SOLVING THE NONHOMOGENEOUS EQUATION

The general 2^{nd} order constant coefficient linear ODE:

$$a y'' + b y' + c y = f(t)$$

Look for a solution in the form

$$y = y_h + y_p$$

We have seen that $y_h = c_1 p(t) + c_2 q(t)$

is the form of the soln to the homogeneous eq. $(f = 0)$.

Now we look at how to find y_p.

Example 1

$$y'' - 5y' + 6y = e^t$$

Look for y_p in the form $y_p = Ae^t$

$$y_p' = Ae^t, \quad y_p'' = Ae^t$$

$$Ae^t - 5(Ae^t) + 6Ae^t = e^t$$

$$2Ae^t = e^t \Rightarrow A = \tfrac{1}{2}$$

$$\therefore y_p = \tfrac{1}{2} e^t$$

Example 2

$$\underbrace{y'' - 5y' + 6y}_{\text{Same as Example 1}} = e^{2t}$$

different RHS
RIGHT HAND SIDE

Try same guess as in Example 1

$$y_p = Ae^{2t}$$
$$y_p' = 2Ae^{2t}$$
$$y_p'' = 4Ae^{2t}$$

$$\underbrace{4Ae^{2t} - 5(2Ae^{2t}) + 6Ae^{2t}}_{0} = e^{2t}$$

What went wrong?

The RHS, e^{2t}, was part of y_h:

$$y_h'' - 5y_h' + 6y_h = 0$$

$$y_h = e^{rt}, \quad r^2 - 5r + 6 = 0$$
$$(r-2)(r-3) = 0$$
$$r = 2, 3$$

$$\therefore \ y_h = c_1 e^{2t} + c_2 e^{3t}$$

Same as the RHS

$\therefore$ Guessing $y_p = Ae^{2t}$ will give ZERO on the LHS

and no choice for A will balance e^{2t} on the RHS.

So the "natural" guess, $y_p = Ae^{2t}$,
fails here.

This is vaguely familiar: In the last lecture
we sought the general solution to this ODE:

$$y'' - 2y' + y = 0$$

We set $y = e^{rt} \Rightarrow r^2 - 2r + 1 = 0$

$$\Rightarrow (r-1)^2 = 0, \quad r = 1, 1$$

We have

$$y = c_1 e^t + c_2 \; ???$$

The choice $y = e^t$ fails here.

We saw that the correct choice was $y = te^t$.

In the case of multiple roots of the characteristic eq.,
the second solution is te^{rt}.

It turns out that the same thing happens here!

In $y'' - 5y' + 6y = e^{2t}$, the correct guess for y_p is

$$y_p = Ate^{2t}$$

Proof: $y_p' = A(2te^{2t} + e^{2t})$

$y_p'' = A(4te^{2t} + 2e^{2t} + 2e^{2t})$

plug in to ODE: $A(4te^{2t} + 4e^{2t}) - 5A(2te^{2t} + e^{2t}) + 6Ate^{2t} = e^{2t}$

gives $A(4t - 10t + 6t)e^{2t} + (4-5)e^{2t}A = e^{2t}$

$\Rightarrow A = -1$ and $\boxed{y_p = -te^{2t}}$

Example 3

$$y'' - 2y' + y = e^t \qquad (1)$$

Here the homogeneous solution is

$$y_h = c_1 e^t + c_2 t e^t$$

So guessing $y_p = Ae^t$ and $y_p = Ate^t$ both fail (because these guesses give ZERO when plugged into the LHS of (1).)

What is the correct form of the guess for y_p?

You guessed it! $y_p = At^2 e^t$

$$y_p' = Ae^t(t^2+2t)$$

$$y_p'' = Ae^t(t^2+4t+2)$$

$$y_p'' - 2y_p' + y_p = 2Ae^t = e^t \Rightarrow A = \frac{1}{2}$$

$$\boxed{y_p = \frac{1}{2}t^2 e^t}$$

Example 4

$$y'' - 2y' + y = te^t$$

Based on example 3, let's try

$$y_p = At^3e^t$$

$$y_p' = Ae^t(t^3+3t^2)$$

$$y_p'' = Ae^t(t^3+6t^2+6t)$$

$$y_p'' - 2y_p' + y_p = 6Ate^t = te^t \Rightarrow A = \frac{1}{6}$$

$$\boxed{y_p = \frac{1}{6}t^3e^t}$$

Example 5

Returning to examples 1 and 2,

$$y'' - 5y' + 6y = te^{2t}$$

What shall we try?

Based on examples 2, 3, 4, we guess

$$y_p = At^2e^{2t}$$
$$y_p' = Ae^{2t}(2t^2+2t)$$
$$y_p'' = Ae^{2t}(4t^2+8t+2)$$

which gives

$$y_p'' - 5y_p' + 6y_p = \underbrace{Ae^{2t}(-2t+2)}_{\text{BAD NEWS!}}$$

No choice of A will yield te^{2t}

It turns out the correct form for y_p is

$$y_p = At^2e^{2t} + Bte^{2t}$$
$$y_p' = e^{2t}\left(A(2t^2+2t) + B(2t+1)\right)$$
$$y_p'' = e^{2t}\left(A(4t^2+8t+2) + B(4t+4)\right)$$
$$y_p'' - 5y_p' + 6y_p = e^{2t}\left(A(-2t+2) - B\right) = te^{2t}$$
$$\Rightarrow \quad -2A = 1, \quad 2A - B = 0$$
$$\Rightarrow \quad A = -\tfrac{1}{2}, \quad B = -1$$

So $\boxed{y_p = e^{2t}\left(-\tfrac{1}{2}t^2 - t\right)}$

The general rule for choosing the form of y_p is given in the text on p. 141.

Here is a summary: For the ODE

$$a y'' + by' + cy = t^N e^{\alpha t}$$

the correct choice for y_p is

$$y_p = (At^N + Bt^{N-1} + \cdots + Kt + L)t^M e^{\alpha t}$$

where M = number of times α appears as a root of the characteristic eq.

This form will always work, but some of the coefficients may be zero.

Example 5 revisited

$$y'' - 5y' + 6y = te^{2t}$$

Here $N=1$ and $M=1$ $\Rightarrow$

$$y_p = (At + B)te^{2t}$$

which agrees with our answer of

$$y_p = \left(-\tfrac{1}{2}t^2 - t\right)e^{2t}$$

Example 4 revisited

$$y'' - 2y' + y = te^t$$

Here $N = 1$, $M = 2$ (because $r = 1$ is a double root of $r^2 - 2r + 1 = 0$)

Thus

$$y_p = (At + B)\,t^2 e^t$$

which agrees with our solution of

$$y_p = \frac{1}{6} t^3 e^t$$

where B turns out to be zero.