REPEATED ROOTS OF THE CHARACTERISTIC EQUATION

We have seen that the general solution of the 2^{nd} order linear homogeneous ODE

$$a(t)\, y'' + b(t)\, y' + c(t)\, y = 0 \qquad (1)$$

[not necessarily constant coefficients]

is of the form

$$y = c_1 f(t) + c_2 g(t)$$

where c_1 and c_2 are arbitrary constants, and where $f(t)$ and $g(t)$ are linearly <u>independent</u> solutions of (1).

[Two solutions $f(t)$ and $g(t)$ are said to be linearly independent if one of them is <u>NOT</u> a constant multiple of the other.]

<u>Example</u>

$$y'' - 2y' + y = 0$$

Seek a solution of this <u>constant coefficient</u> ODE

in the form $\quad y = e^{rt}$

giving $\quad r^2 - 2r + 1 = 0$

which factors to $(r-1)(r-1) = 0 \Rightarrow r = 1, 1$. repeated root

Here we have only 1 linearly independent solution

$$f(t) = e^t$$

$$y = c_1 e^t + c_2 \; ?$$

There are various methods available for finding a 2^{nd} linearly independent solution. We will look at two such methods.

<u>Method 1</u> Let's work backwards and design an ODE that has solutions e^t and $e^{(1+\varepsilon)t}$) and then let's take the limit as $\varepsilon \to 0$.

$$r = 1, 1+\varepsilon$$

$$(r-1)(r-1-\varepsilon) = r^2 - (2+\varepsilon)r + 1 + \varepsilon = 0$$

$$\Rightarrow y = c_1 e^t + c_2 e^{(1+\varepsilon)t}$$

is the general solution to $y'' - (2+\varepsilon)y' + (1+\varepsilon)y = 0$

Let's add some IC to this ODE:

$$y(0)=0, \quad y'(0)=1$$

Now let's use these IC to solve for c_1, c_2:

$$y(t) = c_1 e^t + c_2 e^{(1+\varepsilon)t}$$

$$y'(t) = c_1 e^t + c_2(1+\varepsilon) e^{(1+\varepsilon)t}$$

$$y(0) = c_1 + c_2 = 0 \qquad (2)$$

$$y'(0) = c_1 + c_2(1+\varepsilon) = 1 \qquad (3)$$

Subtract (2) from (3) $\Rightarrow$ $c_2 \varepsilon = 1 \Rightarrow c_2 = 1/\varepsilon$

Then (2) gives $c_1 = -c_2 = -1/\varepsilon$

We obtain $y(t) = \dfrac{-e^t + e^{(1+\varepsilon)t}}{\varepsilon}$

Now in the limit as $\varepsilon \to 0$, the numerator becomes

$$-e^t + e^t e^{\varepsilon t}$$
$$= e^t(-1 + e^{\varepsilon t}) \to 0$$

and the denominator $\to 0$.

Using L'Hospital's Rule in ε we get

$$\lim_{\varepsilon \to 0} y(t) = \lim_{\varepsilon \to 0} \frac{\frac{d}{d\varepsilon}\left(-e^t + e^{(1+\varepsilon)t}\right)}{\frac{d}{d\varepsilon}\, \varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{e^{(1+\varepsilon)t} \frac{d}{d\varepsilon}(1+\varepsilon)t}{1} = te^t$$

Thus te^t is a second linearly independent solution, in addition to e^t.

The general solution is $y = c_1 e^t + c_2 t e^t$

to the ODE $\quad y'' - 2y' + y = 0.$

Method 2

We have that $y'' - 2y' + y = 0$ (4)

has the solution $y = e^t$

So let's look for a second linearly independent solution in the form

$$y = e^t v(t) \qquad (5)$$

where $v(t)$ is to be found.

We find
$$y' = e^t v + e^t v' = e^t(v+v')$$
$$y'' = e^t(v+v') + e^t(v'+v'')$$
$$= e^t(v+2v'+v'')$$

Substituting into eq.(4)

$$e^t(v''+2v'+v-2v-2v'+v) = 0$$

$$e^t v'' = 0 \Rightarrow \frac{d^2v}{dt^2} = 0$$

$$\Rightarrow v = c_1 + c_2 t$$

Plugging this back into eq.(5), we get

$$y = e^t(c_1 + c_2 t)$$

This method, called "reduction of order" also works on equations with variable coefficients.

Example

The ODE $t^2 y'' - t(t+2)y' + (t+2)y = 0$ (6)

has the solution $y(t) = t$

Find a second linearly independent solution.

Look for soln in form $y = t\,v(t)$ (7)

$$y' = tv' + v$$

$$y'' = tv'' + 2v'$$

Substituting into (6):

$$t^2[tv'' + 2v'] - (t^2 + 2t)(tv' + v) + (t+2)tv = 0$$

$$t^3v'' + \cancel{2t^2v'} - t^3v' - \cancel{t^2v} - \cancel{2t^2v'} - \cancel{2vt} + \cancel{t^2v} + \cancel{2tv} = 0$$

$$t^3(v'' - v') = 0$$

So the unknown function $v(t)$ in eq. (7) must satisfy

$$v'' - v' = 0$$

Look for a solution in the form $v = e^{rt}$

$$r^2 - r = 0, \quad r = 0, 1 \Rightarrow v = 1, e^t$$

$\therefore$ a second linearly independent soln is

$$y = te^t$$

and the general solution is

$$y = c_1 t + c_2 te^t$$