VARIATION OF PARAMETERS

Recall that in section 3.4, page 131, we had a method of determining a second, linearly independent solution to a homogeneous linear ODE (given a solution), the method known as "reduction of order".

Example
$$y'' - 2y' + y = 0 \qquad (1)$$

Given that $y = e^t$ is a solution, find a second solution

Answer: Set
$$y = e^t v(t) \qquad (2)$$

where $v(t)$ is as yet unknown, to be found.

Substituting (2) into (1), we get

$$y' = e^t v + e^t v' = e^t(v+v')$$
$$y'' = e^t(v+v') + e^t(v'+v'')$$
$$= e^t(v+2v'+v'')$$

$$y''-2y'+y=0 \Rightarrow$$
$$e^t\left[(v+2v'+v'')-2(v+v')+v\right]=0$$

Thus $v(t)$ must satisfy $v''=0 \Rightarrow v=c_1+c_2t$

And this gave us the result that $y=te^t$ was a solution to (1) [repeated root case].

This lecture concerns a method of determining a particular solution to a nonhomogeneous linear ODE, given the general solution of the associated homogeneous ODE. The method is called "variation of parameters", and is similar to reduction of order.

The method

We are given that the following linear ODE

$$y'' + b(t)\,y' + c(t)\,y = g(t)$$

has a known general solution to the homogeneous eq:

$$y_h = c_1 y_1(t) + c_2 y_2(t)$$

The method gives a formula for y_p:

$$y_p = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

where $u_1(t)$ and $u_2(t)$ are functions which depend on y_1, y_2:

(See p. 145 in text)

$$u_1(t) = -\int \frac{y_2(t)\, g(t)}{W(t)}\, dt\,, \quad u_2(t) = \int \frac{y_1(t)\, g(t)}{W(t)}$$

where

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$= \text{Wronskian}$$

Example

$$y'' - 2y' + y = te^t$$

We saw (earlier in this lecture) that

$$y_h = c_1 \underset{\substack{\shortparallel \\ y_1(t)}}{e^t} + c_2 \underset{\substack{\shortparallel \\ y_2(t)}}{te^t}$$

$$y_p = u_1 y_1 + u_2 y_2$$

where $W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{vmatrix}$

$$= e^{2t}(1+t) - te^{2t} = e^{2t}$$

$$u_1(t) = -\int \frac{y_2 g}{W} dt = -\int \frac{(te^t)(te^t)}{e^{2t}} dt = -\int t^2 dt = -\frac{t^3}{3}$$

$$u_2(t) = \int \frac{y_1 g}{W} dt = \int \frac{(e^t)(te^t)}{e^{2t}} dt = \int t\, dt = \frac{t^2}{2}$$

Therefore a particular solution is

$$y_p = u_1 y_1 + u_2 y_2 = -\frac{t^3}{3} e^t + \frac{t^2}{2}(te^t)$$

$$\boxed{y_p = \frac{t^3}{6} e^t}$$

(which agrees with results previously obtained by the method of undetermined coefficients.)

Where do the formulas come from?

Given $$y'' + b(t)\, y' + c(t)\, y = g(t) \qquad (3)$$

with $y_h = c_1 y_1(t) + c_2 y_2(t)$

Assume an expression for y_p in the form:

$$y_p = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

Then $$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}$$

The method involves assuming this $= 0$

$$u_1' y_1 + u_2' y_2 = 0 \qquad (4)$$

whereupon $$y_p' = u_1 y_1' + u_2 y_2'$$

Differentiating y_p', we get

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Plugging into the ODE (3),

$$\left[u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' \right]$$
$$+ b(t) \left[u_1 y_1' + u_2 y_2' \right]$$
$$+ c(t) \left[u_1 y_1 + u_2 y_2 \right] = g(t)$$

That is $u_1(t) \left[y_1'' + b(t)\, y_1' + c(t)\, y_1 \right] \leftarrow$ this term is zero since y_1 is in y_h

$+ u_2(t) \left[y_2'' + b\, y_2' + c y_2 \right] \leftarrow$ this term is zero too

$$+ u_1' y_1' + u_2' y_2' = g(t)$$

So we have

$$u_1' y_1 + u_2' y_2 = 0 \qquad \text{(from eq. (4))}$$
$$u_1' y_1' + u_2' y_2' = g$$

Multiply the first of these by y_2' and the second by y_2 and subtract

$$u_1' \underbrace{(y_1 y_2' - y_2 y_1')}_{W(t)} = -y_2 g$$

Thus we get $\quad u_1' = \dfrac{du_1}{dt} = \dfrac{-y_2(t) g(t)}{W(t)}$

Integrating, $\quad u_1(t) = -\displaystyle\int \frac{y_2(t) g(t)}{W(t)} dt$

and a similar expression for $u_2(t)$.

Another example p. 146 #4

$$y'' + y = \tan t$$

Here $y_h = \underset{\substack{\| \\ y_1}}{c_1 \sin t} + \underset{\substack{\| \\ y_2}}{c_2 \cos t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -\sin^2 t - \cos^2 t$$

$\therefore W = -1$

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1 = -\int \frac{y_2 g}{W} = \int (\cos t)(\tan t)\,dt = \int \sin t\,dt = -\cos t$$

$$u_2 = \int \frac{y_1 g}{W} = -\int (\sin t)(\tan t)\,dt$$

$$= -\int \frac{\sin^2 t}{\cos t}\,dt = -\int \frac{1-\cos^2 t}{\cos t}\,dt$$

$$= \int \left(\cos t - \frac{1}{\cos t}\right)\,dt$$

$$= \sin t - \log\left(\tan t + \frac{1}{\cos t}\right)$$

where we use $\int \frac{dt}{\cos t} = \log\left(\tan t + \frac{1}{\cos t}\right)$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= -\cos t \sin t + \left[\sin t - \log\left(\tan t + \frac{1}{\cos t}\right)\right]\cos t$$

$$\boxed{y_p = -\cos t \log\left(\tan t + \frac{1}{\cos t}\right)}$$