COMPLEX ROOTS OF THE CHARACTERISTIC EQUATION

The general topic of this and the next lecture is the homogeneous ODE:

$$a y'' + b y' + c y = 0$$

An example of a 2nd order linear constant coefficient homogeneous ODE.

Look for a solution in the form:

$$y = e^{rt}$$

which gives the "characteristic equation":

$$a r^2 + b r + c = 0$$

In this lecture we shall be interested in the case that r is complex. In the next lecture we will be interested is the case that r is repeated.

Example

$$y'' + y = 0$$

"The Simple Harmonic Oscillator"

$$y = e^{rt}$$

$$r^2 + 1 = 0$$

$$r = \pm\sqrt{-1} = \pm i$$

$$\Rightarrow \quad y = c_1 e^{it} + c_2 e^{-it}$$

(the general solution)

Use Euler's formula

$$e^{it} = \cos t + i\sin t$$

$$e^{-it} = \cos t - i\sin t$$

$$y = c_1(\cos t + i\sin t) + c_2(\cos t - i\sin t)$$

$$= (c_1 + c_2)\cos t + (ic_1 - ic_2)\sin t$$

Let us rename the arbitrary constants c_1, c_2 so that

$$y = A\cos t + B\sin t$$

where we expect A, B to be real

(since the original problem $y'' + y = 0$, plus two IC,

$$y(0) = y_0, \quad y'(0) = y_0'$$

involve all real quantities.)

This requires that

$$c_1 + c_2 = A, \quad \underbrace{ic_1 - ic_2 = B}_{c_1 - c_2 = -iB}$$

which gives
$$c_1 = \frac{1}{2}(A - iB)$$
$$c_2 = \frac{1}{2}(A + iB)$$

Thus in the original form of the solution,

$$y = c_1 e^{it} + c_2 e^{-it}$$

both c_1 and c_2 need to be complex,

and in fact complex conjugates.

(This so that the final expression for y is real.)

Thus when we encounter the ODE

$$y'' + y = 0$$

we may go directly to the real expression

$$y = A\cos t + B\sin t$$

where A, B are real arbitrary constants.

<u>Another example</u>

$$y'' + y' + y = 0$$

$$y = e^{rt}$$

$$r^2 + r + 1 = 0 \quad \text{characteristic equation}$$

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

general solution: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$$= c_1 e^{-\frac{1}{2}t + \frac{\sqrt{3}}{2}it} + c_2 e^{-\frac{1}{2}t - \frac{\sqrt{3}}{2}it}$$

$$= e^{-\frac{1}{2}t}\left[c_1 e^{\frac{\sqrt{3}}{2}it} + c_2 e^{-\frac{\sqrt{3}}{2}it}\right]$$

Note that $e^{-\frac{1}{2}t}$ is real, and the rest of the solution will be real if we take c_1 and c_2 to be complex conjugates (just as in the previous example)

So if we write $c_1 = \frac{1}{2}(A - iB)$, $c_2 = \frac{1}{2}(A + iB)$,

we get $\quad y = e^{-\frac{1}{2}t}\left(A\cos\frac{\sqrt{3}}{2}t + B\sin\frac{\sqrt{3}}{2}t\right)$

in which each quantity is real.

An example with initial conditions

page 125, problem 16

$$3y'' - y' + 2y = 0, \quad y(0) = 2, \; y'(0) = 0$$

$$y = e^{rt}$$

$$3r^2 - r + 2 = 0$$

$$r = \frac{1 \pm \sqrt{1-24}}{6} = \frac{1}{6} \pm \frac{\sqrt{23}}{6} i$$

Based on the previous example, we can go directly to the general solution

$$y = e^{\frac{t}{6}}\left(A \cos \frac{\sqrt{23}}{6} t + B \sin \frac{\sqrt{23}}{6} t\right)$$

Now use the IC to find the arbitrary constants A, B:

$$y(0) = 2 = A$$

To apply $y'(0) = 0$ we need to compute $y'(t)$:

$$y'(t) = \frac{1}{6} e^{t/6}\left(A \cos \frac{\sqrt{23}}{6} t + B \sin \frac{\sqrt{23}}{6} t\right)$$
$$+ e^{t/6} \frac{\sqrt{23}}{6}\left(-A \sin \frac{\sqrt{23}}{6} t + B \cos \frac{\sqrt{23}}{6} t\right)$$

$$y'(0) = 0 = \frac{1}{6} A + \frac{\sqrt{23}}{6} B$$

But $A = 2 \Rightarrow B = \frac{-2}{\sqrt{23}}$

so $$\boxed{y(t) = e^{\frac{t}{6}}\left(2 \cos \frac{\sqrt{23}}{6} t - \frac{2}{\sqrt{23}} \sin \frac{\sqrt{23}}{6} t\right)}$$

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>> syms y t
>> y=exp(t/6)*(2*cos(sqrt(23)*t/6)-2/sqrt(23)*sin(sqrt(23)*t/6));
>> ezplot(y,[0 25 -110 110]);
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