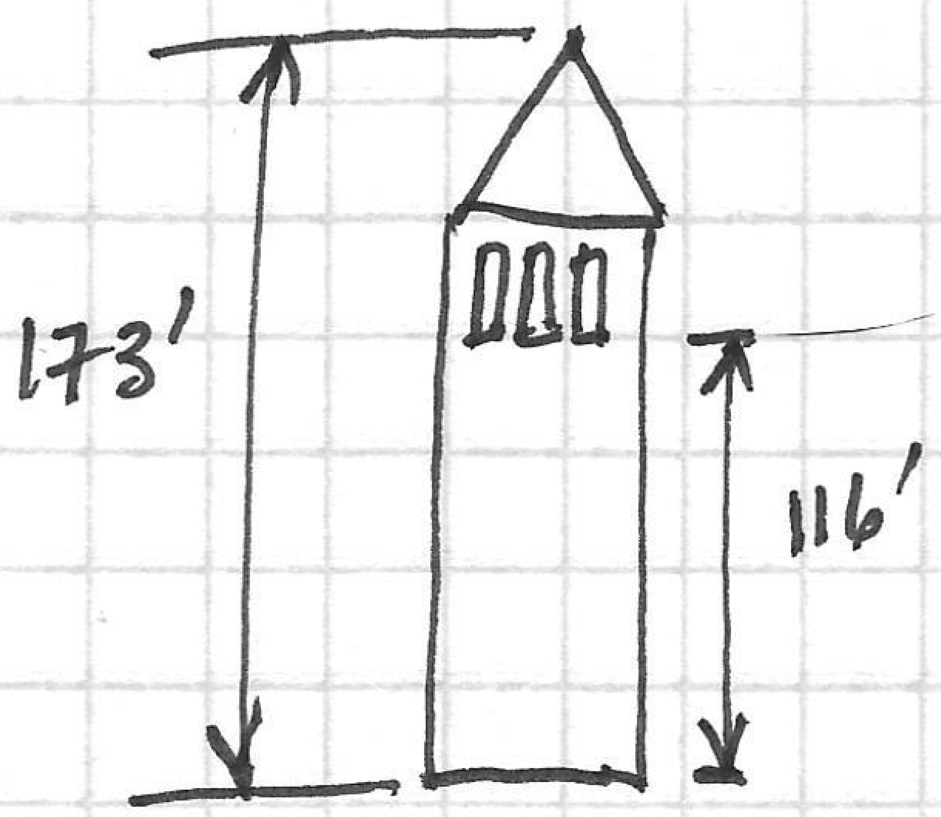
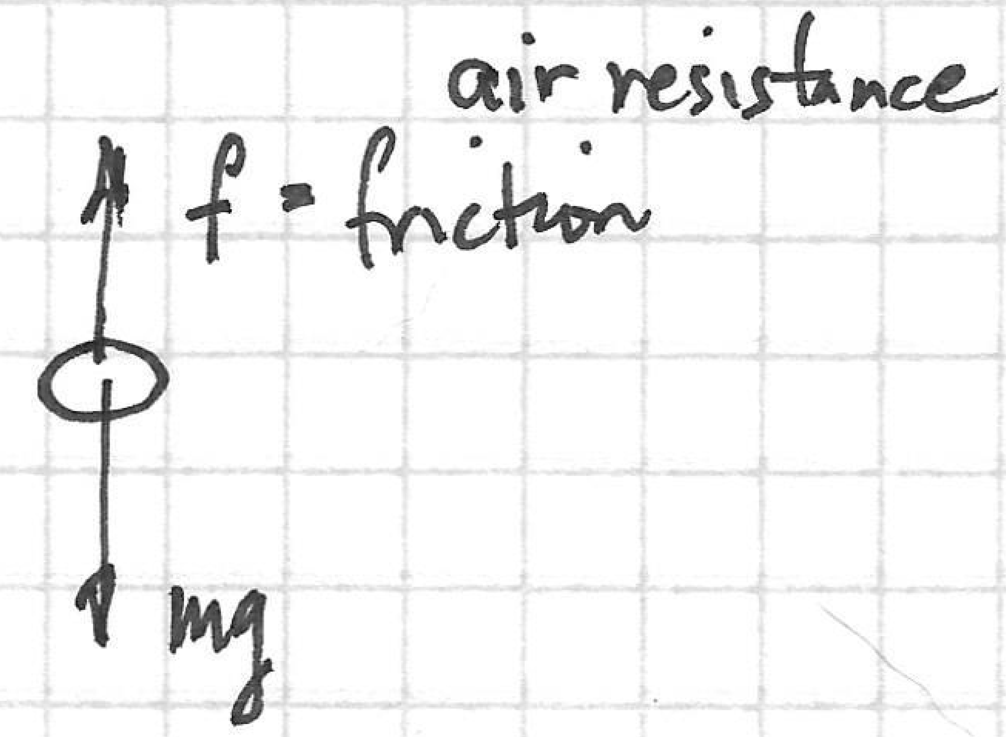
An example from physics

A penny is dropped off of the Cornell Bell Tower. How long does it take to reach the ground?



McGraw Tower

Free body



Newton's 2nd Law

$$F = mg - f = ma$$

$a = \frac{dv}{dt}$, a = acceleration, v = velocity

We suppose that f = friction force $= \gamma v$

$$m\frac{dv}{dt} = mg - \gamma v$$

$$\frac{dv}{dt} = g - \frac{\gamma}{m} v$$

Let $a = \frac{\gamma}{m}$ $\Rightarrow$ $\frac{dv}{dt} = g - av$

Associated with this d.e. (= differential equation) is an <u>initial condition</u> (IC):

at $t=0$, $v=0$ (released from rest)

We use a technique called separation of variables and write the d.e. in the form

$$\frac{dv}{g-av} = dt$$

Integrate both sides

$$\int \frac{dv}{g-av} = \int dt = t + C$$

arbitrary constant

$$= \frac{1}{-a}\int \frac{d(g-av)}{g-av}$$

$$= -\frac{1}{a}\ln(g-av)$$

That is, $-\frac{1}{a}\ln(g-av) = t + C$

To find C we use the IC: $t=0, v=0$

Substituting, we get

$$-\frac{1}{a}\ln g = C \Rightarrow C = -\frac{1}{a}\ln g$$

Substituting the value for C, we get

$$-\frac{1}{a}\ln(g-av) = t - \frac{1}{a}\ln g$$

multiply by $-a$ $\Rightarrow$ $\underbrace{\ln(g-av) - \ln g}_{\ln\left(\frac{g-av}{g}\right)} = -at$

$$\Rightarrow \quad \frac{g-av}{g} = e^{-at}$$

$$1 - \frac{a}{g}v = e^{-at}$$

$$-\frac{a}{g}v = -1 + e^{-at}$$

$$v = \frac{g}{a}(1 - e^{-at})$$

Now we know that $g = 32$ feet/sec^2

But what is $a = \frac{\gamma}{m}$?

Note that as $t \to \infty$, $e^{-at} \to 0$, $v \to \frac{g}{a}$

This value of velocity $= \frac{g}{a}$ is called "terminal velocity"

I looked around on the web and found two values for the terminal velocity of a penny:

a) 25 mph
b) 30 - 50 mph

I will take 30 mph = 44 feet/sec as representative.

So $\frac{g}{a} = 44$ ft/sec, $\quad a = \frac{g}{44 \frac{ft}{sec}} = \frac{32 \text{ ft/sec}^2}{44 \text{ ft/sec}}$

or $a = 0.72$ sec^{-1}

Although we have solved the v equation for $v = v(t)$, we still don't know when the penny hits the ground. We need to use

$$v = \frac{dx}{dt} = \frac{g}{a}(1 - e^{-at})$$

Integrating, $x = \frac{g}{a}t + \frac{g}{a^2}e^{-at} + K$

arbitrary constant

with IC $t=0,\ x=0$

Substituting the IC, $x = \frac{g}{a^2} + K = 0 \Rightarrow K = -\frac{g}{a^2}$

$$x(t) = \frac{g}{a}t + \frac{g}{a^2}\left(e^{-at} - 1\right)$$ where $\frac{g}{a} = 44$ ft/sec and $a = 0.72\ \text{sec}^{-1}$

We want time $t = T$ such that $x = 116$ feet;

$$116 = 44T + \frac{44}{.72}\left(e^{-.72T} - 1\right)$$

where T is in seconds.

numerical evaluation gives

$T \approx 3.95$ sec.