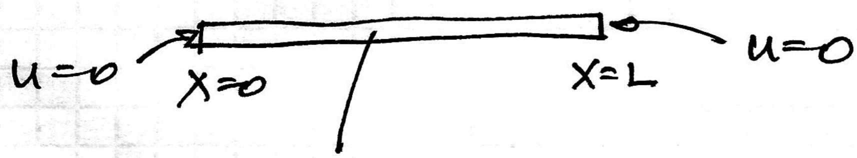
EIGENVALUES AND 2-POINT BOUNDARY VALUE PROBLEMS

In the last lecture we saw how separating variables in a PDE led to a 2-point BVP:

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0$$

[Reminder:



$$u_t = u_{xx}$$

$$X T' = X'' T$$

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda \Rightarrow X'' + \lambda X = 0$$

The BC $u=0$ at $x=0, L$ give rise to the BC

$$X(0) = 0, \quad X(L) = 0$$]

We solved the two point BVP as follows:

general solution:

$$X(x) = C_1 \sin\sqrt{\lambda}\, x + C_2 \cos\sqrt{\lambda}\, x$$

$$X(0) = 0 = C_2$$

$$X(L) = 0 = C_1 \sin\sqrt{\lambda}\, L$$

$$\Rightarrow \sqrt{\lambda}\, L = n\pi, \quad n = 1, 2, 3, \ldots$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

Now what if the example of last time had both ends <u>insulated</u> ($\Rightarrow \frac{\partial u}{\partial x} = 0$ at $x = 0, L$)

$$X(x) = C_1 \sin\sqrt{\lambda}x + C_2 \cos\sqrt{\lambda}\,x$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(X(x)\,T(t)\right) = X'(x)\,T(t)$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 \Rightarrow X'(0)\,T(t) = 0$$

$$\Rightarrow X'(0) = 0$$

Also $X'(L) = 0$

$$X'(x) = C_1\sqrt{\lambda}\cos\sqrt{\lambda}\,x - C_2\sqrt{\lambda}\sin\sqrt{\lambda}\,x$$

$\therefore X'(0) = C_1\sqrt{\lambda} = 0 \Rightarrow$ Either $C_1 = 0$ or $\lambda = 0$

$X'(L) = C_1\sqrt{\lambda}\cos\sqrt{\lambda}L - C_2\sqrt{\lambda}\sin\sqrt{\lambda}L$

If $C_1 = 0$ then $\sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi$, $n = 1, 2, 3, \dots$

$$\lambda = \frac{n^2\pi^2}{L^2}$$

$$X(x) = C_2 \cos\frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots$$

or if $\lambda = 0$ then

$$X(x) = C_2$$

Return to the T equation, $T' + \lambda T = 0$

$$T(t) = C_3 e^{-\lambda t}$$

when $\lambda = \frac{n^2\pi^2}{L^2}$, $T(t) = C_3 e^{-\frac{n^2\pi^2}{L^2}t}$

when $\lambda = 0$, $T(t) = C_3$

So we get, for the PDE

$$u(x,t) = \sum_{n=1,2,3,\dots}^{\infty} a_n \cos\frac{n\pi x}{L}\, e^{-\frac{n^2\pi^2}{L^2}t} + a_0$$

Next example: $x=0$ is kept at $u=0$ degrees C

$x=L$ is insulated $\Rightarrow \frac{\partial u}{\partial x}=0$ at $x=L$

$$X(x) = c_1 \sin\sqrt{\lambda}\,x + c_2 \cos\sqrt{\lambda}\,x$$

$$X'(x) = c_1\sqrt{\lambda}\cos\sqrt{\lambda}\,x - c_2\sqrt{\lambda}\sin\sqrt{\lambda}\,x$$

B.C. $x=0$, $X(0)=0$ and $x=L$, $X'(L)=0$

$$X(0)=0=c_2 \Rightarrow c_2=0$$

$$X'(L)=0=c_1\sqrt{\lambda}\cos\sqrt{\lambda}\,L \Rightarrow$$

Either $c_1=0$ or $\lambda=0$ or $\sqrt{\lambda}L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$

Since $X(x)=c_1\sin\sqrt{\lambda}\,x$ we must reject $c_1=0$ and $\lambda=0$

since these choices give the trivial solution.

$$\therefore \quad \lambda = \frac{\pi^2}{4L^2}, \frac{9\pi^2}{4L^2}, \frac{25\pi^2}{4L^2}, \dots$$

$$u(x,t) = \sum_{\substack{n=1,3,5,\dots \\ \text{(odd only)}}}^{\infty} a_n \sin\frac{n\pi x}{2L}\, e^{-\frac{n^2\pi^2}{4L^2}t}$$