An overview of the rest of this course:

PARTIAL DIFFERENTIAL EQS

and related topics

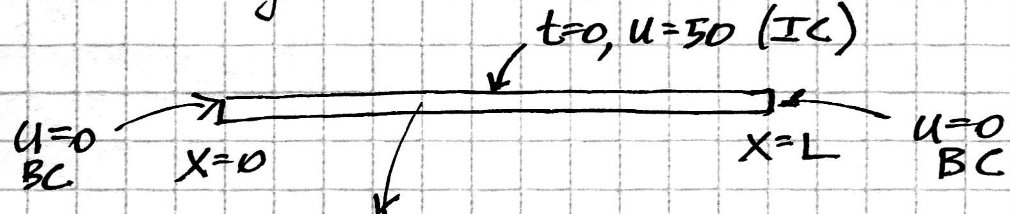
Explanation: The syllabus involves 6 lectures on topics which are used in solving PDE's (two point BVP's = Boundary Value Problems, and Fourier series), followed by 9 lectures on PDE's (and 2 lectures on Reviews).

The idea of this lecture is to give you an overview of how the two point BVP's and Fourier series topics are to be related to the process of solving a PDE.

You are not expected to master everything in one lecture!

Main purpose is to motivate studying two-point BVP's and Fourier series.

Imagine a rod, which goes from $X=0$ to $X=L$, at every point of which a temperature $u(x,t)$ is defined. The ends of the rod are kept at 0 degrees C, and the temperature in the rod is initially (at $t=0$) given as 50 degrees C.



Find $u(x,t)$

Given
$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \text{BC}$$
$$u(x,0) = 50 \leftarrow \text{IC}$$

The governing PDE is "the heat equation":

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Step 1: Separation of variables

$$u(x,t) = X(x)\, T(t)$$

$$\frac{\partial u}{\partial t} = X(x)\, T'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)\, T(t)$$

Where $'$ means differentiation wrt the argument

$$X T' = X'' T$$

$$\underbrace{\frac{X''}{X}}_{\text{fn of } x \text{ only}} = \underbrace{\frac{T'}{T}}_{\text{fn of } t \text{ only}}$$

Since X and T are fns of independent quantities x, t, the only way the two sides of the last equation can be equal is if each is really a constant, call it $-\lambda$

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

gives 2 ODE's: $X'' + \lambda X = 0$, $T' + \lambda T = 0$

The BC $u(0,t) = 0$ and $u(L,t) = 0$ become

$$X(0)T(t) = 0 \text{ and } X(L)T(t) = 0$$

Since $T(t)$ cannot be identically zero, we get

$$X(0) = 0, \quad X(L) = 0$$

Step 2 The system $X'' + \lambda X = 0$, $X(0) = X(L) = 0$

is called a 2-point BVP

in which λ is an unknown constant called an eigenvalue

We have $X(x) = c_1 \sin\sqrt{\lambda}\, x + c_2 \cos\sqrt{\lambda}\, x$

$$X(0) = 0 = c_1 \cdot 0 + c_2 \cdot 1 = c_2$$

$$X(L) = 0 = c_1 \sin\sqrt{\lambda}\, L$$

$$\sin\sqrt{\lambda}\, L = 0 \Rightarrow \sqrt{\lambda}\, L = n\pi, \quad n = 1, 2, 3, \ldots$$

$$\lambda = \frac{n^2\pi^2}{L^2}$$

$$X(x) = c_1 \sin\sqrt{\lambda}\, x = c_1 \sin \frac{n\pi x}{L}$$

Next we look at the T equation:

$$T' + \lambda T = 0$$

that is, $$\frac{dT}{dt} + \frac{n^2\pi^2}{L^2} T = 0$$

The general solution is

$$T(t) = c_3 e^{-\frac{n^2\pi^2}{L^2}t}$$

We have found

$$u(x,t) = X(x)\,T(t) = a_n \sin\frac{n\pi x}{L} e^{-\frac{n^2\pi^2}{L^2}t}, \quad n = 1,2,3,\ldots$$

where $a_n = c_1 c_3 =$ an arbitrary constant

Step 3 The general solution to the PDE is a superposition of each of the terms for $n = 1,2,3,\ldots$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-\frac{n^2\pi^2}{L^2}t}$$

Now we must select the coefficients a_n so that the IC is satisfied

$$u(x,0) = 50 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \cancel{e^0}^{=1}$$

This problem is called "Fourier series."

Multiply the series by $\sin\frac{m\pi x}{L}$ where m is a specific (but arbitrary) integer [versus n, which is summed to include <u>every</u> integer.]

$$50 \sin \frac{m\pi x}{L} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}$$

Next you integrate from 0 to L in X:

$$\int_0^L 50 \sin \frac{m\pi x}{L} \, dx = \sum_{n=1}^{\infty} a_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx$$

a trig identity: $\sin a \sin b = \frac{1}{2}\left[-\cos(a+b) + \cos(a-b)\right]$

$$\therefore \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \frac{1}{2} \int_0^L -\cos(n+m)\frac{\pi x}{L} + \cos(n-m)\frac{\pi x}{L} \, dx$$

$$= \frac{1}{2}\left[-\sin(n+m)\frac{\pi x}{L} + \sin(n-m)\frac{\pi x}{L} \Big|_0^L\right]$$

$$= \frac{1}{2}\left[-\sin(n+m)\pi + \sin(n-m)\pi\right]$$

$$= 0 \quad \text{unless } n=m$$

So the series has become a single term:

$$\int_0^L 50 \sin \frac{m\pi x}{L} \, dx = a_m \int_0^L \sin^2 \frac{m\pi x}{L} \, dx$$

we solve for $a_m = \dfrac{\int_0^L 50 \sin \frac{m\pi x}{L} \, dx}{\int_0^L \sin^2 \frac{m\pi x}{L} \, dx}$

and substitute in here:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \, e^{-\frac{n^2\pi^2}{L^2} t}$$