarrow a + ib$, isomorphic to the complex numbers. (Cf. Problem 12.7, Chapter 2.)

(b) Compute the inverse matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1}$$

and show that it corresponds to $(a + bi)^{-1}$ as defined in Property (11).

2.7 Prove the associative law of multiplication by showing that

$[(a + bi)(c + di)](e + fi) = (a + bi)[(c + di)(e + fi)]$.

2.8 For each of the given values of z determine $R(z)$, $I(z)$, $\bar{z}$, $|z|$:

$z = 4 + 3i,\ 1 - i,\ -3,\ 2i,\ 0$.

2.9 Show that $\overline{u + v} = \bar{u} + \bar{v}$; that $|u| = |-u| = |\bar{u}|$.

2.10 (a) Show that $u\bar{u} = |u|^2$.

(b) Show that $\frac{u}{v} = \frac{u\bar{v}}{|v|^2}$, by using part (a).

(c) Reduce to the form $a + ib$ by using part (b) $\frac{2 + i}{1 - i}$; $\frac{i}{2 + i} + \frac{3}{1 - 2i}$.

Answer. $\frac{1}{2} + \frac{3i}{2}$; $\frac{4}{5} + \frac{8i}{5}$.

2.11 Complete the lower line of the table

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
i^n					1	i	-1						

Formulate a general rule for determining the value of i^n if n is an integer. What is i^{531}; i^{-3243}?

2.12 Show that $|a + b| \leq |a| + |b|$ for any complex numbers a and b.

[Hint: Interpret geometrically.]

2.13 For $z = 2 + 3i$ plot the points $z, \bar{z}, -z, -\bar{z}$. For any z describe the geometrical relation between z and $\bar{z}$; between z and $-z$.

2.14 For what values of z is $|z| = z$; $\bar{z} = z$?

2.15 Prove that (a) $z + \bar{z} = 2\mathrm{Re}(z)$.

(b) $z - \bar{z} = 2i\mathrm{Im}(z)$.

(c) $\left|\frac{z}{\bar{z}}\right| = 1$.

(d) $\overline{(uv)} = \bar{u}\,\bar{v}$.

(e) $\left|\frac{1}{z}\right| = \frac{1}{|z|} \qquad (z \neq 0)$.

(f) $|uv| = |u|\,|v|$.

(g) $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}} \qquad (z \neq 0)$.

(h) if $z = \bar{z}$ then z is real.

(i) $|\mathrm{Re}(z)| \leqq |z|$.

2.16 Solve for z.

$$2z + \bar{z} + \frac{1}{3 + i} = 0.$$

3. <u>The Polar Form</u>. In the complex plane it is often convenient to associate a complex number $z = x + iy$ (x and y being real) with the point (x,y) rather than with the vector from the origin to (x,y). Accordingly, we speak of "the point z." Consider

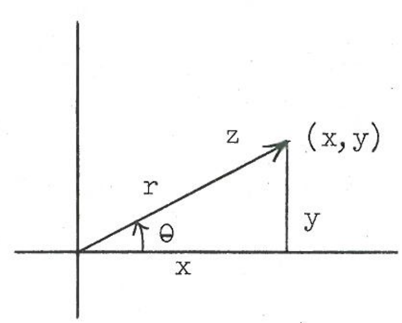

Figure 3.1

the polar coordinates of z. Since (see Figure 3.1)

(3.1) $$x = r \cos \theta, \qquad y = r \sin \theta,$$

we can write z as

(3.2) $$z = r\,(\cos \theta + i \sin \theta) .$$

This is called the polar form of z, and is sometimes abbreviated to $z = r \operatorname{cis} \theta$.

As is well known for polar coordinates, r and θ are not uniquely determined by the point z. We shall exclude negative values of r; then r is uniquely determined by z, and in fact $r = |z|$. Angle θ is called the angle of z or the argument of z and written arg z. It can be taken positive or negative, and is multiple-valued.

Passage from the polar to the rectangular form is effected by equations (3.1). To pass from rectangular to polar form we determine r from the equation:

(3.3) $$r^2 = x^2 + y^2 ,$$

and θ from the equations: $\cos \theta = \frac{x}{r}$, $\sin \theta = \frac{y}{r}$. One is tempted to use

(3.4) $$\theta = \arctan \frac{y}{x} ,$$

but this is not sufficient, since this equation gives extraneous values for θ. If additional information is given-such as $y > 0$, or $|\theta| < \pi/2$, etc. - equation (3.4) can be used.

Example 3.1. Reduce $1 - i$ to polar form. Here $x = 1$, $y = -1$, (see Figure 3.2) and so from (3.3), $r = \sqrt{2}$. Then (3.1) gives

$$\cos\theta = \frac{1}{\sqrt{2}}\,, \qquad \sin\theta = -\frac{1}{\sqrt{2}}\,,$$

so that θ is an angle in the fourth quadrant; $\theta = 315°$, $675°$, ..., $-45°$, $-405°$, ...; or more specifically $\theta = -45° + k \cdot 360°$, where k is any integer. In radian measure, $\theta = -\frac{\pi}{4} + k(2\pi)$.

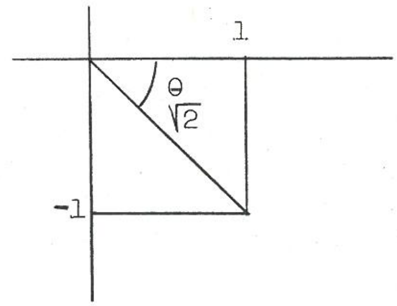

Figure 3.2

Picking one of these values, we have

$$1 - i = \sqrt{2}\,(\cos(-\tfrac{\pi}{4}) + i\sin(-\tfrac{\pi}{4}))\ .$$

Note that, if we had carelessly used (3.4) instead of (3.1) to determine θ, one of the values would be $\theta = \arctan(-1) = 135°$, and $135°$ is not one of the values of $\arg(1 - i)$.

Theorem 3.1. $|zw| = |z|\,|w|$, $\arg zw = \arg z + \arg w$.

Proof. Let

$$z = r(\cos\theta + i\sin\theta), \qquad w = s(\cos\phi + i\sin\phi).$$ (See Figure 3.3.)

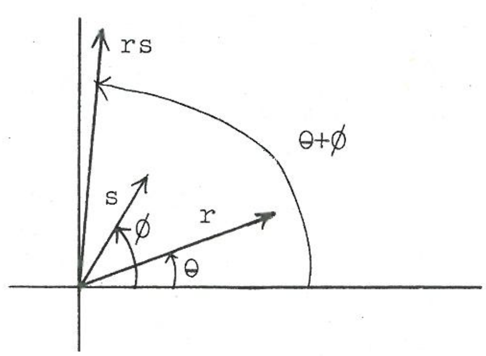

Figure 3.3

Then

$$zw = rs[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)]$$
$$= (rs)[\cos(\theta+\phi) + i\sin(\theta+\phi)] ,$$

which proves the theorem.

Since the argument is a multiple-valued function we must be careful about the meaning of an equation involving arguments. The meaning to be assigned to "arg zw = arg z + arg w" is evidently "Given a value of arg z and a value of arg w, the sum of these two values is one of the values of arg zw."

Corollary 3.1. $|z/w| = |z|/|w|$. Arg z/w = arg z - arg w.

Corollary 3.2. If n is an integer, $|z^n| = |z|^n$, arg z^n = n arg z.

This property of complex numbers is known as DeMoivre's Theorem.

Corollary 3.3. Multiplication by $\cos\theta + i\sin\theta$ rotates a vector through the angle θ, since $|\text{cis }\theta| = 1$.

By virtue of Theorem 3.1 and its corollaries, multiplication and division of complex numbers in polar form is particularly easy. For example,

$$\frac{\sqrt{2}\text{ cis }\frac{\pi}{4}}{4\text{ cis }\frac{5\pi}{6}} = \frac{\sqrt{2}}{4}\text{ cis }(-\frac{7\pi}{12})$$

is an easier process than evaluating the same quotient in the form

$$\frac{1+i}{-2\sqrt{3}+2i} = \frac{(1+i)\ (-2\sqrt{3}-2i)}{12+4} = \frac{1}{8}[(1-\sqrt{3}) - (1+\sqrt{3})i] .$$

Similarly

$$(\sqrt{2}\text{ cis }\frac{\pi}{4})^8 = 16\text{ cis }2\pi = 16$$

is much to be preferred to

$$(1+i)^8 = 1 + 8i + \frac{8\cdot 7}{1\cdot 2} i^2 + \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3} i^3 + \ldots$$
$$= 1 + 8i - 28 - 56i + \ldots$$
$$= 16 .$$

However, one must not infer from this that the best way to multiply or divide two complex numbers in rectangular form is first to put them in polar form. In general the passage from rectangular to polar form, and vice versa, involves the use of tables of sines and cosines, which introduces further computations. (See Problem 3.5.) But often in theoretical work, or if the factors are already in polar form, one can take advantage of this simple way to multiply, divide or raise to powers.

Problems

3.1 (a) Find all values of arg z for each of the given values of z. Express the values in radians, either as multiples of π or to three places of decimals.

$z = i,\ 4,\ -2,\ -5i,\ 1 + i,\ 1 - i,\ \sqrt{3} - i,\ i - \sqrt{3},\ -2 - 3i.$

(b) Reduce each of the above numbers to polar form.

3.2 Is there any relation between arg z and arg $\bar{z}$? State the relation very carefully, remembering that the argument is a multiple-valued function.

3.3 What can you say about arg 0?

3.4 Reduce the complex number $\frac{3 - i}{5} + \frac{4i}{3 + i}$ to polar form. Answer. $\sqrt{2}$ cis $\pi/4$.

3.5 Find $(2 + 3i)(1 - i)$ and $(2 + 3i)/(1 - i)$ by reducing to polar form, using Theorem 3.1 and Corollary 3.1, and returning to rectangular form.

3.6 (a) By Corollary 3.2 squaring a number squares its absolute value and doubles its argument. Show how to work this backward to find the number whose square is i. How many such numbers are there?

(b) Give a general method for finding the square root of a complex number.

(c) Solve the equation $x^2 + 2ix - 2 + i = 0$. Answer. $x = 1.09869 - 1.45509i$ or $-1.09869 - .54491i$.

3.7 Prove Corollary 3.1.

3.8 Describe the curve which is the locus of the points z satisfying each of the following conditions:

(a) $|z| = 2$,

(b) $|z - i| = 2$,

(c) $|z - 1| + |z + 1| = 3$.

(d) $|z| \leqq 4$,

(e) $|z - (1 + i)| \geqq 9$,

(f) $R(z) \geqq 1$,

(g) $-1 < I(z-2) \leqq 3$.

3.9 Verify that

$$|z_2 - z_1|^2 = (z_2 - z_1)(\overline{z_2 - z_1}) = |z_2|^2 - z_1\bar{z}_2 - z_2\bar{z}_1 + |z_1|^2$$

$$= |z_2|^2 + |z_1|^2 - 2R(z_1\bar{z}_2)$$

$$= |z_2|^2 + |z_1|^2 - 2|z_1||z_2| \cos(\theta_1 - \theta_2) .$$

Interpret this as a proof of the law of cosines: $a^2 = b^2 + c^2 - 2bc \cos A$.

(Cf. Figure 3.4.)

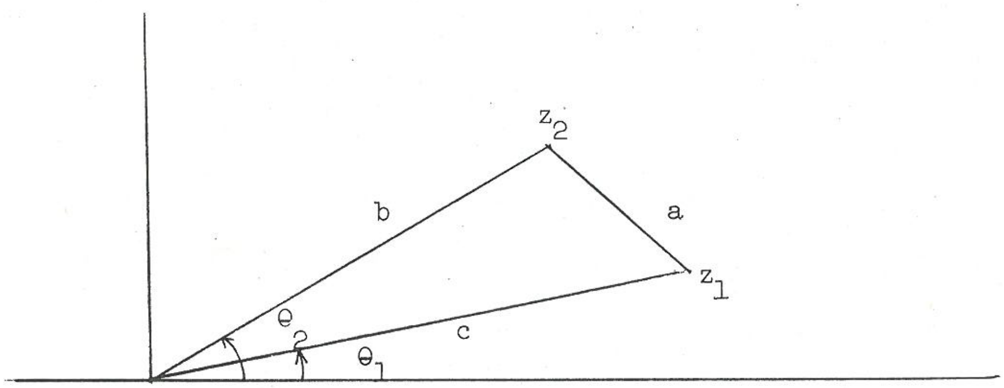

Figure 3.4

3.10 Prove that the sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals by verifying the following algebraic steps and interpreting the result appropriately. (Cf. Figure 3.5.)

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2R(z_1\bar{z}_2) ,$$

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2R(z_1\bar{z}_2) ,$$

and adding,

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

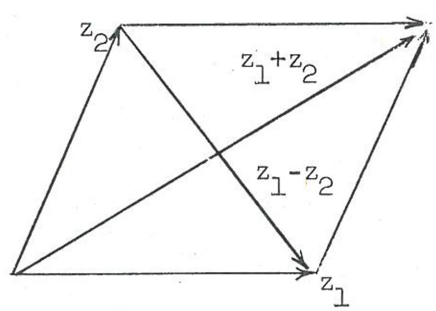

Figure 3.5

4. Complex Algebra. In Section 2 we remarked that the rational operations of elementary algebra can be extended from the real field to the complex field. We mention here a few of the more useful aspects of this process.

A polynomial in one variable, x, is an expression of the form

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n ,$$

the a's being numbers. We can of course allow the a's to be complex numbers. A zero of $P(x)$, or a root of the equation $P(x) = 0$, is a number r such that

$$P(r) = a_0r^n + a_1r^{n-1} + \dots + a_{n-1}r + a_n = 0.$$

The polynomial

$$\bar{a}_0x^n + \bar{a}_1x^{n-1} + \dots + \bar{a}_{n-1}x + \bar{a}_n$$

is called the conjugate of $P(x)$ and is designated by $\bar{P}(x)$.

Theorem 4.1. For any number q,

$$\overline{P(q)} = \bar{P}(\bar{q}) .$$

Proof. We have seen that

$$\overline{a + b} = \bar{a} + \bar{b} \quad \text{and} \quad \overline{ab} = \bar{a}\,\bar{b} .$$

By successive applications of these two properties we can reduce

$$\overline{P(q)} = \overline{a_0q^n + a_1q^{n-1} + \dots + a_{n-1}q + a_n}$$

to

$$\bar{a}_0(\bar{q})^n + \bar{a}_1(\bar{q})^{n-1} + \dots + \bar{a}_{n-1}\bar{q} + \bar{a}_n = \bar{P}(\bar{q}) .$$

Example 4.1. If $P(x) = (2 + i)x^2 + (1 - 2i)x + (3 + 2i)$, then $\overline{P}(x) = (2 - i)x^2 + (1 + 2i)x + (3 - 2i)$. If $x = 1 + i$, then $P(x) = (2 + i)(1 + i)^2 + (1 - 2i)(1 + i) + (3 + 2i) = 4 + 5i$; also $\overline{x} = 1 - i$ and $\overline{P}(\overline{x}) = (2 - i)(1 - i)^2 + (1 + 2i)(1 - i) + (3 - 2i) = 4 - 5i$. That is, $\overline{P(x)} = 4 - 5i = \overline{P}(\overline{x})$.

Corollary 4.1. If r is a root of $P(x) = 0$ then $\overline{r}$ is a root of $\overline{P}(x) = 0$.

Example 4.2. Continuing Example 4.1 we see that $x = 1 + i$ is a root of the equation $(2 + i)x^2 + (1 - 2i)x + (3 + 2i) = 4 + 5i$, i.e. a root of the equation $(2 + i)x^2 + (1 - 2i)x - (1 + 3i) = 0$. Corollary 4.1 asserts that $x = 1 - i$ is a root of $(2 - i)x^2 + (1 + 2i) - (1 - 3i) = 0$. This can also be verified from the evaluation of $\overline{P}(\overline{x})$ computed in Example 4.1.

Corollary 4.2. If $P(x)$ has real coefficients and r is a root of $P(x) = 0$, then $\overline{r}$ is a root of $P(x) = 0$.

Example 4.3. The equation $x^3 - 3x^2 + 4x - 2 = 0$ has a root $x = 1 + i$. By virtue of Corollary 4.2 it follows that it also has a root $x = 1 - i$. This fact is easily verified. Thus $(1 - i)^3 - 3(1 - i)^2 + 4(1 - i) - 2 = - 2(1 + i) + 3(2i) + 4(1 - i) - 2 = 0$.

Corollary 4.2 is a well-known and useful result. Note the condition that the coefficients be real. For example, i is a root of $x - i = 0$ but $-i$ is not.

The so-called Fundamental Theorem of Algebra says that every polynomial equation (with $n \geqq 1$ and $a_0 \neq 0$) has a root. Many different proofs of this theorem have been found in the past 165 years but none of them is simple enough to present here. Nor will we give the proof of the following theorems; these are fairly easy and can be found in a good high school algebra text or a "college algebra."

Theorem 4.2. If r is a root of $P(x) = 0$ then $x - r$ is a factor of $P(x)$; that is, there is a polynomial $Q(x)$ such that $P(x) = (x - r)Q(x)$.

This is known as the Factor Theorem.

Example 4.4. In Example 4.3 we saw that $x = 1 - i$ was a root of $x^3 - 3x^2 + 4x - 2 = 0$. According to Theorem 4.2 it follows that $x - (1 - i)$ is a factor of $x^3 - 3x^2 + 4x - 2$. The student may carry out the required long division to verify that this is so. (See also Example 4.5 below, however.)

Corollary 4.3. Given

$$P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n ,$$

with $a_0 \neq 0$, $n > 1$; there is a set of n numbers $r_1, r_2, \dots, r_n$, not necessarily distinct, such that

$$P(x) = a_0(x - r_1)(x - r_2) \dots (x - r_n) . \tag{4.1}$$

We sometimes express this theorem by saying that an equation of degree n has n roots.

Example 4.5. Continuing Examples 4.3 and 4.4 we know that the equation $x^3 - 3x^2 + 4x - 2 = 0$ has roots $x = 1 + i$, $x = 1 - i$. It also has the root $x = 1$. Consequently, it follows from Corollary 4.3 that $x^3 - 3x^2 + 4x - 2 \equiv (x - (1 + i))(x - (1 - i))(x - 1)$. The student can verify this identity by direct multiplication.

<u>Corollary 4.4</u>. If $P(x)$ has real coefficients and $a + bi$ is a root, where a and b are real and $b \neq 0$, then $(x^2 - 2ax + a^2 + b^2)$ is a factor of $P(x)$.

<u>Example 4.6</u>. Corresponding to a pair of roots $a \pm ib$ we have the pair of factors $[x - (a + ib)][x - (a - ib)] = [(x - a) - ib][(x - a) + ib]$ $= (x - a)^2 + b^2 = x^2 - 2ax + a^2 + b^2$, as asserted. For example from the factorization in Example 4.5 we obtain $(a = b = 1)$ $x^3 - 3x^2 + 4x - 2 = (x^2 - 2x + 2)(x - 1)$, which the student may also verify by direct multiplication.

<u>Corollary 4.5</u>. Any polynomial with real coefficients can be factored into real factors each of which is either linear or quadratic.

The numbers $r_1, r_2, \ldots, r_n$ appearing in (4.1) might not be distinct. In that case the equation $P(x) = 0$ is said to have <u>multiple roots</u>; the number of times a given root appears in the list $r_1, r_2, \ldots, r_n$ is called the <u>algebraic multiplicity</u> of the root. Thus if the distinct roots are $s_1, s_2, \ldots, s_m$, with multiplicities $n_1, n_2, \ldots, n_m$ then the factorization in (4.1) can also be given in the form

$$(4.2) \qquad P(x) = a_0(x - s_1)^{n_1}(x - s_2)^{n_2}\ldots(x - s_m)^{n_m}$$

where $\sum_{k=1}^{m} n_k = n$.

<u>Example 4.7</u>. The equation $4x^3 + 20x^2 + 12x - 36 = 0$ has the roots $-3, -3, 1$. Hence we have the factorization

$$4x^3 + 20x^2 + 12x - 36 = 4(x + 3)^2(x - 1) .$$

In many practical problems it is necessary to determine the roots of a polynomial equation numerically. Newton's method (Thomas, Sec. 9-3) will still work, since the analytic derivation can be extended to the complex case. The algebra involved in computing $f(x)$ and $f'(x)$ when x is complex is somewhat messy, but the flow diagram remains the same.

Newton's Method has the advantage of working even if the given polynomial has complex coefficients. If the coefficients are real there are somewhat simpler methods available. Those due to Bainstow and to Lin are the most common; they can be found in books on Numerical Analysis (see, for instance, F.B. Hildebrand, Introduction to Numerical Analysis, McGraw-Hill Book Co., New York, 1956, pp. 468-476).

Problems

4.1 Show that $(1 + 7i)^2 = -48 + 14i$. Solve the equation $(1 - i)z^2 - (9 - 5i)z + 26 = 0$. [Each solution should be expressed in the form $z = a + ib$, with a, b real.] [Ans. $z_1 = 2 + 3i$, $z_2 = 5 - i$] .

4.2 The roots of

$$x^4 - (4 + i)x^3 + (6 + 2i)x^2 - 6x + 4 = 0$$

are to be found among the numbers

$$-1, 2, -i, 2i, 1 + i, 1 - 2i .$$

Find them.

4.3 Some of the roots of

$$F(x) = x^6 + 2x^3 + x^2 + 2x + 2 = 0$$

are to be found among the numbers

$$-1,\ 2,\ -i,\ 2i,\ 1 + i,\ 1 - 2i\ .$$

Find them and factor $F(x)$ into real linear and quadratic factors.

4.4 (a) Using the initial value $x = .5 + .5i$ carry out one step of Newton's Method for the solution of $x^3 - x + 1 = 0$. Answer. $.691 + .538i$.

(b) Program Newton's Method for this equation and solve to six decimal places with the above initial value. Answer. $.662359 + .562280i$.

5. <u>Complex Analysis</u>. We define the limit of a sequence of complex numbers in exact analogy to the definition of the limit of a sequence of real numbers. That is, the complex number **A** is the limit of the sequence $\{a_1, a_2, \ldots, a_k, \ldots\}$ if, for any given positive (real) number ϵ, we can find an integer N such that

$$|a_k - A| < \epsilon \qquad \text{whenever} \quad k > N\ .$$

Roughly this means that as k gets larger the points a_k are confined to a shrinking circle about the point A (Figure 5.1).

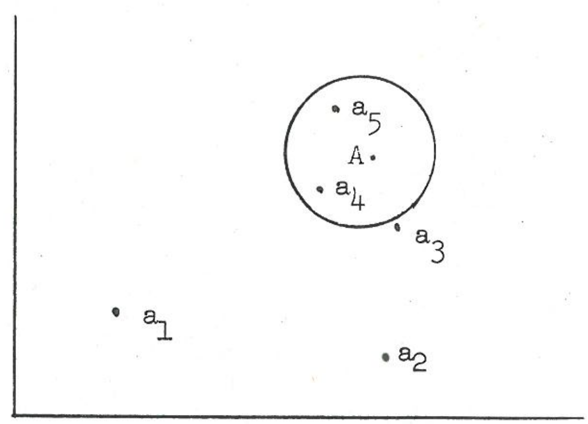

Figure 5.1

<u>Theorem 5.1</u>. The sequence $\{a_k\}$ converges if and only if the two sequences $\{R(a_k)\}$ and $\{I(a_k)\}$ converge. (Cf. Thomas, Sec. 17-6.)

<u>Proof</u>. $|a_k - A|^2 = [R(a_k) - R(A)]^2 + [I(a_k) - I(A)]^2$. From this the theorem follows readily.

Similarly if $w = f(z)$ is a function of the complex variable z, we say

$$\lim_{z \to a} f(z) = L$$

provided that for each given positive number ϵ there is a positive number δ such that

$$|f(z) - L| < \epsilon \quad \text{whenever} \quad |z - a| < \delta .$$

Note that both z and w may be complex. Figure 5.2 illustrates the condition; whenever z is within δ of a, $f(z)$ must be within ϵ of L.

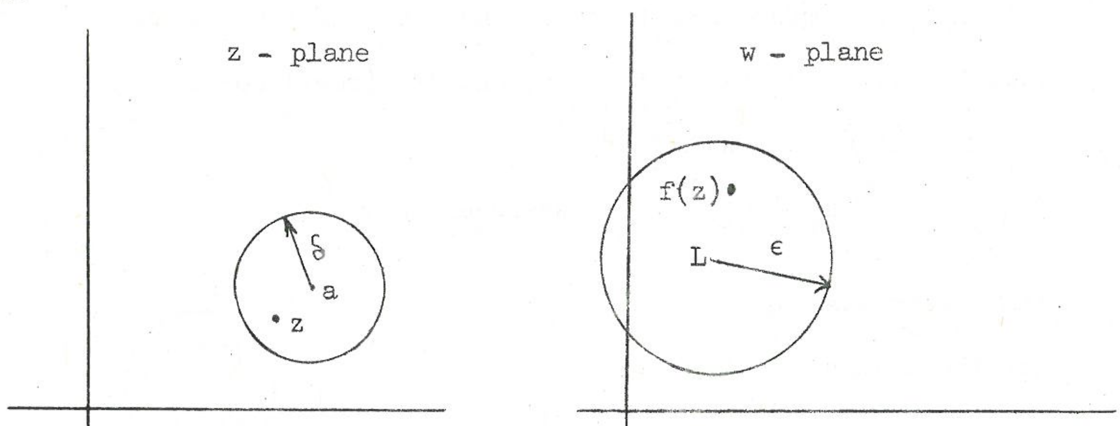

Figure 5.2

The basic theorems regarding sums, products and quotients hold for limits of complex quantities just as for real quantities. Continuity of a function can be defined as follows: $f(z)$ is continuous at $z = a$ if

$$\lim_{z \to a} f(z) = f(a) .$$

Similarly, the derivative, $f'(z)$ or dw/dz, of the function $w = f(z)$ is defined by

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} ,$$

where h is a complex variable. The chain rule and the standard formulas for differentiating polynomials and rational functions can be proved in the same manner as for real functions. From these follow the corresponding properties of indefinite integrals (or "antiderivatives"). It can also be shown that the only function whose derivative is identically zero is a constant, and hence that two indefinite integrals of the same function differ at most by an additive constant.

Definite integrals of real functions, on the other hand, depend strongly on the 1-dimensionality of the real number system, and their theory must be changed considerably when we pass to the complex case. We shall do nothing with them in this course.

Since we can handle sequences we can also handle series; $\sum_{k=1}^{\infty} a_k$ is defined to be

$$\lim_{n \to \infty} \sum_{k=1}^{n} a_k .$$

From Theorem 5.1 we get the corollary,

Corollary 5.1. $\sum_{1}^{\infty} a_k$ converges if and only if two series $\sum_{1}^{\infty} R(a_k)$ and $\sum_{1}^{\infty} I(a_k)$ converge.

A series of complex constants $\sum_{k=1}^{\infty} a_k$ is said to be <u>absolutely convergent</u> if the series of absolute values, $\sum_{k=1}^{\infty} |a_k|$, is convergent. It can also be proved that if a series is absolutely convergent then it is convergent. (See Problem 5.1.)

We can also consider power series $\sum_{n=0}^{\infty} a_n(z-c)^n$. Similarly to the real case there are three possibilities for the convergence properties of such a series. The proofs of the following two theorems are similar to those of the corresponding theorems of Chapter 3. (See also Problems 5.2, 5.3, 5.4.)

<u>Theorem 5.2</u>. If $c, a_0, a_1, a_2, \ldots$ are complex numbers the infinite series

$$a_0 + a_1(z-c) + a_2(z-c)^2 + \ldots$$

either

(i) converges for every value of z;

(ii) converges for every z within a certain circle with center c and diverges for every z outside this circle;

(iii) diverges for every z not equal to c.

In case (ii) the circle is called the "circle of convergence." Case (i) can be included in case (ii) by allowing an "infinite" circle of convergence. Case (iii) is of no interest to us in this course, and we shall assume that we have a circle of convergence whose radius is greater than zero and may be infinite.

The next theorem is essentially a restatement for complex power series of the properties of real power series stated in Section 5 of Chapter 3. The proofs are the same in both cases.

Theorem 5.3. Let

$$f(z) = a_0 + a_1 z + \dots + a_n z^n + \dots \quad \text{for } |z| < R_1,$$
$$g(z) = b_0 + b_1 z + \dots + b_n z^n + \dots \quad \text{for } |z| < R_2.$$

Then

(1) $f(z) + g(z) = (a_0 + b_0) + (a_1 + b_1)z + \dots + (a_n + b_n)z^n + \dots$

for $|z| < \min(R_1, R_2)$;

(2) $c\,f(z) = ca_0 + ca_1 z + \dots + ca_n z^n + \dots \quad \text{for } |z| < R_1$;

(3) $f(z)g(z) = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots + (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0)z_n$

$+ \dots$ for $|z| < \min(R_1, R_2)$;

(4) $f'(z) = a_1 + 2a_2 z + \dots + (n+1)a_{n+1} z^n + \dots \quad \text{for } |z| < R_1$;

(5) $\int f(z)dz = C + a_0 z + \frac{1}{2} a_1 z^2 + \dots + \frac{1}{n} a_{n-1} z^n + \dots \quad \text{for } |z| < R_1$.

The series for the derivative and the integral have the same radius of convergence as the original series.

Corollary 5.2. Throughout the circle of convergence the function $w = f(z)$ has an n-th derivative

$$f^{(n)}(z) = (n!)a_n + \frac{(n+1)!}{1} a_{n+1}(z-c) + \frac{(n+2)!}{2!} a_{n+2}(z-c)^2 + \dots ,$$

obtained by termwise differentiation of the series.

Corollary 5.3. $f^{(n)}(c)/n! = a_n$, and so

$$f(z) = f(c) + f'(c)(z-c) + \frac{1}{2!} f''(c)(z-c)^2 + \dots .$$

In short, the usual properties of real Taylor Series hold for the complex case, with the interval of convergence replaced by the circle of convergence.

Note that we have said nothing about what happens if the point z lies <u>on</u> the circle of convergence. This case can be very complicated, and we shall not pursue it further here.

In the preceding discussion we started with a power series and then examined the function to which it converged. Next, just as in Chapter 3, let us turn things around and start instead with a given function, $f(z)$, and consider the possibility of expanding it in a Taylor series about a given point, c.

We need two definitions. We say that $f(z)$ is <u>regular</u>, or <u>holomorphic</u>, at c if there is some circle with center at c such that the derivative $f'(z)$ exists at every point z within that circle. If $f(z)$ is not regular at the point c then c is called a <u>singular</u> point of $f(z)$.

For example, if $f(z) = \frac{1}{z+1}$ then $f(z)$ is regular at every point except $z = -1$. That is, there is precisely one singular point, $z = -1$.

<u>Theorem 5.4</u>. If $f(z)$ is regular at c, then

(i) All derivatives $f^{(n)}(c)$ exist, $n = 1, 2, 3, \ldots$;

(ii) The Taylor series of $f(z)$ at c converges to $f(z)$ at all points inside the circle of convergence, which has a radius greater than zero (and is possibly infinite) (i.e., case (iii) of Theorem 5.2 cannot occur).

(iii) If $f(z)$ has no singular points the Taylor series converges for all values of z (i.e., case (i) of Theorem 5.2 holds).

(iv) If $f(z)$ has one or more singular points then the radius of convergence of the Taylor series is the distance from c to the nearest singular point. Consequently the circle of convergence passes through at least one singular point of $f(z)$, but has only regular points in its interior.

Theorem 5.4 is one of the most basic parts of the theory of functions of a complex variable. Its proof can be found in any thorough treatment of this theory.

Example 5.1. In Chapter 3 we found that the Maclaurin Series representation of the function $\frac{1}{1+x^2}$,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots ,$$

converged only for $|x| < 1$, although the function $\frac{1}{1+x^2}$ shows no peculiar properties at any real value of x. Now we can see what is happening. The function $\frac{1}{1+z^2}$ (which coincides with $\frac{1}{1+x^2}$ when z is real) is undefined at $z = \pm i$ and is singular at these points. At all other points it is regular. It follows that the circle of convergence is: $|z| = 1$. Hence the series $\sum_{n=0}^{\infty} (-1)^n z^{2n}$ diverges for all $|z| > 1$. Consequently the real series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ diverges for $|x| > 1$, even though the real function $\frac{1}{1+x^2}$ is well behaved in this region. Similarly from Figure 5.3 we can see that the Taylor series of $1/(1+z^2)$ about the point $2 + 2i$, which looks like

$$\frac{1}{1 + (2+2i)^2} - \frac{2(2+2i)}{(1 + (2+2i)^2)^2}(z - (2+2i)) + \dots ,$$

converges for $|z - (2+2i)| < \sqrt{5}$.

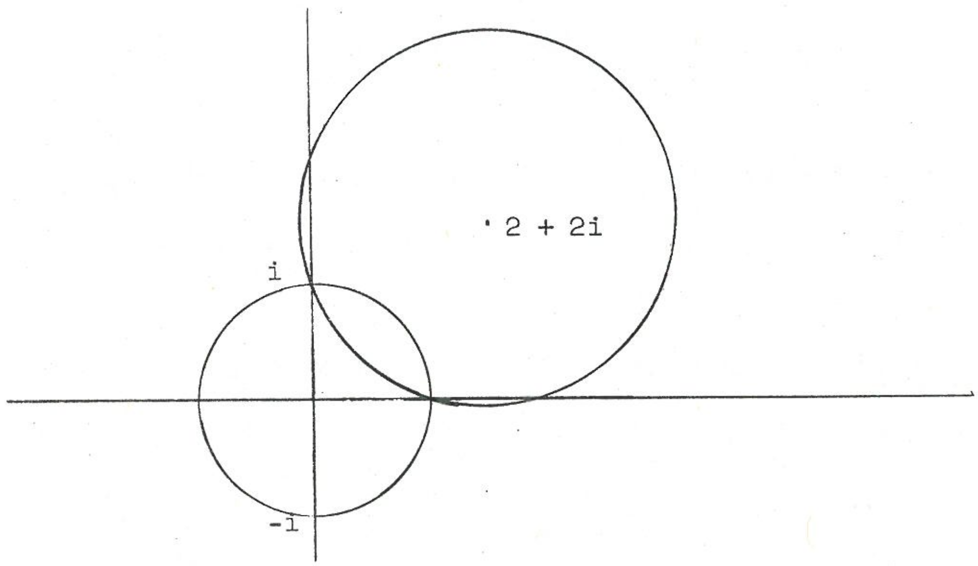

Figure 5.3

Problems

5.1 Theorem. An absolutely convergent series is convergent.

Partial Proof. Let $\sum |a_n|$ converge. Since $|R(a_n)| \leqq |a_n|$, (why?), it follows that $\sum R(a_n)$ converges, (why?). Similarly $\sum I(a_n)$ converges. Hence $\sum a_n$ converges (why?).

Fill in all the gaps in this proof.

5.2 Theorem. If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists and is less than 1, then $\sum a_n$ converges absolutely.

Prove as for Theorem 14.1 of Chapter 3.

5.3 Theorem. If $\sum a_n z^n$ converges for $z = z_1$ then it converges absolutely for $z = z_2$ if $|z_2| < z_1$.

Prove as for Theorem 14.2 of Chapter 3.

5.4 Theorem. If $\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = R$, then $\sum a_n z^n$ converges for $|z| < R$ (including the case $R = \infty$).

Prove as for Theorem 14.3 of Chapter 3.

5.5 In Problem 5.5 of Chapter 3 we showed that the generating function of the Fibonacci numbers, $F(t) = \sum_{n=0}^{\infty} f_n t^n$, was $(1 - t - t^2)^{-1}$. Interpreting t as a complex number, use Theorem 5.4(iv) to find the radius of convergence of $F(t)$. Then use the theorem in Problem 5.4 to show that if $\lim_{n \to \infty} \frac{f_n}{f_{n+1}}$ exists then this limit is $(\sqrt{5} - 1)/2$. Check this by computing the same limit from the formula for f_n derived in the earlier problem.

6. Elementary Functions. Functions of complex variables frequently arise by "extending" functions of a real variable. This is easily done for rational functions;* for example the function

$$y = \frac{x^2 - 1}{x^2 + 1}$$

of the real variable x can be extended to the function

$$w = \frac{z^2 - 1}{z^2 + 1}$$

* Recall that a rational function is the quotient of two polynomials.

of the complex variable z. Similarly any rational function of the real variable x can be extended to the corresponding rational function of the complex variable z. In general, by an extension of a function $y = f(x)$ of a real variable we mean a function $w = F(z)$ such that if z takes any real value x we have $F(x) = f(x)$. If $f(x)$ is a function with interesting properties -- say $f(x)$ is differentiable, or satisfies some interesting identities -- we would like $F(z)$ to have similar properties. For instance, what shall we mean by $\sin z$ and $\cos z$? Whatever definitions we give, the values should reduce to the customary ones for $\sin x$ and $\cos x$ (if $z = x$ is real) and for any complex z we should have

$$\frac{d}{dz} \sin z = \cos z, \quad \sin^2 z + \cos^2 z = 1, \quad \text{etc.}$$

If $\sin x$ and $\cos x$ could be expressed as rational functions the extension would be straightforward. It is not possible, however, to represent $\sin x$ and $\cos x$ as rational functions. Let us express them instead as power series. This will be our method of attack on the elementary functions.

It is most convenient to start our investigations with the exponential function $y = e^x$. For real x we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots ,$$

and the series converges for all values of x. Hence if we define

(6.1) $$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

the function $w = e^z$ has the following properties:

(A) e^z is defined for every complex value of z. The circle of convergence of (6.1) contains the entire real axis, hence case(i) of Theorem 5.2 applies.

(B) $\frac{d}{dz} e^z = e^z$. This follows at once on differentiating (6.1). The usual "chain rule" then gives

$$\frac{d}{dz} e^u = e^u \frac{du}{dz} .$$

(C) $e^u e^v = e^{u+v}$. In Example 5.7 of Chpater 3 we established this identity for real values of u and v by multiplying power series. The same proof holds if u and v are complex.

Important special cases are

$$e^z e^{-z} = 1, \quad \text{or} \quad e^{-z} = (e^z)^{-1};$$

$$(e^z)^2 = e^{2z}, \quad (e^z)^3 = (e^z)^2(e^z) = e^{2z}e^z = e^{3z},$$

and in general $(e^z)^n = e^{nz}$ for any positive integer n. From these it follows that $(e^z)^m = e^{mz}$ for any integer m, positive, negative or zero.

(D) e^z is never zero, for any value of z. This follows from the fact that, since $e^z e^{-z} = 1$, we cannot have $e^z = 0$.

We now define $\cos z$ and $\sin z$ in a similar manner:

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots ,$$

(6.2)

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots .$$

Here also the series converge for every value of z.

(E) $e^{iz} = \cos z + i \sin z$. This is proved by substituting iz for z in (6.1) and comparing with the sum of the two series $\cos z$ and $i \sin z$.

This unexpected relation between the exponential and the trigonometric functions is one of the most striking results in elementary function theory. It is generally known as Euler's Formula.

(F) $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, $\quad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$.

To get these we first note that from (6.2) we have

$$\cos(-z) = \cos z, \qquad \sin(-z) = -\sin z.$$

Then from (E),

$$e^{-iz} = e^{i(-z)} = \cos(-z) + i \sin(-z)$$
$$= \cos z - i \sin z.$$

Solving this simultaneously with (E) gives the desired result.

Many of the identities connecting trignometric functions can be derived from (F). For example

$$\cos u \cos v + \sin u \sin v$$
$$= \frac{e^{iu}+e^{-iu}}{2} \cdot \frac{e^{iv}+e^{-iv}}{2} + \frac{e^{iu}-e^{-iu}}{2i} \cdot \frac{e^{iv}-e^{-iv}}{2i}$$
$$= \frac{1}{4}[e^{i(u+v)} + e^{i(u-v)} + e^{i(-u+v)} + e^{i(-u-v)} - e^{i(u+v)} + e^{i(u-v)} + e^{i(-u+v)} - e^{i(-u-v)}]$$
$$= \frac{1}{2}[e^{i(u-v)} + e^{-i(u-v)}] = \cos(u-v).$$

For other useful applications see Problems 6.2, 6.3 and 6.10 below.

(G) If θ is a real number, $\text{cis}\,\theta = e^{i\theta}$. Therefore we can dispense with the function $\text{cis}\,\theta$ and use $e^{i\theta}$ instead. Thus the polar form of z becomes $re^{i\theta}$, where $r = |z|$ and $\theta = \arg z$. From this Theorem 3.1 follows at once, since

$$(re^{i\theta})(se^{i\phi}) = rs\, e^{i(\theta+\phi)}.$$

(H) As special cases of (E) we have

$$e^{2\pi i} = 1, \quad e^{\pi i} = -1, \quad e^{\pi i/2} = i .$$

From the first of these, and (C), follows $e^{z+2\pi i} = e^z$, or, more generally, $e^{z+2n\pi i} = e^z$ for any integer n. That is, e^z is a periodic function of period $2\pi i$.

(I) If $z = x + iy$, where x and y are real, then

$$e^z = e^x \cos y + ie^x \sin y.$$

Since e^x, $\sin y$, and $\cos y$ can be looked up in tables this enables us to evaluate e^z for any complex z.

(J) $\cos(iz) = \frac{1}{2}(e^z + e^{-z})$, $\qquad \sin(iz) = \frac{i}{2}(e^z - e^{-z})$.

These follow from (F). It is useful to introduce the so-called "hyperbolic functions" for the combinations of exponentials that occur here; that is,

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \frac{1}{2}(e^z - e^{-z}).$$

("Cosh" is read as "hyperbolic cosine"; "sinh" is read as "hyperbolic sine". There is a connection with hyperbolas, but it is of historic interest only.) Then (J) becomes

$$\cos(iz) = \cosh z, \qquad \sin(iz) = i \sinh z.$$

Any identity connecting trignometric functions can be made into an identity connecting hyperbolic functions. For example, from

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

we get

$$\cos(iu-iv) = \cos(iu) \cos(iv) + \sin(iu) \sin(iv),$$

or

$$\cosh(u-v) = \cosh u \cosh v - \sinh u \sinh v,$$

by (J).

(K) If $z = x + iy$, where x and y are real, then

$$\cos z = \cos x \cosh y - i \sin x \sinh y,$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y.$$

These equations enable us to evaluate $\sin z$ and $\cos z$ for any complex value of z.

Problems

6.1 Evaluate to three decimal places

(a) e^{i}, e^{2-i}

(b) $\sin i$, $\sin(1+i)$.

6.2 Derive

$$\int e^{ax} \sin bx\, dx = e^{ax} \frac{(a \sin bx - b \cos bx)}{a^2 + b^2}$$

by first expressing $\sin bx$ in terms of imaginary exponentials.

6.3 By expressing $\sin x$ in terms of exponentials and using the binomial theorem show that

$$\sin^5 x = \frac{1}{16} (\sin 5x - 5 \sin 3x + 10 \sin x).$$

6.4 Graph carefully $y = \sinh x$, $y = \cosh x$, and $y = \tanh x$ (= $\sinh x/\cosh x$), on the same set of axes, from $x = -2$ to $x = 2$.

6.5 Prove the following relations

(a) $\cosh^2 z - \sinh^2 z = 1.$

(b) $\cosh 2z = \cosh^2 z + \sinh^2 z.$

(c) $\sinh 2z = 2 \sinh z \cosh z.$

(d) $\frac{d}{dz} \sinh z = \cosh z$, $\frac{d}{dz} \cosh z = \sinh z.$

(e) $\cosh iz = \cos z$, $\sinh iz = i \sin z.$

6.6 (a) By making the substitution $x = a \cosh u$ show that

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \text{arccosh}\, \frac{x}{a} + c.$$

(b) Evaluate $\int \frac{dx}{\sqrt{x^2 + a^2}}$ by a similar substitution.

6.7 Derive the series expansions in powers of z for $\cosh z$ and $\sinh z$.

6.8 If $z = x + iy$, where x and y are real, show that

$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}.$$

6.9 (a) Show that if n is an integer

$$\int_0^{2\pi} e^{inx}dx = \begin{cases} 2\pi & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases}$$

(b) What do we get if n is a real number not an integer, in particular, if $n = 1437.7$?

6.10 (a) Evaluate $\int \sin mx \sin nx dx$.

(b) Evaluate $\int_0^{2\pi} \sin mx \sin nx dx$ if m and n are integers.

6.11 (a) Prove that in any of the function spaces C^n, the pair of functions $\{e^{ax}, e^{-ax}\}$ span the same subspace as the pair $\{\sinh ax, \cosh ax\}$.

(b) Illustrate by expressing $e^{2x} + 3e^{-2x}$ in the form $A \sinh 2x + B \cosh 2x$, and by expressing $2 \sinh 3x + \cosh 3x$ in the form $Ae^{3x} + Be^{-3x}$.

(c) Similarly prove that any linear combination of the functions $\{e^{iax}, e^{-iax}\}$ can be expressed as a linear combination, with complex coefficients, of the functions $\{\sin ax, \cos ax\}$, and conversely.

These facts will be useful in Chapter 6.

6.12 Show that $\cosh z_1 + \cosh z_2 = 2 \cosh \frac{z_1 - z_2}{2} \cosh \frac{z_1 + z_2}{2}$.

6.13 Let $C = 1 + \frac{\cos \theta}{2} + \frac{\cos 2\theta}{2^2} + \frac{\cos 3\theta}{2^3} + \ldots + \frac{\cos n\theta}{2^n} + \ldots$

and

$$S = \frac{\sin \theta}{2} + \frac{\sin 2\theta}{2^2} + \frac{\sin 3\theta}{2^3} + \ldots + \frac{\sin n\theta}{2^n} + \ldots .$$

(a) Show that

$$C + iS = 1 + \frac{e^{i\theta}}{2} + \frac{e^{i2\theta}}{2^2} + \frac{e^{i3\theta}}{2^3} + \ldots + \frac{e^{in\theta}}{2^n} + \ldots .$$

(b) Sum the series on the right in part (a) to obtain

$$C + iS = \frac{2}{2 - e^{i\theta}} .$$

(c) Equating the real and imaginary parts on each side of the equation given in part (b), find the formulas for C and S.

6.14 Find a real formula for $\sum_{n=1}^{N} \cos n\theta$ by noting that $\cos n\theta = R(e^{in\theta})$, hence $\sum_{n=1}^{N} \cos n\theta = R \sum_{n=1}^{N} e^{in\theta}$, summing the geometric series and separating real and imaginary parts. Similarly find a formula for $\sum_{n=1}^{N} \sin n\theta$.

Answer. $\sum_{n=1}^{N} \cos n\theta = \cos(N+1)\theta/2 \, \frac{\sin N\theta/2}{\sin \theta/2}$, $\sum_{n=1}^{N} \sin n\theta = \sin(N+1)\theta/2 \, \frac{\sin N\theta/2}{\sin \theta/2}$.

6.15 Prove the identity

$$\frac{\sin 4\theta}{2 \sin \theta} = \cos \theta + \cos 3\theta$$

by expressing the trignometric functions in terms of exponentials.

6.16 In each of the following problems, t is a real number. Find two forms for each answer, as indicated in (a).

(a) $R(\frac{1}{3+2i}e^{2it})$. Answer. $\frac{1}{13}(3 \cos 2t + 2 \sin 2t)$.

$\frac{1}{\sqrt{13}} \cos (2t - \theta)$, $\tan \theta = \frac{2}{3}$, θ in quadrant I.

(b) $I(\frac{1}{3+2i} e^{2it})$.

(c) $R(\frac{2i}{1-i} e^{3it})$.

(d) $I(\frac{2}{a+bi} e^{\omega it})$.

7. <u>Multivalued Functions</u>. To find the n-th root of a non-zero complex number z, where n is a positive integer, we must solve the equation $w^n = z$ for w. Let z have the polar form $z = ae^{i\alpha}$ and w the form $w = re^{i\theta}$. Then we wish to find r and θ so that

$$ae^{i\alpha} = (re^{i\theta})^n = r^n e^{in\theta}.$$

This means that $r^n = a$, so that $r = \sqrt[n]{a}$; there is no ambiguity here since a and r are both positive real numbers. However, from $e^{i\alpha} = e^{in\theta}$ we cannot conclude that $n\theta = \alpha$ but only (see (H) of Section 6) that $n\theta = \alpha + 2\pi k$, where k is an integer. That is,

(7.1) $$w = \sqrt[n]{a}\, e^{i(\alpha + 2\pi k)/n}.$$

The values $k = 0, 1, 2, \ldots, n-1$, all give different values for w, but $k = n$ gives the same w as $k = 0$, and similarly for other values of k. Hence the complex number z has n distinct n-th roots, given by (7.1) for $k = 0, 1, 2, \ldots, n-1$ (or any other n consecutive values of k).

<u>Example 7.1.</u> Evaluate $(1 + i)^{1/3}$.

We first write $(1 + i) = \sqrt{2}\, e^{i(\frac{\pi}{4} + 2k\pi)}$. (See Figure 7.1.)

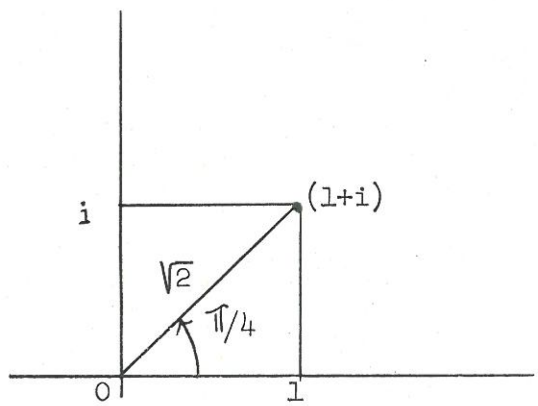

Figure 7.1

Next

$$(1 + i)^{1/3} = \sqrt[6]{2}\, e^{i(\frac{\pi}{12} + \frac{2k}{3}\pi)} .$$

This gives,

for $k = 0$, $(1 + i)^{1/3} = \sqrt[6]{2}\, e^{i\,\pi/12}$;

for $k = 1$, $(1 + i)^{1/3} = \sqrt[6]{2}\, e^{i9\,\pi/12}$;

for $k = 2$, $(1 + i)^{1/3} = \sqrt[6]{2}\, e^{i17\,\pi/12}$;

(See Figure 7.2.) Other integer values of k yield these same roots. For example if $k = 3$, we get $\sqrt[6]{2}\, e^{i(\frac{\pi}{12} + \frac{6\pi}{3})} = \sqrt[6]{2}\, e^{i\,\pi/12}$, for $k = 4$ we get $\sqrt[6]{2}\, e^{i(\frac{\pi}{12} + \frac{8\pi}{3})} = \sqrt[6]{2}\, e^{i9\,\pi/12}$ and so on. In fact increasing k by unity advances the point in Figure 7.2 by an angle of $\frac{2\pi}{3}$ around the circle of radius $\sqrt[6]{2}$. Thus only three distinct points are ever reached. These are the three cube roots of $(1 + i)$, i.e. the three values of $(1 + i)^{1/3}$.

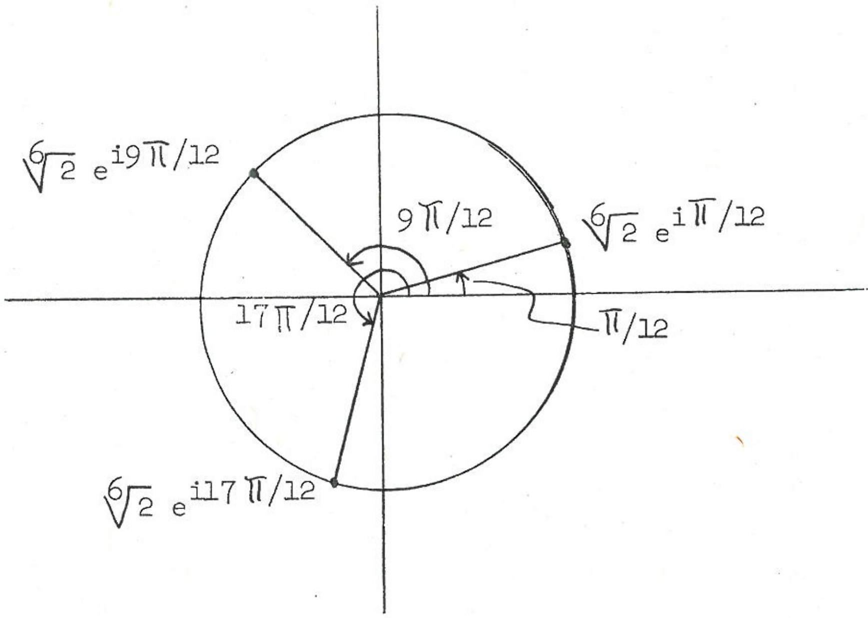

Figure 7.2

Since the power of any non-zero number is never zero it is evident that the n-th root of zero is zero and nothing else.

A similar argument works for logarithms. By $\log z$ we mean a number w such that $e^w = z$. Write z in polar form and w in rectangular form $w = x + iy$. Then

$$z = ae^{i\alpha} = e^{x+iy} = e^x e^{iy} .$$

This requires that $a = e^x$, or $x = \log a$. If $z \neq 0$, a is positive and has a unique real logarithm x. To get y we solve $e^{iy} = e^{i\alpha}$, or, as before,, $y = \alpha + 2\pi k$ for any integer k. Here, however, each k gives a different w, and the logarithm is an infinitely multiple-valued function, (7.2) $$w = \log a + i(\alpha + 2\pi k), \qquad k = 0, \pm 1, \pm 2, \ldots .$$

Because of this one must be careful in interpreting such identities as $\log(uv) = \log u + \log v$. If each of the three terms can have an infinite number of values just what can such an equation mean? Perhaps the best way to interpret it is to say that if any two of the terms are assigned specific values there is a value of the remaining term for which the equality holds.

Since e^w is never zero (see (D) of Section 6) the logarithm of zero does not exist.

Since the trigonometric functions are expressible in terms of the exponential we might expect that the inverse trigonometric functions are expressible in terms of logarithms. This is easily established. Suppose we wish to find w so that $\sin w = z$. We have then

$$z = \frac{e^{iw} - e^{-iw}}{2i} .$$

Multiplying both sides by $2ie^{iw}$ and transposing gives

$$e^{2iw} - 2ize^{iw} - 1 = 0.$$

This is a quadratic equation in e^{iw}, whose solution is

$$e^{iw} = iz \pm \sqrt{1 - z^2} .$$

Hence

(7.3) $$w = \frac{1}{i} \log(iz \pm \sqrt{1 - z^2})$$

is the expression for $\arcsin z$.

Problems

7.1 Compute each of the following:

(a) $\sqrt{3+4i}$, $\sqrt[3]{i}$, $\sqrt[5]{1}$.

(b) $\log(3+4i)$, $\log i$, $\log(-1)$.

(c) $\arcsin 2$.

7.2 Show that the values of $\sqrt[n]{z}$ are equally spaced around the circumference of a circle with center at the origin. (n is a positive integer). What similar statement can be made about the values of $\log z$?

7.3 If u and v are complex numbers we define u^v to be $e^{v \log u}$. Find the values of i^i.

7.4 Derive the formulas

(a) $\arccos z = \frac{1}{i} \log(z \pm \sqrt{z^2-1})$.

(b) $\operatorname{arcsinh} z = \log(z \pm \sqrt{z^2+1})$.

(c) $\arctan z = \frac{1}{2i} \log(\frac{1+iz}{1-iz})$.

7.5 Using the formula (7.3) for $z = \frac{1}{2}$ show that one gets the customary values for $\arcsin(\frac{1}{2})$.

7.6 (a) Obtain a power series expansion for $\log(1+z)$ by expanding $1/(1+z)$ by long division and integrating.

(b) Use this result to get a power series for $\arctan z$ from

$$\arctan z = \frac{1}{2i} \log \frac{1+iz}{1-iz}$$

$$= \frac{1}{2i} [\log(1+iz) - \log(1-iz)].$$

(c) Expand $\arctan z$ in a power series by integrating the series for $1/(1+z^2)$. Compare with (b).

7.7 By solving the equation $z = \dfrac{e^w + e^{-w}}{2}$ for w, show that

$$\text{arc cosh } z = \log (z \pm \sqrt{z^2-1}).$$

Use the fact that $R(\log z) = \log |z|$ to obtain R (arc cosh i).

7.8 (a) Compute the four fourth roots of -1 and show them graphically.

(b) Write 2^{1+i} in polar form $r(\cos \theta + i \sin \theta)$.

(c) Compute $\log(1 - \sqrt{3}\, i)$ and show the result graphically.

7.9 (a) Compute the values of $(-16)^{1/4}$, put them in the form $a + ib$ and show them graphically.

(b) Do the same for $(-16)^{1/5}$, $(-16)^{1/6}$.

(c) Do the same for $(1-i)^{1/2}$, $(1-i)^{1/3}$, $(1-i)^{1/4}$.

(d) Do the same for $(-1+\sqrt{3}\, i)^{1/3}$.

(e) Do the same for $(2+3i)^{1/2}$. (Use two decimal place accuracy.)

8. Applications.

Example 8.1. Linkages. The design of machines, computers, engines, instruments and many other devices requires the study of the motion of linkages, cams, gears and combinations of these components. Many interesting and important mechanisms are obtained by using components which are constrained to move in a plane.

Shown in Figure 8.1 is rigid link 1 which is pinned to a fixed support at **A** and is thus free to rotate in the plane. Its instantaneous angular position is given by Θ_1 and its instantaneous angular velocity by $\frac{d\Theta_1}{dt} = \dot{\Theta}_1 = \omega_1$. The position of any point B on the link is given by the vector z_1 which originates at point **A** and extends to B; thus

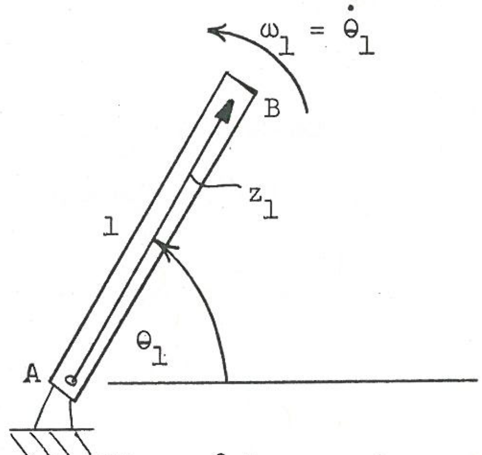

Figure 8.1

$$(8.1) \qquad z_1 = r_1 e^{i\Theta_1} .$$

The velocity vector of point B relative to point **A** is given by the time rate of change of the position vector z_1. When (8.1) is differentiated it yields

$$(8.2) \qquad \frac{dz_1}{dt} = v_1 = ir_1\dot{\Theta}_1 e^{i\Theta_1}$$

because for a rigid link the distance between points, r_1, is constant. Equation (8.2) can be written as

$$(8.3) \qquad v_1 = iz_1\omega_1$$

for $\dot{\Theta}_1 = \omega_1$ and $z_1 = r_1 e^{i\Theta_1}$. Equation (8.3) shows that the velocity vector v_1 has a magnitude $r_1\omega_1$ and a direction given by a +90° rotaof the position vector z_1 (see Section 2). This result is to be expected for in a rigid link having angular velocity ω_1 point B moves on a circle of radius r_1 and thus has a velocity of magnitude $r_1\omega_1$ directed along the tangent to the circle. The instantaneous position and velocity vectors are shown in Figure 8.2.

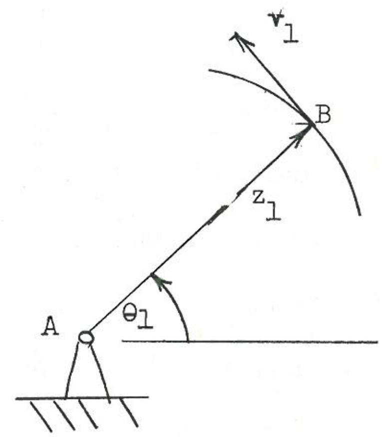

Figure 8.2

In addition to the determination of positions and velocities present in links and mechanisms it is often necessary to know the forces which act so that the components may be proportioned to be sufficiently strong. Through Newton's Second Law the forces depend on the accelerations. Thus, a knowledge of the forces requires an investigation of the accelerations. The acceleration vector of point B relative to point A is given by the time rate of change of the velocity vector v_1. Differentiation of (8.3) gives

$$(8.4) \qquad \frac{dv_1}{dt} = a_1 = i\omega_1 \frac{dz_1}{dt} + iz_1 \frac{d\omega_1}{dt} .$$

If the the angular acceleration $\frac{d\omega_1}{dt} = \frac{d^2\theta_1}{dt^2}$ is denoted by α_1 and if (8.3) is used for $\frac{dz_1}{dt}$, then equation (8.4) becomes

$$a_1 = i^2 z_1 \omega_1^2 + iz_1\alpha_1$$

or

(8.5) $$a_1 = -z_1\omega_1^2 + iz_1\alpha_1 .$$

Equation (8.5) shows that the acceleration vector of point B has two components. The first, $-z_1\omega_1^2$, has a magnitude $r_1\omega_1^2$ and a direction given by a 180° rotation of the position vector z_1; this is the centrifugal component of acceleration of a point moving on a circle. The second component, $iz_1\alpha_1$, has a magnitude $r_1\alpha_1$ and a direction given by a +90° rotation of the position vector z_1; this is the tangential acceleration component of a point moving on a circle. Figure 8.3 shows these vectors.

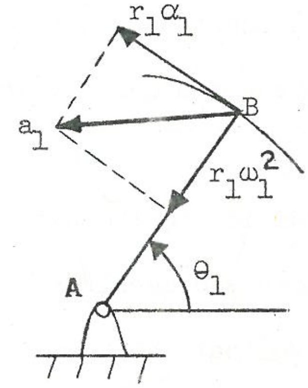

Figure 8.3

Example 8.2. Slider - Crank Mechanism. A simple and widely used mechanism which can convert rotational motion to motion along a straight line is the slider - crank mechanism shown in Figure 8.4. Crank AB rotates about A and is linked to the slider by connecting rod BC. The slider, the piston in an internal combustion engine, is constrained to move along a straight line. Smooth bearings at A, B and C permit the

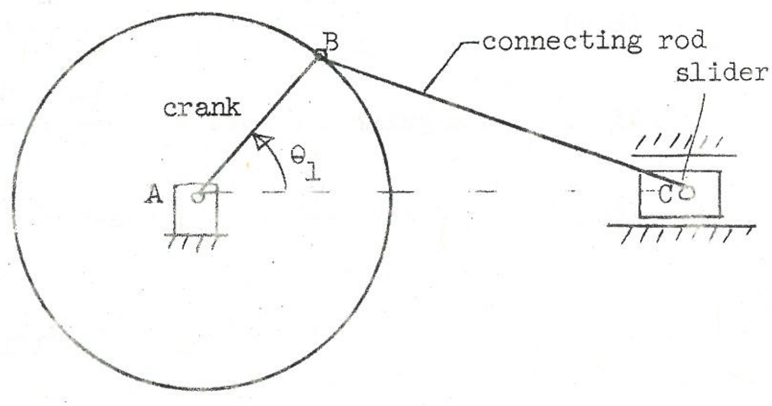

Figure 8.4

links to freely pivot at these points.

Assume crank AB rotates with constant angular velocity $\omega_1 = \dot{\theta}_1$ and that it is required to find the position and velocity of the slider. Using the notation of Example 8.1, denote the position of B by vector z_1, the position of C with respect to B by vector z_2, and the position of the slider by vector z_3. See Figure 8.5.

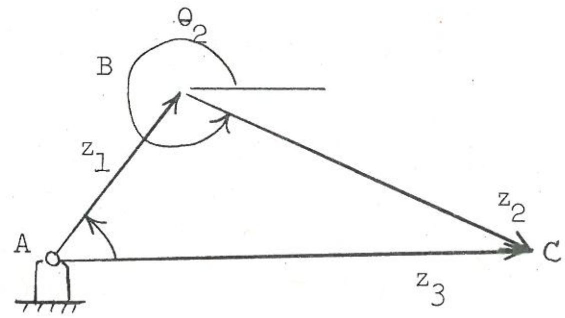

Figure 8.5

The position vectors are related by the vector equation

$$z_3 = z_1 + z_2 \tag{8.6}$$

which when written in terms of complex numbers becomes

$$r_3 e^{i\theta_3} = r_1 e^{i\theta_1} + r_2 e^{i\theta_2}. \tag{8.7}$$

Radii r_1 and r_2 are the known lengths of links AB and BC; θ_1 and θ_2 are the angular positions of links AB and BC with respect to the real axis direction AC. Angle $\theta_3 = 0$ because of the constraint on the slider. Taking the real and imaginary parts of (8.7) gives the two equations

$$r_3 = r_1 \cos \theta_1 + r_2 \cos \theta_2 , \tag{8.8}$$

$$0 = r_1 \sin \theta_1 + r_2 \sin \theta_2 . \tag{8.9}$$

In equation (8.9), r_1 and r_2 are of known length and $\theta_1 = \omega_1 t$ so that (8.9) is a transcendental equation for the angular position θ_2 of link BC

(8.10) $$\sin \theta_2 = - \frac{r_1}{r_2} \sin \omega_1 t \ .$$

With $\sin\theta_2$ determined by (8.10), equation (8.8) gives the position of the slider as

(8.11) $$r_3 = r_1 \cos \omega_1 t + r_2 \sqrt{1 - \left(\frac{r_1}{r_2}\right)^2 \sin^2 \omega_1 t} \ .$$

(The negative square root is ruled out by the physical situation.) Now, the crank and connecting rod are frequently proportioned so that $r_1/r_2 \leqq 1/4$. As crank AB rotates, the $\sin \omega_1 t$ varies in magnitude between 0 and 1, so that (8.10) gives the $\sin \theta_2$ varying in magnitude between 0 and 1/4. Thus, equation (8.11) for slider position r_3 can be simplified by replacing the radical by the first two terms of its series expansion. We obtain

(8.12) $$r_3 = r_1 \cos \omega_1 t + r_2 \left\{ 1 - \frac{1}{2} \left(\frac{r_1}{r_2}\right)^2 \sin^2 \omega_1 t + \ldots \right\} .$$

Further, we observe that the mechanism can to a first approximation produce simple harmonic oscillation of the slider if the $\frac{1}{2} \left(\frac{r_1}{r_2}\right)^2 \sin^2 \omega_1 t$ term is negligible. Thus, if we proportion links AB and BC so that $r_1/r_2 \ll 1$ then (8.10) shows that θ_2 remains small and (8.12) gives

$$r_3 \cong r_2 + r_1 \cos \omega_1 t \ .$$

The velocities of the mechanism are determined by differentiating (8.7) with respect to time. We obtain

(8.13) $$\mathring{r}_3 = ir_1\mathring{\theta}_1 e^{i\theta_1} + ir_2\mathring{\theta}_2 e^{i\theta_2} .$$

Using the notation $\mathring{\theta}_1 = \omega_1$ and $\mathring{\theta}_2 = \omega_2$, the real and imaginary parts of (8.13) give

(8.14) $$\mathring{r}_3 = - r_1\omega_1 \sin \theta_1 - r_2\omega_2 \sin \theta_2 ,$$

(8.15) $$0 = r_1\omega_1 \cos \theta_1 + r_2\omega_2 \cos \theta_2 .$$

With θ_2 determined by (8.10), equation (8.15) is an expression for the instantaneous angular velocity ω_2 of link BC. Equation (8.14) then gives the velocity of the slider. Approximations similar to (8.12) can be found when $r_1/r_2 << 1$. Another differentiation would yield the accelerations of the mechanism which, when used in Newton's Law, give the forces present.

Example 8.3. Response of Linear Systems to Harmonic Excitation. First order differential equations with constant coefficients of the type

(8.16) $$\frac{dy}{dt} + py = q_1(t)$$

were solved in Chapter 1 by adding the complementary solution Ce^{-pt} to a particular solution, $y_b(t)$.

When $q_1(t) = E \cos \omega t$, equation (8.16) is the mathematical description of simple mechanical or electrical systems which frequently arise in engineering. Equation (8.16) becomes

(8.17) $$\frac{dy}{dt} + py = E \cos \omega t .$$

If we replace the function $E \cos \omega t$ by $Ee^{i\omega t}$ then equation (8.17) becomes the new differential equation

(8.18) $$\frac{dy}{dt} + py = Ee^{i\omega t} .$$

It is seen that the $q_1(t) = Ee^{i\omega t}$ of equation (8.18) contains the right hand side of (8.17), $E \cos \omega t$, as well as an imaginary term $iE \sin \omega t$. To obtain a particular solution of (8.18) assume a form

(8.19) $$y_1(t) = Ae^{i\omega t} .$$

When solution (8.19) is substituted in (8.18) and the undetermined coefficient A adjusted so that (8.18) is satisfied, we obtain

$$(i\omega+p)\ Ae^{i\omega t} = Ee^{i\omega t}$$

so that

(8.20) $$A = \frac{E}{(i\omega+p)} .$$

The particular solution, $y_1(t)$, of equation (8.18) becomes

(8.21) $$y_1(t) = \frac{E}{Z} e^{i\omega t} = \frac{E}{|Z|} e^{i(\omega t-\theta)}$$

where Z is the complex constant

(8.22) $$Z = i\omega + p$$

and

(8.23) $$|Z| = \sqrt{\omega^2 + p^2} \ ; \quad \tan \theta = \frac{\omega}{p} , \quad \theta \text{ in first quadrant} .$$

Particular solution (8.21) is seen to be a complex function which can be written in terms of a real and an imaginary part in the form

(8.24) $$y_1(t) = R_1(t) + iS_1(t)$$

where

$$(8.25) \qquad R_1(t) = \frac{E}{|Z|} \cos(\omega t - \theta) \; ; \quad S_1(t) = \frac{E}{|Z|} \sin(\omega t - \theta) \; .$$

If solution (8.24) is substituted into equation (8.18) we obtain

$$(8.26) \qquad \frac{dR_1}{dt} = pR_1 + i\left(\frac{dS_1}{dt} + pS_1\right) = Ee^{i\omega t} \; .$$

Equating real and imaginary parts on both sides of (8.26) gives R_1 as the solution to the differential equation

$$(8.27) \qquad \frac{dR_1}{dt} + pR_1 = E \cos \omega t$$

and S_1 as the solution to the differential equation

$$(8.28) \qquad \frac{dS_1}{dt} + pS_1 = E \sin \omega t \; .$$

We conclude that if the function $q_1(t)$ is simple harmonic either $E \cos \omega t$ or $E \sin \omega t$, it is possible to replace it by $Ee^{i\omega t}$ and the real and imaginary parts of the solution, (8.24), are the solutions to the original equation having $q_1(t)$ either $E \cos \omega t$ or $E \sin \omega t$.

Consider the electric circuit shown in Figure 8.6 which has resistance R, inductance L, and is excited by an oscillatory voltage $E \sin \omega t$. Kirchhoff's Law for the circuit gives

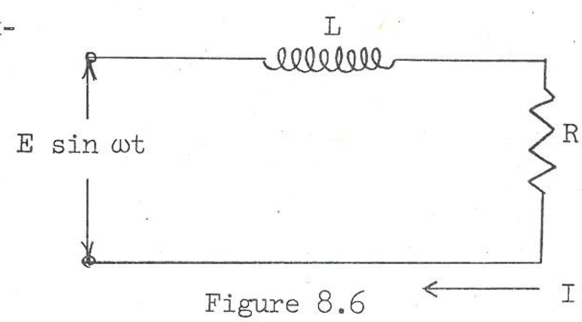

Figure 8.6

$$(8.29) \qquad L \frac{dI}{dt} + RI = E \sin \omega t$$

where I is the instantaneous current. If we replace the applied voltage $E \sin \omega t$ by the vector voltage $V = Ee^{i\omega t}$ and seek a particular solution of the form

$$(8.30) \qquad I = Ae^{i\omega t}$$

then (8.29) becomes

$$(8.31) \qquad i\omega LI + RI = V = Ee^{i\omega t} .$$

The differential equation has become an algebraic equation. The voltage drop across the resistor is RI. The voltage across the inductance is $i\omega LI$ and is seen to "lead" the drop across the resistor by 90°. Equation (8.31) states that the sum of the vector voltages RI and $i\omega LI$ equals the applied vector voltage V. These voltages are shown in Figure 8.7 as vectors rotating with angular velocity ω. The analysis of complicated electrical circuits excited by oscillatory disturbances is greatly facilitated by this approach.

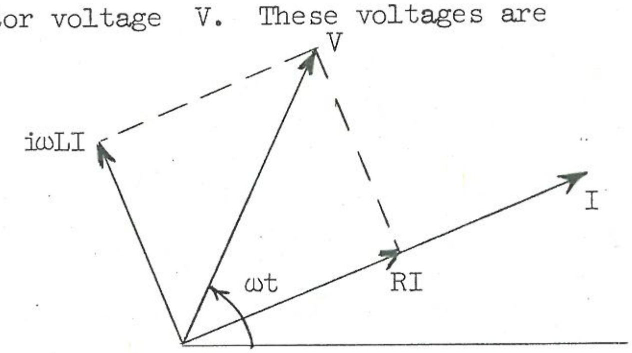

Figure 8.7

To obtain the amplitude A of the vector current, substitute (8.30) in equation (8.31) and solve for A. The results are clearly the same as those obtained previously in equations (8.20), (8.21), (8.22) and (8.23) only with ωL replacing ω. Hence, the particular solution (8.30) is

$$(8.32) \qquad I = \frac{E}{Z} e^{i\omega t} = \frac{E}{|Z|} e^{i(\omega t - \theta)}$$

where

$$|Z| = \sqrt{(\omega L)^2 + R^2} \quad ; \quad \tan \theta = \frac{\omega L}{R}, \ \theta \text{ in first quadrant.}$$

Vector Z is called the impedance of the circuit and clearly $ZI = V$. To find the actual steady-state current in the given circuit, the particular solution of equation (8.29) must be obtained. The imaginary part of (8.32) gives the particular solution (8.29) as

(8.33) $$I = \frac{E}{|Z|} \sin (\omega t - \theta) \ .$$

Problems

8.1 The slider - crank of an internal combustion engine has a crank length AB = 2.0 inches and a connecting rod length BC = 8.0 inches. The crank speed of the engine is constant at 3000 rpm (314 radians/sec.). When the crank angle is $\theta_1 = 30°$, determine (a) the angular position of connecting rod BC, (b) the angular velocity ω_2 of the connecting rod, (c) the displacement of the slider from its position when $\theta_1 = \theta_2 = 0$, and (d) the velocity of the slider.

8.2 In the slider - crank mechanism of Example 8.2, determine the equations for the angular acceleration $d\omega_2/dt = \alpha_2$ of connecting rod BC and the acceleration of the slider in terms of known or previously determined quantities.

8.3 An offset slider - crank mechanism is one in which the axis of sliding of the slider is not through the crank axis **A** as shown in Figure 8.8. Determine general equations for (a) the angular position of connecting rod BC, (b) the horizontal position of the slider, (c) the angular velocity, $\omega_2 = \dot{\theta}_2$, of the connecting rod, and (d) the velocity of the slider. Assume the crank rotates with constant angular velocity, $\omega_1 = \dot{\theta}_1$.

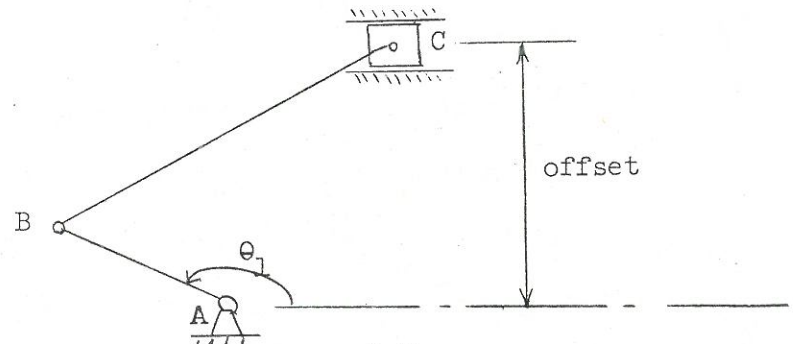

Figure 8.8

If AB = 10 inches, BC = 30 inches, ω_1 = 1 rad./sec., $\theta_1 = 180°$, and the offset is 12 inches, determine the quantities (a) - (d) for these conditions.

8.4 If the circuit of Figure 8.6 is excited by an oscillatory voltage E cos ωt, determine the differential equation for the current I. Find the vector voltage drops across R and L and draw the vector voltage diagram similar to Figure 8.7. Show ωt and θ on this diagram. Find the expression for the actual steady state current in the circuit.

8.5 Analyze the low pass filter circuit of Example 7.4 of Chapter 1 using the complex number approach of Example 8.3. Thus, verify equation (7.39) of Chapter 1. Draw the vector voltage diagrams similar to Figure 8.7 and show its form for low and high frequencies.

8.6 If capacitor C is put in the circuit of Figure 8.6 in place of inductor L, determine the steady state current by an analysis similar to that given in Example 8.3. Include a vector voltage diagram showing drops V, RI and $-\frac{i}{\omega C} I$, and angles ωt and θ.

8.7 The motion of a point in a plane can be described by the vector $z = re^{i\theta}$, where r and θ, and hence z, are functions of time t.

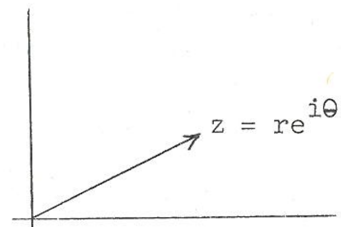

Figure 8.9

(a) Show that the acceleration of the point is

$\ddot{z} = (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})ie^{i\theta}$.

(The two terms, $-r\dot{\theta}^2 e^{i\theta}$ and $2\dot{r}\dot{\theta}ie^{i\theta}$, that involve only first derivatives are known respectively as centripetal acceleration and Coriolis acceleration.)

(b) For a particle moving under the action of a central force only, we must have $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$. On multiplying this by r show that it can be integrated to give $r^2\dot{\theta} = h$, a constant. The constant h is the angular momentum of a particle of unit mass about the center.

(c) Assume the central force is an attraction of magnitude kr^{-2}, i.e. gravitation. Then

(8.34) $$\ddot{r} - r\dot{\theta}^2 = -kr^{-2} .$$

To simplify this we let $r = u^{-1}$ and eliminate t, using $\dot{\theta} = hr^{-2} = hu^2$, thus:

$$\dot{r} = \frac{dr}{dt} = -u^{-2}\frac{du}{dt} = -u^{-2}\frac{du}{d\theta}\dot{\theta} = -h\frac{du}{d\theta} ;$$

$$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2} .$$

Show that (8.34) becomes

(8.35) $$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2} .$$

A particular solution of (8.35) is $u = \frac{k}{h^2}$, and the general solution of the homogeneous equation $\frac{d^2u}{d\theta^2} + u = 0$ is given in Problem 5.6 of Chapter 1 as $u = A \sin(\theta + B)$. Hence (cf, Section 11 of Chapter 2) the general solution of (8.35) is

$$u = \frac{k}{h^2} + A \sin(\theta + B),$$

and so

(8.36) $$r = \frac{1}{\frac{k}{h^2} + A \sin(\theta + B)} .$$

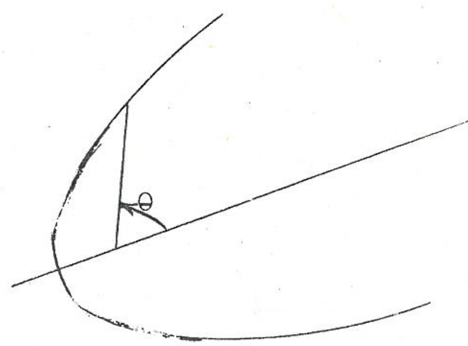

Figure 8.10

r has its minimum value when $\theta + B = \frac{\pi}{2}$. Choose axes (see Figure 8.10) so that this minimum occurs when $\theta = \pi$. Then $B = -\frac{\pi}{2}$ and (8.36) becomes

(8.37) $$r = \frac{h^2/k}{1 - e \cos \theta} .$$

By Thomas, Section 11-3, this is a conic of eccentricity e with the focus at the center of attraction.

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