FOURIER SERIES (continued)

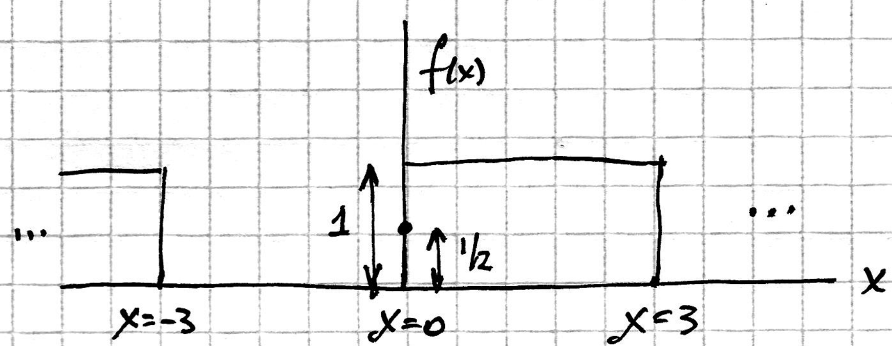
In the last set of notes we showed that

$$f(x) = \begin{cases} 1, & 0 \le x < 3 \\ 0, & -3 \le x < 0 \end{cases}$$

had the following expansion

$$f(x) = \frac{1}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi x}{3}$$

Note that $f(x)$ has a discontinuity at $x=0$:

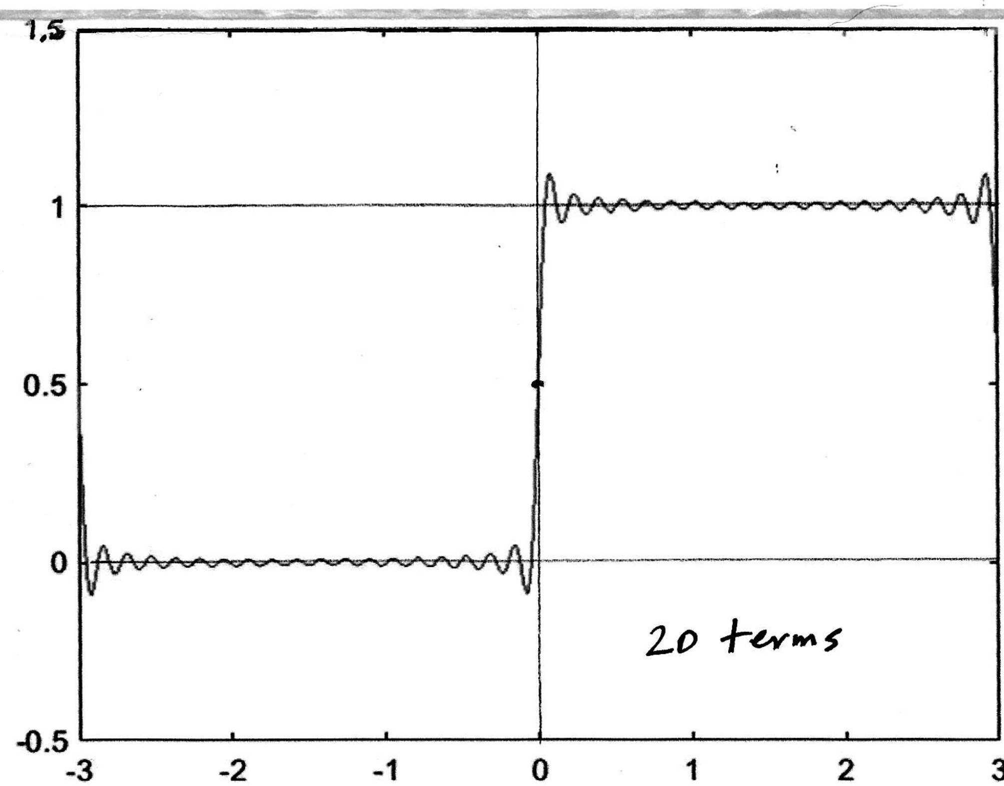


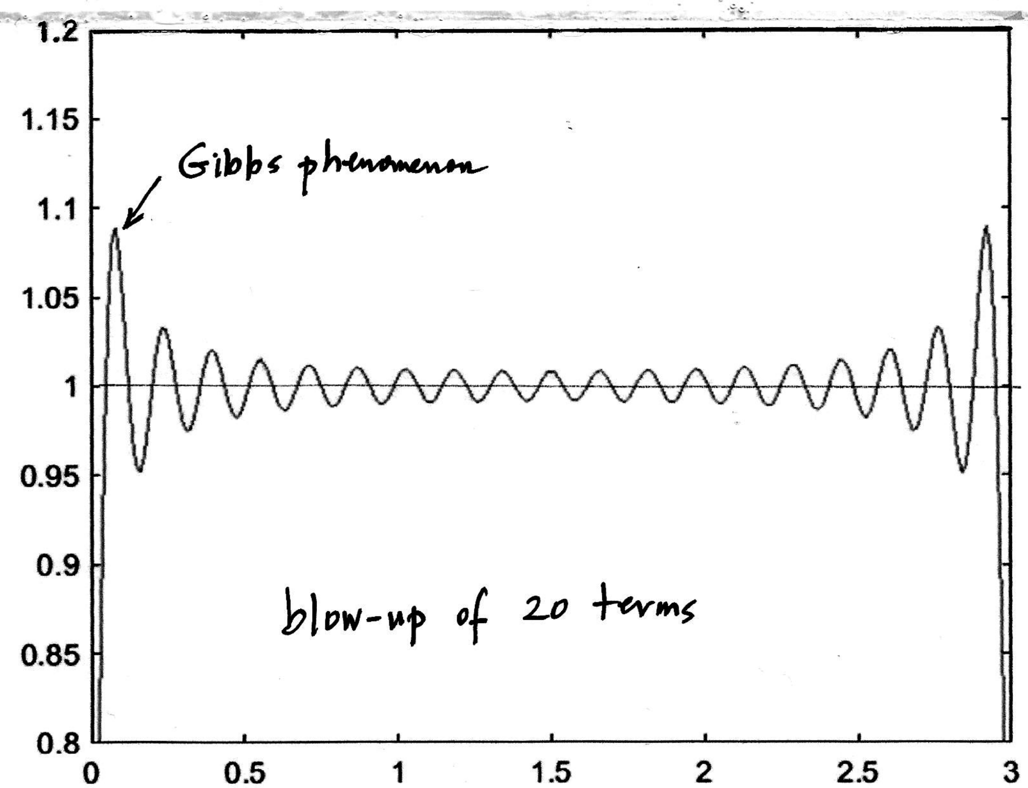
And note that at $x=0$, the series gives

$$f(0) = \frac{1}{2} + 0$$

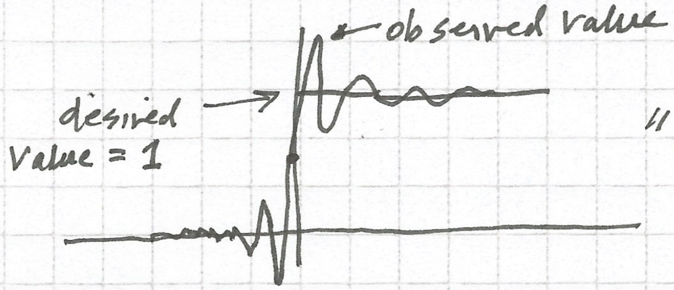
Thus the Fourier series converges to the average value of $f(x)$ at a discontinuity (a jump) $= \frac{1}{2}(\underbrace{f(0^-)}_{=0} + \underbrace{f(0^+)}_{=1})$

$$= \frac{1}{2}$$


1.5
1
0.5
0
-0.5
-3
-2
-1
0
1
2
3
20 terms


1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
0.8
0
0.5
1
1.5
2
2.5
3
Gibbs phenomenon
blow-up of 20 terms

Fourier series exhibit a peculiar behavior in the neighborhood of a discontinuity.
The series overshoots the correct value:



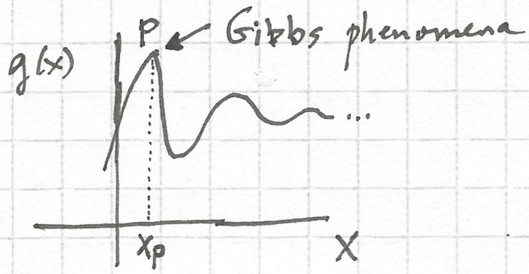
"Gibbs phenomenon"

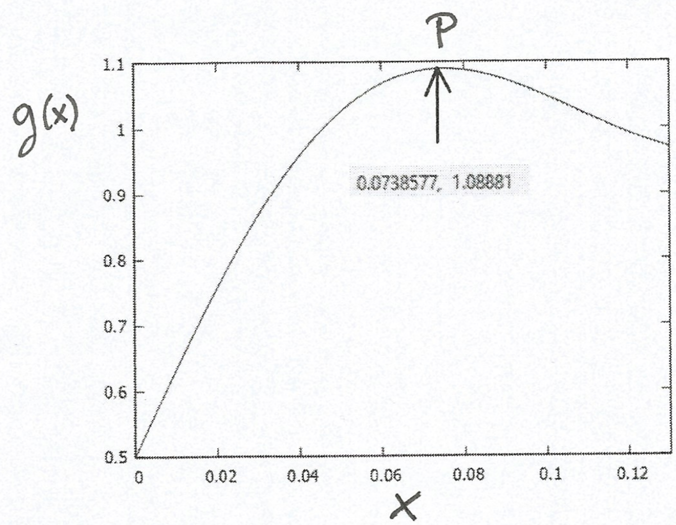
How can we approximate the size of the overshoot?

Let $g(x)$ be a truncation of the Fourier series

For example, let $g(x)$ be a 20-term approximation:

$$g(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin\frac{\pi x}{3} + \frac{1}{3}\sin\frac{3\pi x}{3} + \frac{1}{5}\sin\frac{5\pi x}{3} + \cdots + \frac{1}{37}\sin\frac{37\pi x}{3} \right)$$





To find point P, we blow up the graph around point P and locate the maximum:

$$x_p = 0.0738$$

$$g(x_p) = 1.0888$$

Thus there is about a 9% overshoot.

Questions of Convergence

Recall from your earlier studies of infinite series, in particular of power series, there are many convergence tests involved in determining whether a given series converges or not: ratio test, comparison test, etc.

As an example, take the geometric series, e.g.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

The ratio test shows that this is not valid if $|x| > 1$.

$\boxed{\text{In contrast to the behavior of power series, Fourier series convergence questions do not involve any convergence tests!}}$

The convergence theorem for Fourier Series is stated on p. 478. If $f(x)$ and $f'(x)$ are

"piecewise continuous" on $-L \le x < L$

(and $2L$ periodic), then the Fourier series AUTOMATICALLY converges to $f(x)$.