A closer look at Variation of Parameters

In a previous lecture we studied Variation of Parameters as a way of obtaining a particular solution to an ODE of this form:

$$y'' + by' + cy = g(t) \qquad (1)$$

where $y_h = c_1 y_1(t) + c_2 y_2(t)$ is given

where

$$y_h'' + b y_h' + c y_h = 0$$

The idea is to look for a solution in the form:

$$y_p = u_1(t) y_1(t) + u_2(t) y_2(t) \qquad (2)$$

where $u_1(t)$ and $u_2(t)$ are to be found.

The procedure is to substitute eq. (2) into eq. (1).

This will give a single equation on the two unknowns u_1 and u_2.

In order to solve for u_1 & u_2 we need a second equation (as in "two equations in two unknowns.")

The question I want to examine in these notes is how to get the second equation.

The standard approach (but <u>not</u> the <u>only</u> approach) is to take the derivative of y_p:

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}$$

and then choose this chunk to be zero

Then you continue to substitute y_p into eq. (1) and you get an equation on u_1 and u_2, which together with the above,

$$u_1' y_1 + u_2' y_2 = 0 \qquad (3)$$

gives you 2 eqs in 2 unknowns.

This is the way the text does it. See p. 145.

They show that after you substitute y_p into eq. (1), you get eq. (25) on p. 145:

$$u_1' y_1' + u_2' y_2' = g(t) \qquad (4)$$

You solve eqs. (3), (4) for u_1' and u_2' and then you integrate to go from u_1' to u_1 etc.

This procedure gives the formulas (27) on p. 145

$$u_1(t) = -\int \frac{y_2(t)\,g(t)}{W(t)}\,dt \qquad (5)$$

etc. for $u_2(t)$ (where $W = y_1 y_2' - y_2 y_1'$)

OK fine, the foregoing has been a review of the method of variation of parameters.

BUT as a student wrote to me in an email:

WHY DO WE CHOOSE

$$u_1' y_1 + u_2' y_2 = 0 \quad ?$$

Well first of all, it gives the nice formulas (5) (top of this page).

And secondly, it has stood the test of time, having been invented by LAGRANGE, one of the world's great mathematicians, who managed to live through and survive the French revolution.

BUT IT IS NOT THE ONLY CHOICE!

To illustrate that there are other ways to choose the 2nd equation on u_1 and u_2, let's look at an example:

$$y'' - y = e^t \qquad (6)$$

We know the homogeneous solution is:

$$y_h = c_1 e^t + c_2 e^{-t}$$

So we look for a particular solution in the form:

$$y_p = u_1(t) e^t + u_2(t) e^{-t}$$

Then $y_p' = u_1 e^t - u_2 e^{-t} + \underbrace{u_1' e^t + u_2' e^{-t}}$

WE DO NOT SET THIS EQUAL TO ZERO (that's done in the standard approach)

We find $y_p'' = u_1 e^t + u_2 e^{-t} + 2(u_1' e^t - u_2' e^{-t})$
$+ u_1'' e^t + u_2'' e^{-t}$

(Check this last equation out if you don't believe me.)

Now we plug the last 3 expressions, namely for

y_p, y_p' and y_p''

into the original equation (6) (see previous page)

and we get (after cancelling some terms):

$$u_1'' e^t + u_2'' e^{-t} + 2(u_1' e^t - u_2' e^{-t}) = \underset{= e^t}{g(t)}$$

So this is the "1 equation in 2 unknowns"

that we have been talking about.

$\boxed{\text{We need another equation.}}$

I'm going to get one by equating to zero the coefficient of e^{-t}:

$$(u_1'' + 2u_1')e^t + \underbrace{(u_2'' - 2u_2')}_{\text{Set this } = 0} e^{-t} = e^t$$

This gives the two equations:

$$u_2'' - 2u_2' = 0 \qquad (7)$$

and

$$u_1'' + 2u_1' = 1 \qquad (8)$$

$$u_2'' - 2u_2' = 0 \qquad (7)$$

$$u_1'' + 2u_1' = 1 \qquad (8)$$

Let's solve these equations for $u_1(t)$ and $u_2(t)$ and show that they give an expression for y_p which is equivalent to that obtained by the text's method (which is based on eqs. (5).)

the "formulas"

To solve eq. (7), turn it into a first order ODE by setting $p_2 = u_2'$ so that (7) becomes

$$p_2' - 2p_2 = 0$$

which is easily solved by separating variables,

$$p_2(t) = e^{2t}$$

Therefore $u_2' = e^{2t}$ and $u_2 = \frac{e^{2t}}{2}$ (omit the arbitrary constants)

Similarly let $p_1 = u_1'$ so eq. (8) becomes

$$p_1' + 2p_1 = 1$$ Multiply by e^{2t} (an integrating factor)

$$\frac{d}{dt}\left(e^{2t} p_1\right) = e^{2t}$$

$$e^{2t} p_1 = \frac{e^{2t}}{2}$$ (omit the arb. constant)

$$p_1 = \frac{1}{2} \Rightarrow u_1' = \frac{1}{2}, \quad u_1 = \frac{t}{2}$$

So we have found that $u_1 = \frac{t}{2}$, $u_2 = \frac{e^{2t}}{2}$

giving $y_p = u_1 y_1 + u_2 y_2$

$$= \frac{t}{2} e^{t} + \frac{e^{2t}}{2} e^{-t}$$

or $= \frac{t}{2} e^{t} + \underbrace{\frac{1}{2} e^{t}}$

this is part of the homogeneous solution

So we can say that

$$y_p = \frac{t}{2} e^{t}$$

and the general solution of eq.(6) is

$$y = y_h + y_p = c_1 e^{t} + c_2 e^{-t} + \frac{t}{2} e^{t}$$

Comparison with text approach based on formulas (5)

$$y_1 = e^t, \quad y_2 = e^{-t}$$

$$W = y_1 y_2' - y_2 y_1' = e^t(-e^{-t}) - e^{-t}(e^t) = -2$$

$$u_1(t) = -\int \frac{y_2 g}{W}\, dt = -\int \frac{e^{-t} e^{t}}{-2}\, dt = \frac{t}{2} \quad \text{(omit constant)}$$

$$u_2(t) = \int \frac{y_1 g}{W}\, dt = \int \frac{e^t e^t}{-2}\, dt = -\frac{e^{2t}}{4}$$

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \frac{t}{2} e^t - \frac{e^{2t}}{4} e^{-t} \\ &= \frac{t}{2} e^t - \underbrace{\frac{1}{4} e^t}_{\text{part of the homogeneous solution}} \end{aligned}$$

Agrees with other approach