Available online at www.sciencedirect.com
ScienceDirect
Communications in Nonlinear Science and Numerical Simulation 12 (2007) 859–864
Communications in Nonlinear Science and Numerical Simulation
www.elsevier.com/locate/cnsns

Short communication

Hopf bifurcation formula for first order differential-delay equations

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Received 22 August 2005; accepted 22 August 2005
Available online 28 September 2005

Abstract

This work presents an explicit formula for determining the radius of a limit cycle which is born in a Hopf bifurcation in a class of first order constant coefficient differential-delay equations. The derivation is accomplished using Lindstedt's perturbation method.

PACS: 02.30.Ks; 02.30.Oz

Keywords: Hopf bifurcation; Differential delay equation; DDE; Retarded functional differential equation; RFDE

1. Introduction

This work concerns the Hopf bifurcation scenario in which an equilibrium point changes its stability due to a change in parameters, giving rise to the birth of a periodic motion called a limit cycle. The most familiar setting in which this scenario occurs is the phase plane of a pair of first order ordinary differential equations (ODEs) (see for example [3,6]):

$$\frac{dx}{dt} = -y + \mu x + a_1x^2 + a_2xy + a_3y^2 + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3 \tag{1}$$

$$\frac{dy}{dt} = x + \mu y + c_1x^2 + c_2xy + c_3y^2 + d_1x^3 + d_2x^2y + d_3xy^2 + d_4y^3 \tag{2}$$

As μ passes through zero, a limit cycle is generically born. It can be written in the approximate form:

$$x = A\cos\omega t, \quad y = A\sin\omega t \tag{3}$$

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1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2005.08.005

where $\omega = 1 + O(\mu)$, and where the amplitude A is given by the Hopf bifurcation formula:

$$A^2 = \frac{-8}{S}\mu \tag{4}$$

where

$$S = 3d_4 + d_2 + 3b_1 + b_3 + 2a_3c_3 + a_2(a_1 + a_3) - 2a_1c_1 - c_2(c_1 + c_3) \tag{5}$$

In Eq. (4), A is real so that $A^2 > 0$, which means that μ must have the opposite sign as S.

In this work, we present a comparable formula for the differential-delay equation (DDE):

$$\frac{dx}{dt} = \alpha x + \beta x_d + a_1x^2 + a_2xx_d + a_3x_d^2 + b_1x^3 + b_2x^2x_d + b_3xx_d^2 + b_4x_d^3 \tag{6}$$

where $x = x(t)$ and $x_d = x(t - T)$. Here T is the delay. Associated with (6) is a linear DDE

$$\frac{dx}{dt} = \alpha x + \beta x_d \tag{7}$$

We assume that (7) has a critical delay T_{cr} for which it exhibits a pair of pure imaginary eigenvalues $\pm\omega i$ corresponding to the solution

$$x = c_1 \cos \omega t + c_2 \sin \omega t \tag{8}$$

Then for values of delay T which lie close to T_{cr},

$$T = T_{cr} + \mu \tag{9}$$

the nonlinear Eq. (6) may exhibit a periodic solution which can be written in the approximate form:

$$x = A \cos \omega t \tag{10}$$

where the amplitude A can be obtained from the following expression for A^2:

$$A^2 = \frac{P}{Q}\mu \tag{11}$$

where

$$P = 4\beta^3(\beta - \alpha)(\beta + \alpha)^2(-5\beta + 4\alpha) \tag{12}$$

$$\begin{aligned} Q = {} & 15b_4\beta^6T_{cr} + 5b_2\beta^6T_{cr} + 3\alpha b_4\beta^5T_{cr} - 15\alpha b_3\beta^5T_{cr} + \alpha b_2\beta^5T_{cr} - 15\alpha b_1\beta^5T_{cr} - 22a_3^2\beta^5T_{cr} \\ & - 7a_2a_3\beta^5T_{cr} - 14a_1a_3\beta^5T_{cr} - 3a_2^2\beta^5T_{cr} - 7a_1a_2\beta^5T_{cr} - 4a_1^2\beta^5T_{cr} - 12\alpha^2b_4\beta^4T_{cr} - 3\alpha^2b_3\beta^4T_{cr} \\ & + 6\alpha^2b_2\beta^4T_{cr} - 3\alpha^2b_1\beta^4T_{cr} + 12a_3^2\alpha\beta^4T_{cr} + 37a_2a_3\alpha\beta^4T_{cr} + 30a_1a_3\alpha\beta^4T_{cr} + 7a_2^2\alpha\beta^4T_{cr} \\ & + 19a_1a_2\alpha\beta^4T_{cr} + 18a_1^2\alpha\beta^4T_{cr} + 12\alpha^3b_3\beta^3T_{cr} + 2\alpha^3b_2\beta^3T_{cr} + 12\alpha^3b_1\beta^3T_{cr} + 4a_3^2\alpha^2\beta^3T_{cr} \\ & - 20a_2a_3\alpha^2\beta^3T_{cr} - 16a_1a_3\alpha^2\beta^3T_{cr} - 12a_2^2\alpha^2\beta^3T_{cr} - 26a_1a_2\alpha^2\beta^3T_{cr} - 8a_1^2\alpha^2\beta^3T_{cr} - 8\alpha^4b_2\beta^2T_{cr} \\ & - 4a_2a_3\alpha^3\beta^2T_{cr} + 8a_2^2\alpha^3\beta^2T_{cr} + 8a_1a_2\alpha^3\beta^2T_{cr} + 5b_3\beta^5 + 15b_1\beta^5 - 15\alpha b_4\beta^4 + \alpha b_3\beta^4 - 15\alpha b_2\beta^4 \\ & + 3\alpha b_1\beta^4 - 4a_3^2\beta^4 - 9a_2a_3\beta^4 - 18a_1a_3\beta^4 - a_2^2\beta^4 - 9a_1a_2\beta^4 - 18a_1^2\beta^4 - 3\alpha^2b_4\beta^3 + 6\alpha^2b_3\beta^3 \\ & - 3\alpha^2b_2\beta^3 - 12\alpha^2b_1\beta^3 + 26a_3^2\alpha\beta^3 + 19a_2a_3\alpha\beta^3 + 30a_1a_3\alpha\beta^3 + 11a_2^2\alpha\beta^3 + 33a_1a_2\alpha\beta^3 + 12a_1^2\alpha\beta^3 \\ & + 12\alpha^3b_4\beta^2 + 2\alpha^3b_3\beta^2 + 12\alpha^3b_2\beta^2 - 8a_3^2\alpha^2\beta^2 - 32a_2a_3\alpha^2\beta^2 - 12a_1a_3\alpha^2\beta^2 - 14a_2^2\alpha^2\beta^2 \\ & - 18a_1a_2\alpha^2\beta^2 - 8\alpha^4b_3\beta - 8a_3^2\alpha^3\beta + 8a_2a_3\alpha^3\beta + 4a_2^2\alpha^3\beta + 8a_2a_3\alpha^4 \end{aligned} \tag{13}$$

In Eq. (11), A is real so that $A^2 > 0$, which means that μ must have the same sign as $\frac{P}{Q}$.

Eq. (13) depends on μ, α, β, a_i, b_i and T_{cr}. This equation may be alternately written with T_{cr} expressed as a function of α and β. This relationship may be obtained by considering the linear DDE (7). Substituting Eq. (10) into Eq. (7) and equating to zero coefficients of $\sin(\omega t)$ and $\cos(\omega t)$, we obtain the two equations:

$$\beta \sin(\omega T_{cr}) = -\omega, \quad \beta \cos(\omega T_{cr}) = -\alpha \tag{14}$$

Squaring and adding these we obtain

$$\omega = \sqrt{\beta^2 - \alpha^2} \tag{15}$$

Substituting (15) into the second of (14), we obtain the desired relationship between $T_{\rm cr}$ and α and β:

$$T_{\rm cr} = \frac{\arccos\left(\frac{-\alpha}{\beta}\right)}{\sqrt{\beta^2 - \alpha^2}} \tag{16}$$

2. Example 1

As an example, we consider the following DDE:

$$\frac{dx}{dt} = -x - 2x_d - xx_d - x^3 \tag{17}$$

This corresponds to the following parameter assignment in Eq. (6):

$$\alpha = -1, \quad \beta = -2, \quad a_1 = a_3 = b_2 = b_3 = b_4 = 0, \quad a_2 = b_1 = -1 \tag{18}$$

The associated linearized equation (7) is stable for zero delay. As the delay T is increased, the origin first becomes unstable when $T = T_{\rm cr}$, where Eq. (16) gives that

$$T_{\rm cr} = \frac{\arccos\left(\frac{-1}{2}\right)}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}} \tag{19}$$

Substituting (18) and (19) into (11)–(13), we obtain:

$$A^2 = \frac{648\mu}{40\sqrt{3}\pi + 171} = 1.667\mu \tag{20}$$

where we have set

$$T = T_{\rm cr} + \mu = \frac{2\pi}{3\sqrt{3}} + \mu = 1.2092 + \mu \tag{21}$$

Thus the origin is stable for $\mu < 0$ and unstable for $\mu > 0$. In order for A^2 in (20) to be positive, we require that $\mu > 0$. Therefore, the limit cycle is born out of an unstable equilibrium point. Since the stability of the limit cycle must be the opposite of the stability of the equilibrium point from which it is born, we may conclude that the limit cycle is stable and that we have a *supercritical* Hopf. This result is in agreement with numerical integration of Eq. (17).

3. Derivation

In order to derive the result (11)–(13), we use Lindstedt's method. We begin by introducing a small parameter ϵ via the scaling

$$x = \epsilon u \tag{22}$$

The detuning μ of Eq. (9) is scaled like ϵ^2:

$$T = T_{\rm cr} + \mu = T_{\rm cr} + \epsilon^2\hat{\mu} \tag{23}$$

Next we stretch time by replacing the independent variable t by τ, where

$$\tau = \Omega t \tag{24}$$

This results in the following form of Eq. (6):

$$\Omega\frac{du}{d\tau} = \alpha u + \beta u_d + \epsilon(a_1u^2 + a_2uu_d + a_3u_d^2) + \epsilon^2(b_1u^3 + b_2u^2u_d + b_3uu_d^2 + b_4u_d^3) \tag{25}$$

where $u_d = u(\tau - \Omega T)$. We expand Ω in a power series in ϵ, omitting the $O(\epsilon)$ term for convenience, since it turns out to be zero:

$$\Omega = \omega + \epsilon^2 k_2 + \cdots \tag{26}$$

Next we expand the delay term u_d:

$$u_d = u(\tau - \Omega T) = u(\tau - (\omega + \epsilon^2 k_2 + \cdots)(T_{cr} + \epsilon^2 \hat{\mu})) \tag{27}$$

$$= u(\tau - \omega T_{cr} - \epsilon^2 (k_2 T_{cr} + \omega\hat{\mu}) + \cdots) \tag{28}$$

$$= u(\tau - \omega T_{cr}) - \epsilon^2 (k_2 T_{cr} + \omega\hat{\mu}) u'(\tau - \omega T_{cr}) + O(\epsilon^3) \tag{29}$$

Finally, we expand $u(\tau)$ in a power series in ϵ:

$$u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \cdots \tag{30}$$

Substituting and collecting terms, we find:

$$\omega \frac{du_0}{d\tau} - \alpha u_0(\tau) - \beta u_0(\tau - \omega T_{cr}) = 0 \tag{31}$$

$$\omega \frac{du_1}{d\tau} - \alpha u_1(\tau) - \beta u_1(\tau - \omega T_{cr}) = a_1 u_0(\tau)^2 + a_2 u_0(\tau) u_0(\tau - \omega T_{cr}) + a_3 u_0(\tau - \omega T_{cr})^2 \tag{32}$$

$$\omega \frac{du_2}{d\tau} - \alpha u_2(\tau) - \beta u_2(\tau - \omega T_{cr}) = \cdots \tag{33}$$

where $\cdots$ stands for terms in u_0 and u_1, omitted here for brevity. We take the solution of the u_0 equation as (cf. Eq. (8) above):

$$u_0(\tau) = \widehat{A} \cos(\tau) \tag{34}$$

We substitute (34) into (32) and obtain the following expression for u_1:

$$u_1(\tau) = m_1 \sin(2\tau) + m_2 \cos(2\tau) + m_3 \tag{35}$$

where m_1 is given by the equation:

$$m_1 = -\frac{\widehat{A}^2 (2a_3\beta + a_2\beta - 2a_1\beta - 2a_3\alpha)\sqrt{\beta^2 - \alpha^2}}{2\beta(\beta + \alpha)(5\beta - 4\alpha)} \tag{36}$$

and where m_2 and m_3 are given by similar equations, omitted here for brevity. In deriving (36), ω has been replaced by $\sqrt{\beta^2 - \alpha^2}$ as in Eq. (15).

Next the expressions for u_0 and u_1, Eqs. (34) and (35), are substituted into the u_2 equation, Eq. (33), and, after trigonometric simplifications have been performed, the coefficients of the resonant terms, $\sin\tau$ and $\cos\tau$, are equated to zero. This results in Eq. (11) for A^2 as well as an expression for k_2 (cf. Eq. (26)) which does not concern us here. (Note that $A = \epsilon\widehat{A}$ from Eqs. (10), (22) and (34), and $\mu = \epsilon^2\hat{\mu}$ from (23). The perturbation method gives $\widehat{A}^2$ as a function of $\hat{\mu}$, but multiplication by ϵ^2 converts to a relation between A^2 and μ.)

4. Example 2

As a second example, we consider the case in which the quantity Q in Eqs. (11) and (13) is zero. In the context of the ODE system (1) and (2) this case corresponds to $S = 0$ in Eq. (4) and has been discussed in [3], Section 7.1. To generate such an example for the DDE (6), we embed the previous example in a one-parameter family of DDE's:

$$\frac{dx}{dt} = -x - 2x_d - x x_d - \lambda x^3 \tag{37}$$

and we choose λ so that $Q = 0$ in Eq. (11). This leads to the following critical value of λ:

$$\lambda = \lambda_{cr} = \frac{4\pi + 3\sqrt{3}}{18(2\pi + 3\sqrt{3})} = 0.0859 \tag{38}$$

Since this choice for λ leads to $Q = 0$, Eq. (11) obviously cannot be used to find the limit cycle amplitude A. Instead we use Lindstedt's method, maintaining terms of $O(\epsilon^4)$. The correct scalings in this case turn out to be (cf. Eqs. (23) and (26)):

$$T = T_{cr} + \mu = \frac{2\pi}{3\sqrt{3}} + \epsilon^4 \hat{\mu} \tag{39}$$

$$\Omega = \omega + \epsilon^2 k_2 + \epsilon^4 k_4 + \cdots \tag{40}$$

We find that the limit cycle amplitude A satisfies the equation:

$$A^4 = -\Gamma\mu \tag{41}$$

where we omit the closed form expression for Γ and give instead its approximate value, $\Gamma = 620.477$.

The analysis of this example has assumed that the parameter λ exactly takes on the critical value given in Eq. (38). Let us consider a more robust model which allows λ to be detuned:

$$\lambda = \lambda_{cr} + \Lambda = \frac{4\pi + 3\sqrt{3}}{18(2\pi + 3\sqrt{3})} + \epsilon^2 \widehat{\Lambda} \tag{42}$$

Using Lindstedt's method we obtain for this case the following equation on A:

$$A^4 + \sigma\Lambda A^2 + \Gamma\mu = 0 \tag{43}$$

where we omit the closed form expression for σ and give instead its approximate value, $\sigma = 342.689$. Eq. (43) can have 0, 1, or 2 positive real roots for A, each of which is a limit cycle in the original system. Thus the number of limit cycles which are born in the Hopf bifurcation depends on the detuning coefficients Λ and μ. Elementary use of the quadratic formula and the requirement that A^2 be both real and positive gives the following results: If $\mu < 0$ then there is one limit cycle. If $\mu > 0$ and $\sigma\Lambda < -2\sqrt{\Gamma\mu}$ then there are two limit cycles. If $\mu > 0$ and $\sigma\Lambda > -2\sqrt{\Gamma\mu}$ then there are no limit cycles.

5. Discussion

Although Lindstedt's method is a formal perturbation method, i.e., lacking a proof of convergence, our experience is that it gives the same results as the center manifold approach, which has a rigorous mathematical foundation. The center manifold approach has been described in many places, for example, [1,4,5,7,8]. Since the DDE (6) is infinite dimensional (for example, the characteristic equation of the linear DDE (7) is transcendental rather than polynomial, and hence has an infinite number of complex roots), the center manifold approach involves decomposing the original function space into a two dimensional center manifold (in which the Hopf bifurcation takes place) and an infinite dimensional function space representing the rest of the original phase space. The center manifold procedure is much more complicated than the Hopf calculation. Stepan refers to the center manifold calculation as "long and tedious" ([8, p. 112]), and Campbell et al. refer to it as "algebraically daunting" ([1, p. 642]). In [7], Chapter 14, 2 pages are spent explaining the application of Lindstedt's method to DDE's, whereas 10 pages are required for explanation of the center manifold approach. Thus, the main advantage of the Hopf calculation is that it is simpler to understand and easier to execute than the center manifold approach.

The idea of using Lindstedt's method on bifurcation problems in DDE goes back to a 1980 paper by Casal and Freedman [2]. That work provided the algorithm but not the Hopf bifurcation formula. It is hoped that having a general expression for the Hopf bifurcation, as in Eqs. (11)–(13), will be a convenience for researchers in DDE.

Acknowledgments

The authors wish to thank Dr. Tamas Kalmar-Nagy for telling us about Ref. [2].

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