5 Euler's Differential Equation

The simplest case of a linear variable coefficient second order ODE is Euler's equation:

$$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + c\,y = 0 \tag{44}$$

We look for a solution with the ansatz:

$$y = x^r \tag{45}$$

Substitution of (45) into (44) gives

$$ar(r-1) + br + c = 0 \qquad \text{that is,} \qquad ar^2 + (b-a)r + c = 0 \tag{46}$$

We may use the quadratic formula to obtain (in general) a pair of complex conjugate roots r_1 and r_2. Thus the general solution may be written

$$y = c_1\, x^{r_1} + c_2\, x^{r_2} \tag{47}$$

If the roots are both real, then eq.(47) suffices. However in the general case in which r_1 and r_2 are complex, say $r_1 = \mu + i\nu$, we obtain the form

$$y = c_1\, x^{\mu+i\nu} + c_2\, x^{\mu-i\nu} = x^\mu(c_1 x^{i\nu} + c_2 x^{-i\nu}) \tag{48}$$

Using Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, and the identity $x = e^{\log x}$, we obtain the real form

$$y = x^\mu(c_1 e^{i\nu\log x} + c_2 e^{-i\nu\log x}) = x^\mu(c_3\cos(\nu\log x) + c_4\sin(\nu\log x)) \tag{49}$$

where c_3 and c_4 are *real* arbitrary constants, and where $\log x$ stands for *natural* logarithms.

In the case that the roots are repeated, $r_1 = r_2 = r$, the general solution to (44) is

$$y = c_1\, x^r + c_2\, x^r \log x \tag{50}$$

Set $x = e^z$ $(z = \ln x)$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$a x^2 y_{xx} + b x y_x + c y = 0$ becomes

$$a(-y_z + y_{zz}) + b y_z + c y = 0$$

$$a y_{zz} + (b-a) y_z + c y = 0$$

with solution $y = e^{rz} = x^r$

Example

$$x^3 y''' + x^2 y'' - 4x y' = 3x^2$$

Replace x by z where $z = \log x$ $(x = e^z)$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = \frac{1}{x}\dot{y} \qquad \left(\dot{y} = \frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$= \frac{1}{x^2}(\ddot{y} - \dot{y})$$

$$\frac{d^3y}{dx^3} = -\frac{2}{x^3}(\ddot{y} - \dot{y}) + \frac{1}{x^2}(\dddot{y} - \ddot{y})\frac{dz}{dx}$$

$$= -\frac{2}{x^3}(\ddot{y} - \dot{y}) + \frac{1}{x^3}(\dddot{y} - \ddot{y})$$

$$= \frac{1}{x^3}(\dddot{y} - 3\ddot{y} + 2\dot{y})$$

$$(\dddot{y} - 3\ddot{y} + 2\dot{y}) + (\ddot{y} - \dot{y}) - 4(\dot{y}) = 3e^{2z}$$

$$\dddot{y} - 2\ddot{y} - 3\dot{y} = 3e^{2z}$$

$$y = y_h + y_p, \quad y_h = e^{rz}$$

$$r^3 - 2r^2 - 3r = 0$$

$$r(r^2 - 2r - 3) = r(r-3)(r+1) = 0$$

$$y_h = c_1 + c_2 e^{-z} + c_3 e^{3z}$$

$$y_p = Ae^{2z} \Rightarrow 8A - 2(4A) - 3A(2) = 3$$

$$\Rightarrow A = -1/2$$

$$y = c_1 + c_2 e^{-z} + c_3 e^{3z} - \frac{1}{2}e^{2z}$$

$$y = c_1 + \frac{c_2}{x} + c_3 x^3 - \frac{1}{2}x^2$$

Check solution using MATLAB

```
>> syms y x c1 c2 c3
>> y=c1+c2/x+c3*x^3-x^2/2
 y =
 c1 + c2/x + c3*x^3 - x^2/2
 >> x^3*diff(y,x,3)+x^2*diff(y,x,2)-4*x*diff(y,x)
 ans =
 x^2*(6*c3*x + (2*c2)/x^3 - 1) + 4*x*(x + c2/x^2 - 3*c3*x^2) + x^3*(6*c3 - (6*c2)/x^4)
 >> expand(ans)
 ans =
 3*x^2
```