COMPLEX NUMBERS

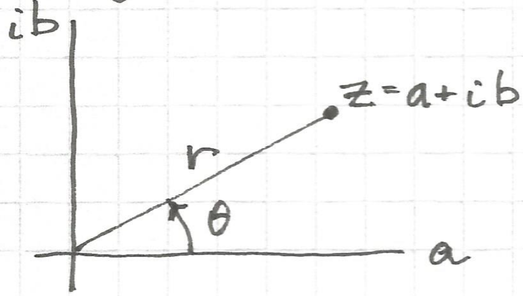
$$i = \sqrt{-1}$$

A complex number is of the form $z = a + ib$

The real part of z is a.
The imaginary part of z is b.

(Note that the imaginary part is real.)

It is frequently useful to use polar coordinates:



$$a = r\cos\theta$$
$$b = r\sin\theta$$

$$z = a + ib = r(\cos\theta + i\sin\theta)$$

The polar representation can be shortened by using Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This formula may be derived by comparing the Taylor series expressions for e^x, $\sin x$ and $\cos x$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Now $e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

In his book "Introductio in Analysin Infinitorum" (which could be translated as "Intro to Infinite Processes," and which is a precalculus treatment of infinite series, infinite products and (infinite) continued fractions), Euler wrote that $\sin^2 z + \cos^2 z = 1$ could be factored as $(\cos z + i\sin z)(\cos z - i\sin z) = 1$

or $e^{iz} \cdot e^{-iz} = 1$

which shows that $\cos z + i\sin z$ follows the law of exponents

(i.e. $e^A \cdot e^B = e^{A+B}$)

and justifies the exponential notation e^{iz}

(since $e^{iz} \cdot e^{-iz} = e^{iz-iz} = e^0 = 1$).

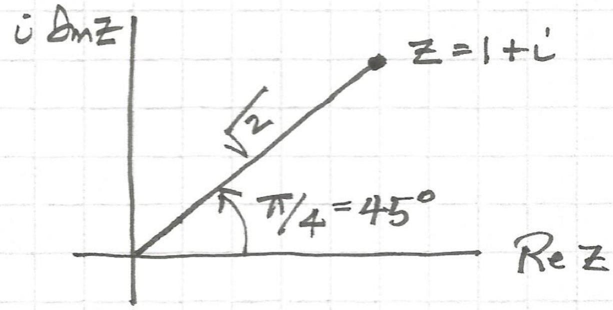
In particular we have the striking results:

$$e^{i\pi} = -1, \quad e^{2\pi i} = 1$$

Now a complex number can be written

$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Example $\qquad z = 1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}}$



Euler's formula can be used to derive various trig identities:

$$e^{i(x+y)} = e^{ix} e^{iy}$$

$$\cos(x+y) + i\sin(x+y) = (\cos x + i\sin x)(\cos y + i\sin y)$$
$$= \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \cos x\sin y)$$

Equating real and imaginary parts,

$\cos(x+y) = \cos x\cos y - \sin x\sin y$, and

$\sin(x+y) = \sin x\cos y + \cos x\sin y$

Euler's formula can be used to evaluate various integrals:

$$I = \int e^x \sin x\, dx = \operatorname{Im} \int e^x e^{ix}\, dx$$

$(\text{since } \sin x = \operatorname{Im} e^{ix})$

$$I = \operatorname{Im} \int e^{(1+i)x}\, dx = \operatorname{Im}\left[\frac{e^{(1+i)x}}{1+i}\right]$$

$$= \operatorname{Im}\left\{\left(\frac{e^{(1+i)x}}{1+i}\right)\left(\frac{1-i}{1-i}\right)\right\} = \operatorname{Im} \frac{e^{(1+i)x} - i e^{(1+i)x}}{2}$$

$$= \operatorname{Im}\left(\frac{e^x}{2}\left[e^{ix} - i e^{ix}\right]\right) = \frac{e^x}{2}(\sin x - \cos x)$$

In the foregoing calculation, we converted a term

$$\frac{1}{1+i}$$, which is not in the form $a+ib$,

to that form by multiplying by its <u>conjugate</u>, $1-i$

$$\frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{2}$$

The conjugate of $z = a+ib$ is denoted $\bar{z} = a-ib$

and we have $\quad z\bar{z} = a^2+b^2$ which is real.

Euler's formula may be used to derive <u>de Moivre's formula</u>:

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

If n is an integer, this formula is straightforward. However, if n is a fraction, then z^n takes on multiple values. For example

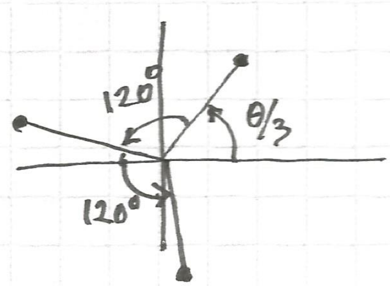
$$z^{\frac{1}{3}} = r^{1/3} e^{i\frac{\theta}{3}}$$

Now if we wanted to write this in the standard form

$$z^{1/3} = Re^{i\phi}$$

then we would have $R = r^{1/3} = \sqrt[3]{r}$ and

$$\phi = \frac{\theta}{3}, \frac{\theta+2\pi}{3}, \frac{\theta+4\pi}{3}$$



<u>logarithms</u> are defined as the inverse of the exponential function:

Let $z = re^{i\theta}$ and $w = u + iv$

Then $w = \log z$ means that $z = e^w$

That is $z = re^{i\theta} = e^{u+iv} = e^w$

$$re^{i\theta} = e^u e^{iv}$$

which gives

$e^u = r$ or $u = \log r$ (these are real, defined in the usual way)

and

$$v = \theta + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots)$$

So we have $\log(re^{i\theta}) = \log r + i(\theta + 2n\pi)$

The log function takes on infinitely many values!

<u>Example</u> Compute $\log(1+i)$

First write $1+i$ in polar form $1+i = \sqrt{2}\, e^{i\frac{\pi}{4}} = re^{i\theta}$

Then $\log(1+i) = \log r + i(\theta + 2n\pi)$

$$= \log\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)$$

<u>Example</u> Compute i^i

Use $i = e^{\log i}$. Then $i^i = e^{i \log i}$

But $\log i = \log e^{i\frac{\pi}{2}} = i\left(\frac{\pi}{2} + 2n\pi\right)$

$$\therefore\ i^i = e^{i^2\left(\frac{\pi}{2} + 2n\pi\right)} = e^{-\left(\frac{\pi}{2} + 2n\pi\right)}$$