Higher Order linear ODE's

Let $L(y) = \frac{d^n y}{dt^n} + p_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1} \frac{dy}{dt} + p_n y$

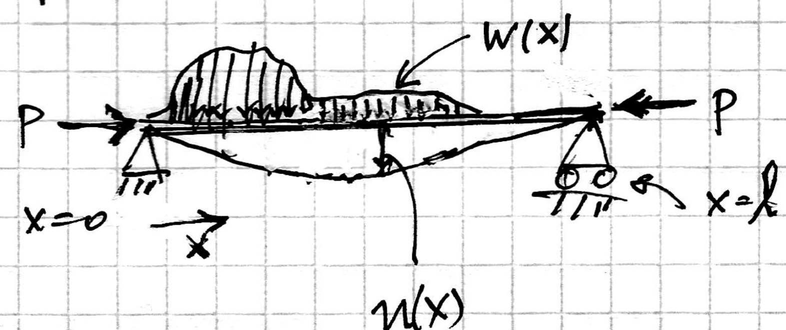
where the p_i's are coefficients.

Then
$$L(y) = g(t)$$
has its solution is of the form
$$y = y_h + y_p$$
where y_h satisfies $L(y_h) = 0$

and y_h includes n arbitrary constants,

and $L(y_p) = g(t)$

<u>Example</u> A beam-column (x instead of t here!)



$$EI \frac{d^4 u}{dx^4} + P \frac{d^2 u}{dx^2} = w(x)$$
EI = bending stiffness

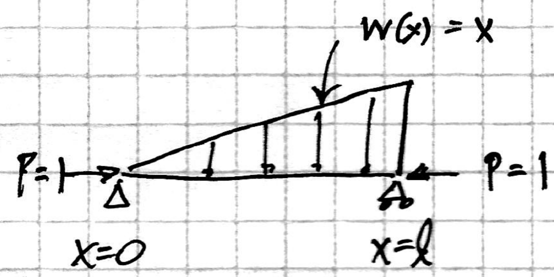
This ode comes with 4 side conditions (boundary conds):

$u = 0$ at $x = 0, l$

$\frac{d^2 u}{dx^2} = 0$ at $x = 0, l$

For convenience, take $EI=1$, $P=1$.

And let $w(x)=x$



$$u'''' + u'' = x$$

BC $x=0,\ u=0,\ u''=0$
$x=l,\ u=0,\ u''=0$

$$u = u_h + u_p$$

For u_h, let $u_h = e^{rx}$

$$u_h'''' + u_h'' = 0$$

$$r^4 + r^2 = 0$$

$$r^2(r^2+1) = 0 \Rightarrow r = 0, 0, \pm i$$

$$\therefore u_h = C_1 + C_2 x + C_3 \sin x + C_4 \cos x$$

For u_p, note that the RHS, $g(x) = x = x e^{0x}$

and that 0 is a double root of the r-equation

$\therefore$ Choose $u_p = Ax^3$

$$u_p' = 3Ax^2,\ u_p'' = 6Ax,\ u_p''' = 6A,\ u_p'''' = 0$$

$$u_p'''' + u_p'' = x \Rightarrow 0 + 6Ax = x \Rightarrow 6A = 1,\ A = \tfrac{1}{6}$$

So $u_p = \frac{1}{6}x^3$

Note that M=2 and N=1, so we should take u_p=(Ax+B)x^2.
But it turns out that B=0.

$$\therefore \; u = u_h + u_p = c_1 + c_2 x + c_3 \sin x + c_4 \cos x + \frac{1}{6} x^3$$

BC

$$\left.\begin{array}{l} x=0, \; u=0 \Rightarrow c_1 + c_4 = 0 \\ x=0, \; u''=0 \Rightarrow -c_4 + 0 = 0 \end{array}\right\} \quad \begin{array}{l} c_4 = 0 \\ c_1 = 0 \end{array}$$

$$x = l, \; u = 0 \Rightarrow \cancel{c_1} + c_2 l + c_3 \sin l + \cancel{c_4} \cos l + \frac{1}{6} l^3 = 0$$

$$x = l, \; u'' = 0 \Rightarrow -c_3 \sin l - \cancel{c_4} \cos l + l$$

$$\therefore \; c_3 = \frac{l}{\sin l}, \quad c_2 = -\frac{l^2}{6} - 1$$

Finally we obtain:

$$u = \left(-1 - \frac{l^2}{6}\right) x + l \frac{\sin x}{\sin l} + \frac{1}{6} x^3$$

The general rule for choosing the form of y_p is given in the text on p. 141.

Here is a summary: For the ODE

$$a y'' + b y' + c y = t^N e^{\alpha t} \qquad (3)$$

the correct choice for y_p is

$$y_p = (At^N + Bt^{N-1} + \cdots + Kt + L)\,t^M e^{\alpha t}$$

where M = number of times α appears as a root of the characteristic eq.

This form will always work, but some of the coefficients may be zero.

First of all notice that this rule applies to the general <u>n^{th} order</u> linear constant coefficient ODE (not just the <u>2^{nd} order</u> version as in eq. (3).)

A practice problem:

The ODE

$$x'' + w^2 x = \sin t \tag{1}$$

has the general solution

$$x(t) = c_1 \sin wt + c_2 \cos wt + \frac{\sin t}{w^2 - 1} \tag{2}$$

The ODE

$$x'' + x = \sin t \tag{3}$$

has the particular solution

$$x(t) = -\frac{t}{2} \cos t \tag{4}$$

Since eq.(1) approaches eq.(3) as $w \to 1$, we would expect that the solution (2) would approach solution (4) as $w \to 1$. And yet this does not seem to be the case. Explain.

The problem is that eq.(2) seems to be singular (i.e. it blows up) as $\omega \to 1$.

The answer is that the arbitrary constants, c_1, c_2 depend on ω.

Choose any two initial conditions for eq.(1),

Say $x(0) = A$, $\dot{x}(0) = B$ (where A, B are given constants)

Then $x(t) = c_1 \sin \omega t + c_2 \cos \omega t + \dfrac{\sin t}{\omega^2 - 1}$

$x(0) = A = c_2 \Rightarrow c_2 = A$

$\dot{x}(t) = c_1 \omega \cos \omega t - c_2 \omega \sin \omega t + \dfrac{\cos t}{\omega^2 - 1}$

$\dot{x}(0) = B = c_1 \omega + \dfrac{1}{\omega^2 - 1} \Rightarrow c_1 = \dfrac{B}{\omega} - \dfrac{1}{\omega(\omega^2 - 1)}$

$$x(t) = \frac{B \sin \omega t}{\omega} + A \cos \omega t + \frac{\omega \sin t - \sin \omega t}{\omega(\omega^2 - 1)}$$

Now let $w \to 1$ and use L'Hospital's Rule

$$\lim_{w\to 1} x(t) = \lim_{w \to 1} \; B\sin t + A\cos t + \frac{\frac{d}{dw}(w\sin t - \sin wt)}{\frac{d}{dw}(w^3 - w)}$$

$$= B\sin t + A\cos t + \lim_{w\to 1} \frac{\sin t - t\cos wt}{3w^2 - 1}$$

$$= B\sin t + A\cos t + \frac{\sin t - t\cos t}{2}$$

which gives the stated $x_p = -\frac{t}{2}\cos t$ for $B = -\frac{1}{2}, A = 0$