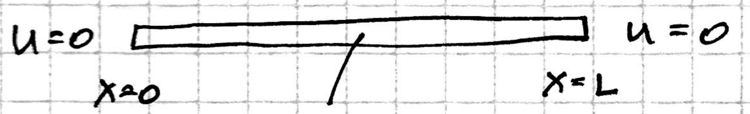
FOURIER SERIES

Review of the PDE:



$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Initial condition: $t=0$, $u=50$

Step 1 Separation of variables $u = X(x)\,T(t)$

Step 2 2 point BVP $X'' + \lambda X = 0$, $X(0) = X(L) = 0$

Solution so far: $$u(x,t) = \sum_{n=1,2,3,\ldots}^{\infty} a_n \sin\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi}{L}\right)^2 t} \qquad (1)$$

Step 3 $t=0$, $u=50$

$$50 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

the procedure is to multiply by $\sin\frac{m\pi x}{L}$

and integrate from 0 to L

$$\int_0^L 50 \sin\frac{m\pi x}{L}\,dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx}_{=0 \text{ unless } m=n}$$

$$= a_m \int_0^L \sin^2\frac{m\pi x}{L}\,dx$$

$$50 \sin\frac{m\pi x}{L} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}$$

Next you integrate from 0 to L in X:

$$\int_0^L 50 \sin\frac{m\pi x}{L}\,dx = \sum_{n=1}^{\infty} a_n \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx$$

a trig identity: $\sin a \sin b = \frac{1}{2}\left[-\cos(a+b) + \cos(a-b)\right]$

$$\therefore \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_0^L \left(-\cos(n+m)\frac{\pi x}{L} + \cos(n-m)\frac{\pi x}{L}\right) dx$$

$$= \frac{1}{2}\left[-\sin(n+m)\frac{\pi x}{L} + \sin(n-m)\frac{\pi x}{L}\Big|_0^L\right]$$

$$= \frac{1}{2}\left[-\sin(n+m)\pi + \sin(n-m)\pi\right]$$

$$= 0 \text{ unless } n=m$$

So the series has become a single term:

$$\int_0^L 50 \sin\frac{m\pi x}{L}\,dx = a_m \int_0^L \sin^2\frac{m\pi x}{L}\,dx$$

we solve for $a_m = \dfrac{\int_0^L 50 \sin\frac{m\pi x}{L}\,dx}{\int_0^L \sin^2\frac{m\pi x}{L}\,dx}$

and substitute in here:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\, e^{-\frac{n^2\pi^2}{L^2}t}$$

$$\therefore a_m = \frac{\int_0^L 50 \sin \frac{m\pi x}{L}\, dx}{\int_0^L \left(\sin \frac{m\pi x}{L}\right)^2 dx}$$

Then we substitute this into eq. (1) (changing m to n), and that is the final answer to the PDE.

In this lecture we focus on "step 3", the Fourier Series.

In section 10.2 in the Text, the problem we solved as part of the PDE (above) is generalized in 3 ways:

$$50 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

1. The "target" function 50 is generalized to be an arbitrary given function $f(x)$.

2. The basis functions $\{\sin \frac{n\pi x}{L}\}$ are generalized to include cosines $\{\sin \frac{n\pi x}{L}, \cos \frac{n\pi x}{L}\}$

$$f(x) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} + \frac{a_0}{2} \qquad (2)$$

3. The integrals are from $-L$ to L instead of 0 to L.

The Fourier coefficients are given by the formulas

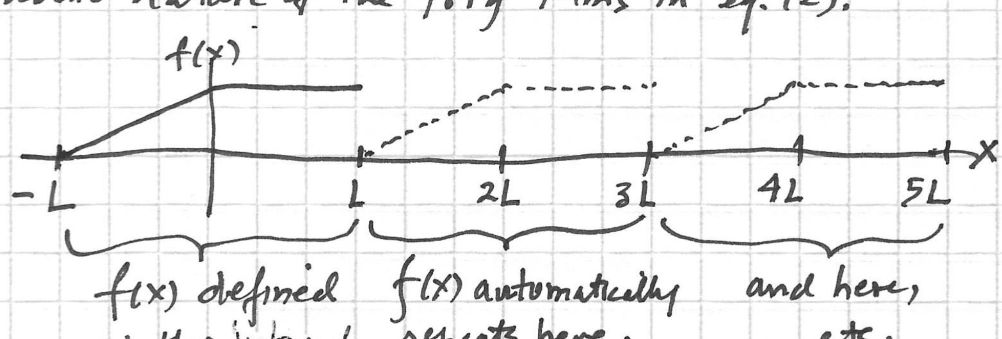
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad (3)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad (4)$$

These formulas are derived by multiplying eq. (2) by $\cos\frac{m\pi x}{L}$, or by $\sin\frac{m\pi x}{L}$, and integrating from $-L$ to L, and then using the trig formulas giving $\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0$ if $m \neq n$,

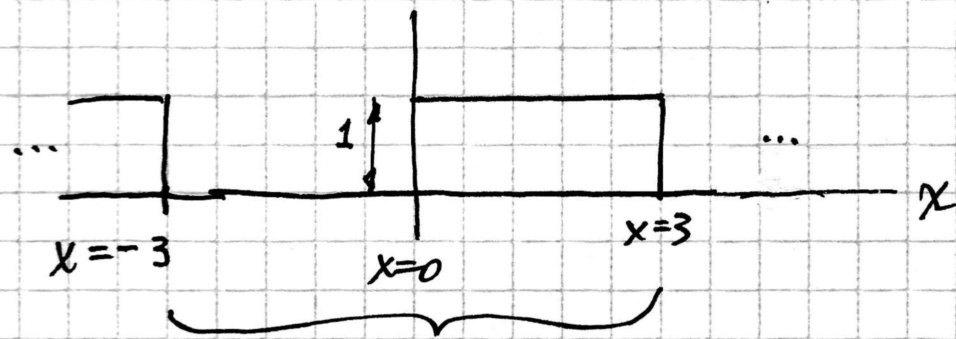
etc., see eqs. (6)-(8) on p. 470 of the text.

Note that $f(x)$ is defined on $(-L, L)$ for the purpose of computing the integrals above. However, once $f(x)$ is so defined, it AUTOMATICALLY becomes PERIODIC, with period $2L$, repeating from L to $3L$, and from $3L$ to $5L$, etc. because of the periodic nature of the trig terms in eq. (2).



Example

$$f(x) = \begin{cases} 1, & 0 \le x < 3 \\ 0, & -3 \le x < 0 \end{cases}$$



$f(x)$ defined on $(-3, 3)$

automatically repeats every 6 units in x.

$f(x)$ has period $= 6$.

Eq. (2) with $L = 3$ becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3} + b_n \sin\frac{n\pi x}{3}$$

and eqs. (3), (4) become

$$a_n = \frac{1}{3}\int_{-3}^{3} f(x)\cos\frac{n\pi x}{3}\,dx$$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\sin\frac{n\pi x}{3}\,dx$$

We find

$$a_n = \frac{1}{3}\int_{0}^{3}\cos\frac{n\pi x}{3}\,dx = \frac{1}{3}\,\frac{3}{n\pi}\,\sin\frac{n\pi x}{3}\Big|_0^3 = 0,\ n > 0$$

NOTE:

$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\,dx$

$= \frac{1}{3}\int_{0}^{3} dx = 1$

$$b_n = \frac{1}{3}\int_{0}^{3}\sin\frac{n\pi x}{3}\,dx = -\frac{1}{3}\,\frac{3}{n\pi}\cos\frac{n\pi x}{3}\Big|_0^3$$

$$= -\frac{1}{n\pi}\left(\cos n\pi - \cos 0\right)$$

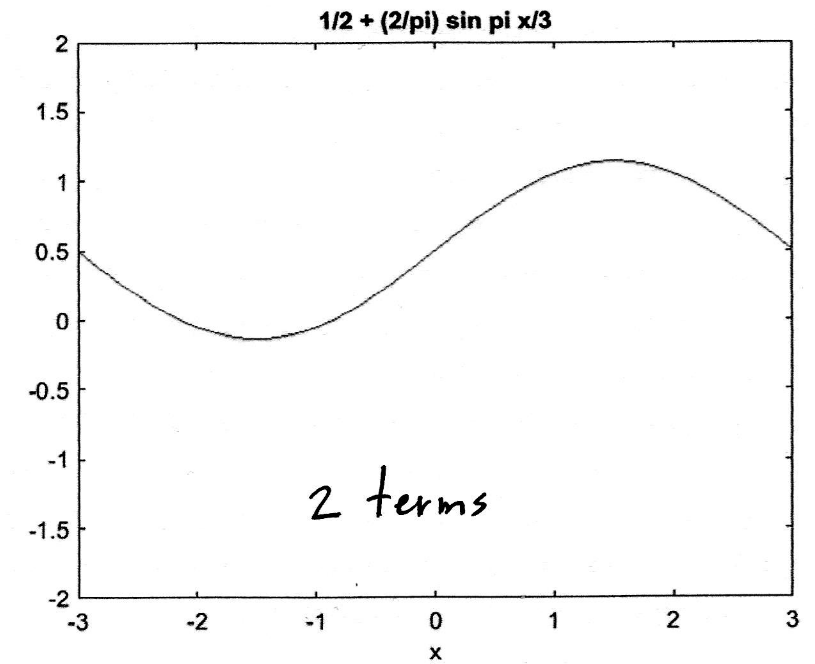
$$b_n = -\frac{1}{n\pi}\left(\underset{(-1)^n}{\underset{\shortparallel}{\cos n\pi}} - \underset{1}{\underset{\shortparallel}{\cos 0}}\right)$$

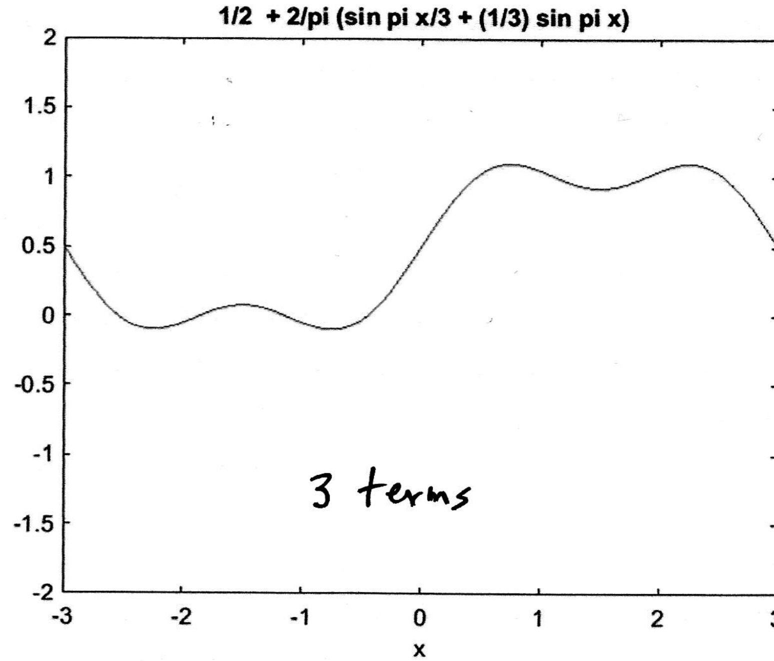
$$b_n = \begin{cases} 0, & n = 2, 4, 6, \ldots \\ \frac{2}{n\pi}, & n = 1, 3, 5, \ldots \end{cases}$$

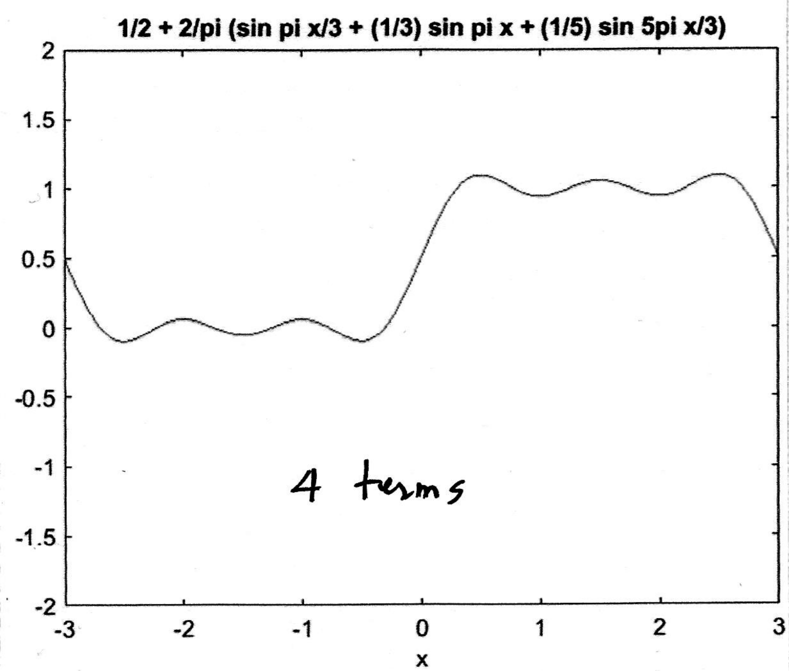
and thus we get

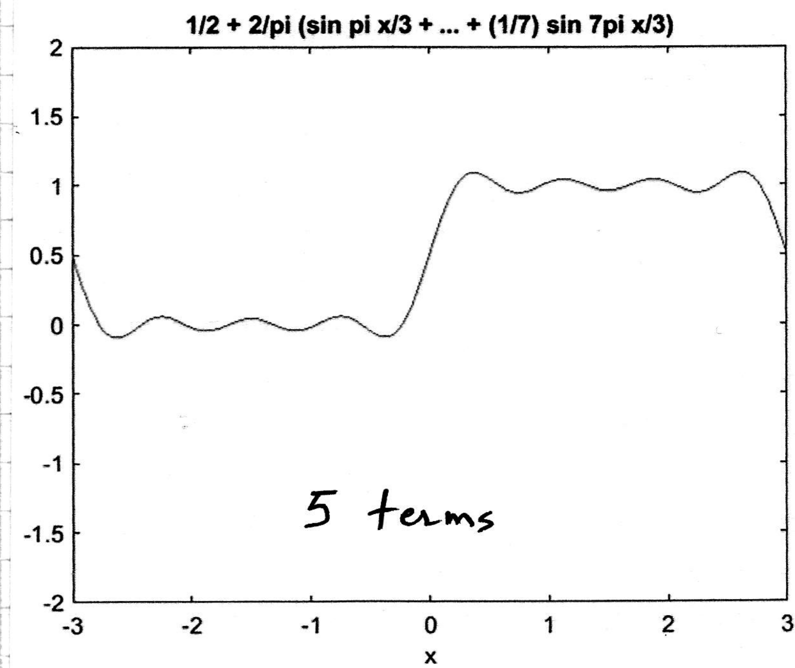
$$f(x) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi x}{3} + \underbrace{\frac{1}{2}}_{\frac{a_0}{2}}$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin \frac{\pi x}{3} + \frac{1}{3} \sin \pi x + \frac{1}{5} \sin \frac{5\pi x}{3} + \cdots\right)$$


1/2 + (2/pi) sin pi x/3
2 terms
x


1/2 + 2/pi (sin pi x/3 + (1/3) sin pi x)
3 terms
x


1/2 + 2/pi (sin pi x/3 + (1/3) sin pi x + (1/5) sin 5pi x/3)
4 terms
x


1/2 + 2/pi (sin pi x/3 + ... + (1/7) sin 7pi x/3)
5 terms
x