FORCED VIBRATIONS WITH NO DAMPING

$$m\ddot{x} + kx = F_0 \cos \omega t \qquad (1)$$

$$x = x_h + x_p$$

$$x_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t \quad , \quad \omega_0 = \sqrt{\frac{k}{m}}$$

$$x_p = A \cos \omega t$$

$$m\ddot{x}_p + kx_p = m(-\omega^2 A)\cos \omega t + kA\cos \omega t = F_0 \cos \omega t$$

$$\Rightarrow (-\omega^2 m + k)A = F_0$$

$$A = \frac{F_0}{m\left(-\omega^2 + \frac{k}{m}\right)} = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$\therefore \ x = x_h + x_p = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

Initial Conditions

Rest solution: $t = 0, \ x = x' = 0$

$$x(0) = c_1 + \frac{F_0}{m(\omega_0^2 - \omega^2)} = 0 \Rightarrow c_1 = \frac{-F_0}{m(\omega_0^2 - \omega^2)}$$

$$x'(t) = -c_1 \omega_0 \sin \omega_0 t + c_2 \omega_0 \cos \omega_0 t - \frac{F_0 \omega}{m(\omega_0^2 - \omega^2)} \sin \omega t$$

$$x'(0) = c_2 \omega_0 = 0 \Rightarrow c_2 = 0$$

$$\boxed{x = \frac{F_0}{m(\omega_0^2 - \omega^2)}\left(\cos\omega t - \cos\omega_0 t\right)}$$

Example Plot this for $\omega = 1$, $\omega_0 = 1.1$

```
>> syms t

>> x=cos(t)-cos(1.1*t)

x =

cos(t) - cos((11*t)/10)

>> ezplot(x,[0 200 -2.5 2.5])
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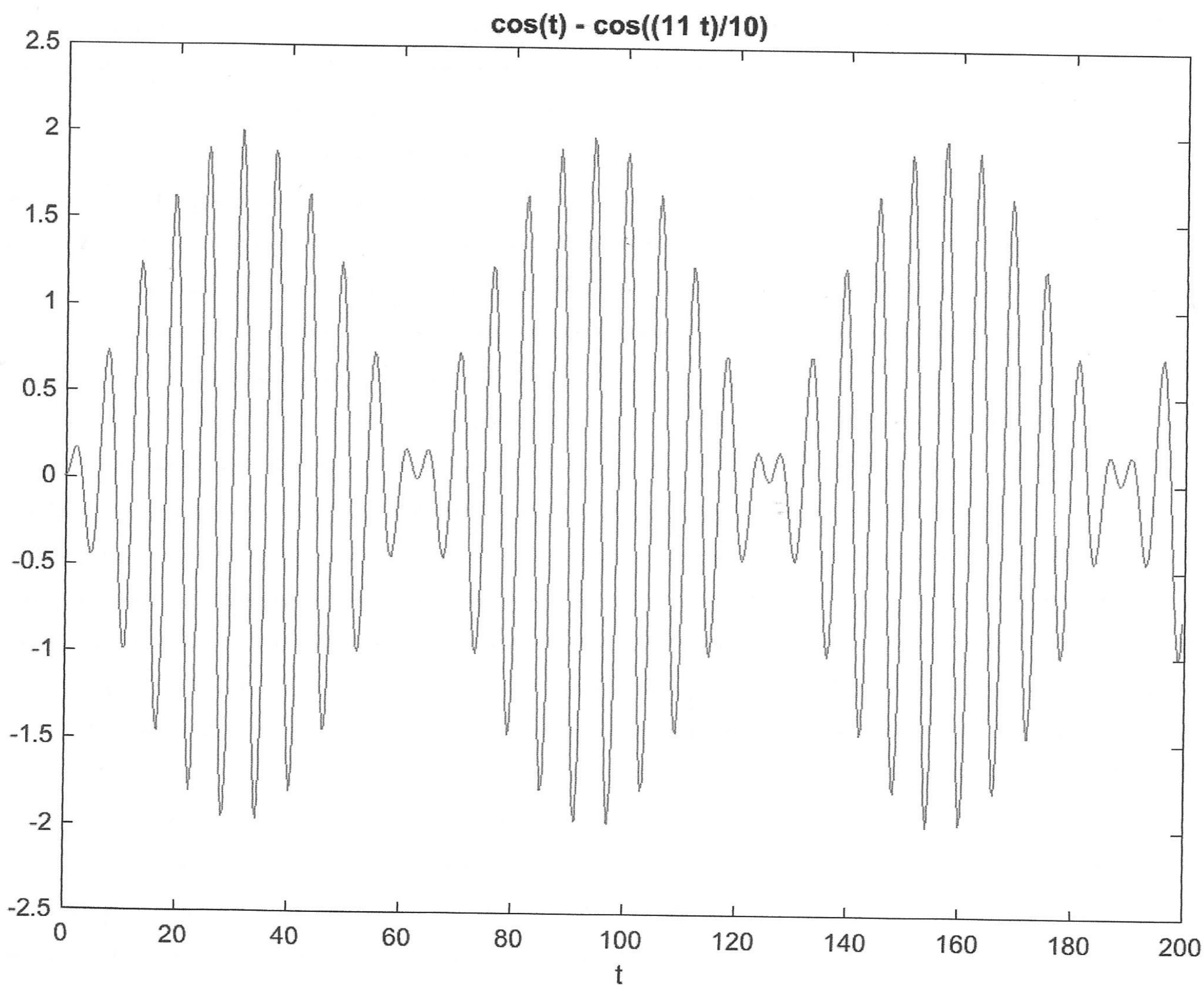


This striking shape is called "beats" and can be explained mathematically by using trig identities:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

Subtracting,

$$\underbrace{\cos(A-B)}_{\omega} - \underbrace{\cos(A+B)}_{\omega_0} = 2 \sin A \sin B$$

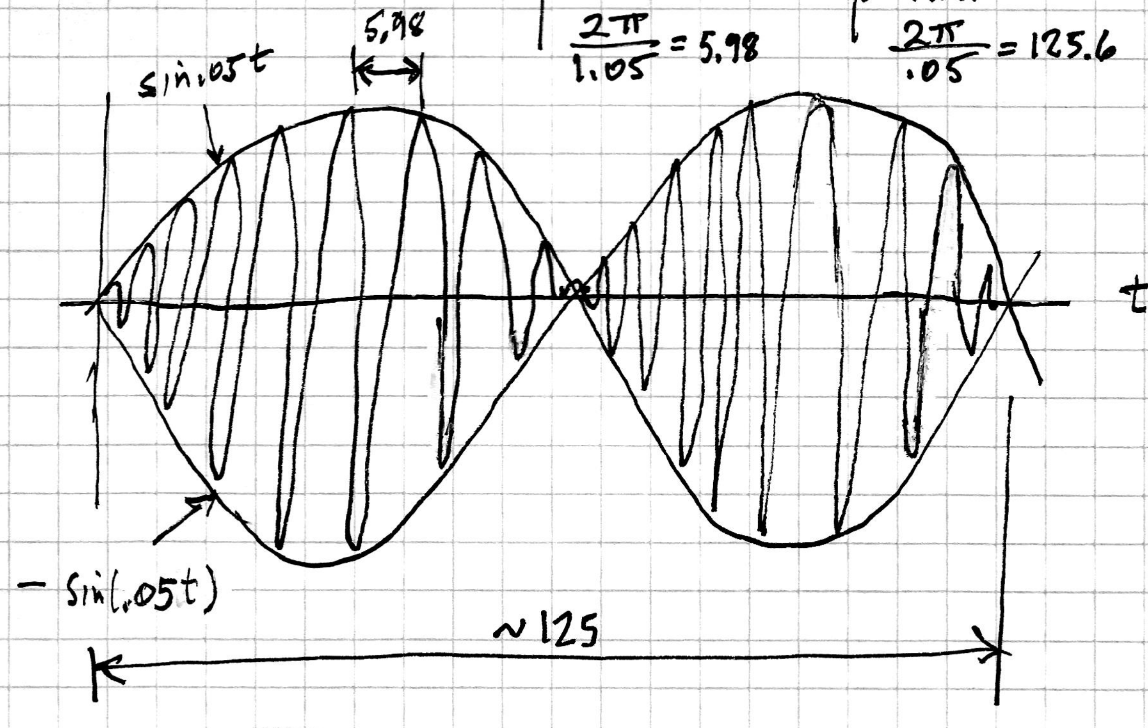
$$\left.\begin{array}{l} A-B=\omega \\ A+B=\omega_0 \end{array}\right\} \quad 2A=\omega+\omega_0, \quad A=\tfrac{1}{2}(\omega+\omega_0)$$

$$B=\omega_0-A=\tfrac{1}{2}(\omega_0-\omega)$$

Therefore, $\cos\omega t - \cos\omega_0 t = 2\sin\left[\frac{(\omega_0+\omega)}{2}t\right]\sin\left[\frac{(\omega_0-\omega)t}{2}\right]$

Example $\quad \omega=1, \ \omega_0=1.1$

$$\cos\omega t - \cos\omega_0 t = 2 \underbrace{\sin 1.05t}_{\substack{\text{period}= \\ \frac{2\pi}{1.05}=5.98}} \ \underbrace{\sin .05t}_{\substack{\text{period}= \\ \frac{2\pi}{.05}=125.6}}$$



The beats effect is more pronounced, the closer ω is to ω_0

So let's take the limit as $\omega \to \omega_0$

$$x = \frac{F_0}{m} \frac{\cos \omega t - \cos \omega_0 t}{\omega_0^2 - \omega^2}$$

In the limit $\omega \to \omega_0$, this is of the form $\frac{0}{0}$.
Use L'Hospital's Rule

$$x \to \frac{F_0}{m} \lim_{\omega \to \omega_0} \frac{\frac{d}{d\omega}(\cos \omega t - \cos \omega_0 t)}{\frac{d}{d\omega}(\omega_0^2 - \omega^2)}$$

$$= \frac{F_0}{m} \lim_{\omega \to \omega_0} \frac{(-\sin \omega t)(t)}{-2\omega}$$

$$= \frac{F_0}{m} \frac{t \sin \omega_0 t}{2\omega_0}$$

Note that this solution can be obtained directly from the ODE (1),

$$m\ddot{x} + kx = F_0 \cos \omega_0 t$$

the RHS is in the homogeneous soln

$$x_h = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

Look for x_p in the form $x_p = A t \cos \omega_0 t + B t \sin \omega_0 t$

$$\dot{x}_p = A \cos \omega_0 t - \omega_0 A t \sin \omega_0 t + B \sin \omega_0 t + \omega_0 B t \cos \omega_0 t$$

$$\ddot{x}_p = -2A\omega_0 \sin \omega_0 t - \omega_0^2 A t \cos \omega_0 t + 2B\omega_0 \cos \omega_0 t - \omega_0^2 B t \sin \omega_0 t$$

Substituting expressions for $x_p, \dot{x}_p, \ddot{x}_p$ into the ODE we see that there are 4 types of terms: $\cos\omega_0 t$, $\sin\omega_0 t$, $t\cos\omega_0 t$, $t\sin\omega_0 t$

$\cos\omega_0 t$: $m(2B\omega_0) + k\cdot 0 = F_0 \implies B = \dfrac{F_0}{2m\omega_0}$

$\sin\omega_0 t$: $m(-2A\omega_0) + k\cdot 0 = 0 \implies A = 0$

$t\cos\omega_0 t$: $m(-\omega_0^2 A) + kA = \underbrace{(-m\omega_0^2 + k)}_{=0} A = 0$ ✓ok

$t\sin\omega_0 t$: $m(-\omega_0^2 B) + kB = \underbrace{(-m\omega_0^2 + k)}_{=0} B = 0$ ✓ok

$$\therefore\ x_p = At\cos\omega_0 t + Bt\sin\omega_0 t$$
$$= 0 + \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t$$

✓OK with limit $\omega \to \omega_0$ method.

This phenomenon is called RESONANCE.