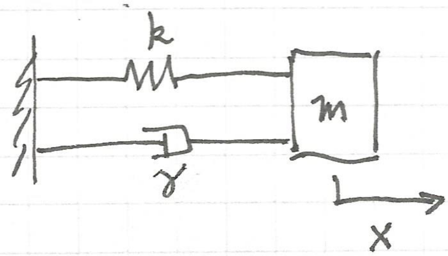
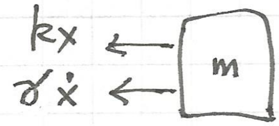
FREE VIBRATIONS



Free body diagram



$F = ma \Longrightarrow \qquad m\ddot{x} = -kx - \gamma\dot{x}$

$$m\ddot{x} + \gamma\dot{x} + kx = 0$$

$$x = e^{rt}$$

$$mr^2 + \gamma r + k = 0$$

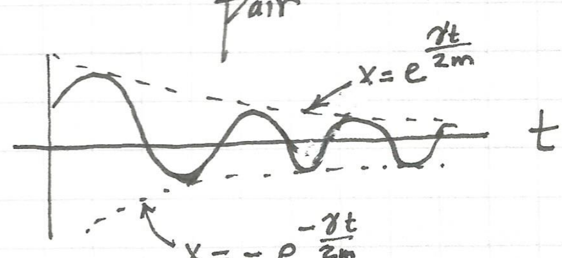
$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

The form of the solution depends on the sign of $\gamma^2 - 4mk$

overdamped $\gamma^2 - 4mk > 0 \Rightarrow$ 2 real roots, $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

critically damped $\gamma^2 - 4mk = 0 \Rightarrow$ double root, $x = (c_1 + c_2 t)\, e^{-\frac{\gamma}{2m} t}$

under-damped $\gamma^2 - 4mk < 0 \Rightarrow$ complex conjugate pair, $x = e^{-\frac{\gamma t}{2m}}(c_1 \sin\omega t + c_2 \cos\omega t)$



$$\omega = \frac{\sqrt{4mk - \gamma^2}}{2m}$$

$$A\cos\omega t + B\sin\omega t = R\cos(\omega t-\theta)$$

$$\cos(\omega t-\theta) = \cos\omega t\cos\theta + \sin\omega t\sin\theta$$

$$\therefore\quad A = R\cos\theta$$

$$B = R\sin\theta$$

$$A^2+B^2=R^2 \text{ since } \cos^2\theta+\sin^2\theta=1$$

$$R=\sqrt{A^2+B^2}$$

$$\tan\theta = \frac{B}{A}$$

$$\theta = \arctan\frac{B}{A}$$

<u>Example</u>

$$\cos t + 2\sin t = R\cos(t-\theta)$$

$$R=\sqrt{1+4} = \sqrt{5}$$

$$\theta = \arctan 2$$

$$mu'' + \gamma u' + ku = F_0 \cos\omega t$$

$$u_p = A\cos\omega t + B\sin\omega t$$

$$u_p' = -A\omega\sin\omega t + B\omega\cos\omega t$$

$$u_p'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$m(-A\omega^2\cos\omega t - B\omega^2\sin\omega t)$$

$$+\gamma(-A\omega\sin\omega t + B\omega\cos\omega t)$$

$$+k(A\cos\omega t + B\sin\omega t) = F_0\cos\omega t$$

Collect terms:

(1) $\cos\omega t: \ -A\omega^2 m + B\omega\gamma + AK = F_0$

(2) $\sin\omega t: \ -B\omega^2 m - A\omega\gamma + BK = 0$

Multiply (1) by A, (2) by B, and add:

(3) $$-R^2\omega^2 m + R^2 K = AF_0$$

Multiply (1) by B, (2) by A, and subtract:

(4) $$R^2\omega\gamma = BF_0$$

Square eq. (3) and add to the square of eq. (4):

$$R^4\left((K-\omega^2 m)^2 + \omega^2\gamma^2\right) = R^2 F_0^2$$

or, cancelling an R^2 and taking the square root:

$$R = \frac{F_0}{\Delta}, \quad \Delta = \sqrt{(K-\omega^2 m)^2 + \omega^2\gamma^2}$$

Thus we have shown that the "steady state" solution of

$$mu'' + \gamma u' + ku = F_0 \cos\omega t$$

is $u = R\cos(\omega t - \theta)$

where $R = \dfrac{F_0}{\Delta}$

where $\Delta = \sqrt{(k-\omega^2 m)^2 + \omega^2\gamma^2}$

Note: "steady state" means the solution after the homogeneous solution has decayed to zero.