MORE ON SOLVING THE NONHOMOGENEOUS EQUATION

Here's a review problem based on the last lecture:

$$y''' - 4y'' + 4y' = t^2 e^{2t} \qquad (1)$$

Find the general solution.

Since it is 3^{rd} order, expect

$$y = y_h + y_p$$

where $y_h = c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t)$

Set

$y_h = e^{rt}$ and obtain

$$r^3 - 4r^2 + 4r = 0$$
$$r(r^2 - 4r + 4) = 0$$
$$r(r-2)^2 = 0$$
$$r = 0, 2, 2$$

$$\therefore \quad y_h = c_1 + c_2 e^{2t} + c_3 t e^{2t}$$

For a particular solution, expect

$$y_p = (At^2 + Bt + C)\, t^2 e^{2t} \qquad (2)$$

Where did this guess come from?

FROM THE END OF THE LAST LECTURE:

The general rule for choosing the form of y_p is given in the text on p. 141.

Here is a summary: For the ODE

$$a y'' + b y' + c y = t^N e^{\alpha t} \qquad (3)$$

the correct choice for y_p is

$$y_p = (At^N + Bt^{N-1} + \dots + Kt + L)\, t^M e^{\alpha t}$$

where M = number of times α appears as a root of the characteristic eq.

This form will always work, but some of the coefficients may be zero.

First of all notice that this rule applies to the general $\underline{n^{th}\text{ order}}$ linear constant coefficient ODE (not just the $\underline{2^{nd}\text{ order}}$ version as in eq. (3).)

For eq. (1), $M = 2$ and $N = 2$

so the above rule gives eq. (2).

Now in order to obtain values for A, B, C in eq. (2), we substitute (2) into (1), which gives:

$$e^{2t}\left[\underbrace{(4C+6B)}_{=0} + t\underbrace{(12B+24A)}_{=0} + t^2\underbrace{(24A)}_{=1}\right] = t^2 e^{2t}$$

which gives $A = \frac{1}{24}$, $B = -2A = -\frac{1}{12}$, $C = -\frac{3}{2}B = \frac{1}{8}$

that is, $\boxed{y_p = \left(\frac{t^2}{24} - \frac{t}{12} + \frac{1}{8}\right) t^2 e^{2t}}$

Another example

$$y'' - 5y' + 6y = \sin t \qquad (4)$$

Look for y_p in the form

$$y_p = A\sin t + B\cos t$$

Then $y_p' = A\cos t - B\sin t$

$$y_p'' = -A\sin t - B\cos t$$

Substituting these results in (4),

$$(-A\sin t - B\cos t) - 5(A\cos t - B\sin t)$$

$$+ 6(A\sin t + B\cos t) = \sin t$$

$$\underbrace{(-A + 5B + 6A)}_{=1}\sin t + \underbrace{(-B - 5A + 6B)}_{=0}\cos t = \sin t$$

We have

$$\left.\begin{array}{l} 5A + 5B = 1 \\ -5A + 5B = 0 \end{array}\right\} \quad A = B = \frac{1}{10}$$

$$\boxed{y_p = \frac{1}{10}\sin t + \frac{1}{10}\cos t}$$

This example may be solved in another way by using Euler's formula: $\sin t = \operatorname{Im} e^{it}$

$$y'' - 5y' + 6y = \operatorname{Im} e^{it}$$

Procedure to find y_p:

1. Drop the Im and solve

$$y'' - 5y' + 6y = e^{it} \quad \text{for } y_p \qquad (5)$$

2. Then take $\operatorname{Im} y_p$

So look for y_p in (5) in the form

$$y_p = A e^{it}$$
$$y_p' = iAe^{it}$$
$$y_p'' = -Ae^{it}$$

$$-Ae^{it} - 5iAe^{it} + 6Ae^{it} = e^{it}$$

$$5(1-i)A e^{it} = e^{it}$$

$$A = \frac{1}{5(1-i)} \frac{1+i}{1+i} = \frac{1+i}{10}$$

So we get

$$y_p = \operatorname{Im}\left(\frac{1+i}{10} e^{it}\right)$$

$$= \operatorname{Im}\left(\frac{1+i}{10}(\cos t + i \sin t)\right)$$

$$= \operatorname{Im}\left[\frac{\cos t - \sin t}{10} + i\left(\frac{\cos t + \sin t}{10}\right)\right]$$

$$\boxed{y_p = \frac{\cos t + \sin t}{10}}$$

which agrees with previous result

Another example

$$y'' - 2y' + 2y = e^t \cos t \qquad (6)$$

$$y = y_h + y_p$$

For y_h, set $y = e^{rt}$

$$y_h'' - 2y_h' + 2y_h = 0$$

$$r^2 - 2r + 2 = 0$$

$$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$y_h = c_1 e^{(1+i)t} + c_2 e^{(1-i)t}$$

$$= e^t \left(c_1 e^{it} + c_2 e^{-it} \right)$$

$$= e^t \left(k_1 \sin t + k_2 \cos t \right)$$

How shall we look for y_p?

Note that the RHS of (6) is in y_h.

Therefore write $y_p = (A e^t \cos t + B e^t \sin t)\, t$

or rather $y_p = \operatorname{Re}\left(C e^{(1+i)t}\, t \right)$

$$\left(\text{Since } e^t \cos t = \operatorname{Re}\left(e^{(1+i)t} \right). \right)$$

So we solve $y_p'' - 2y_p' + 2y_p = e^{(1+i)t}$

in the form $y_p = C e^{(1+i)t}\, t$

and then take the Re part.

$$y_p = c\,t\,e^{(1+i)t}$$

$$y_p' = c\,e^{(1+i)t}\left[t(1+i) + 1\right]$$

$$y_p'' = c\,e^{(1+i)t}\left[(1+i)\left[t+ti+1\right] + (1+i)\right]$$

Substituting into $y_p'' - 2y_p' + 2y_p = e^{(1+i)t}$ gives

$$c\,e^{(1+i)t}\left[(1+i)(t+ti+2) - 2\left[t+ti+1\right] + 2t\right] = e^{(1+i)t}$$

or

$$c\left[t\left[(1+i)^2 - 2(1+i) + 2\right] + \left[2(1+i) - 2\right]\right] = 1$$

$$c\left[t\underbrace{(2i - 2 - 2i + 2)}_{=0} + 2i\right] = 1$$

$$2ic = 1 \Rightarrow c = \frac{1}{2i} = -\frac{i}{2}$$

$$\therefore \quad y_p = \mathrm{Re}\left(-\frac{i}{2}\,t\,\underbrace{e^{(1+i)t}}_{e^t(\cos t + i\sin t)}\right)$$

$$= \mathrm{Re}\left(\frac{te^t}{2}\left[-i\cos t + \sin t\right]\right)$$

$$\boxed{y_p = \frac{te^t \sin t}{2}}$$