FOURIER SERIES (continued)

We have seen that the general story for Fourier series goes like this:

A function $f(x)$ defined on $-L \leq x < L$

Can be represented by the series

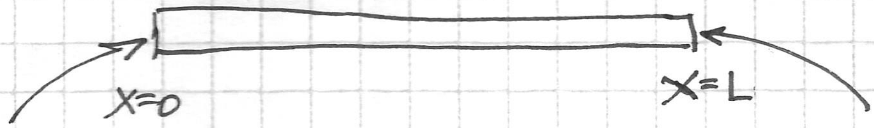
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

where $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Such a representation gives a function $f(x)$ which is periodic with period $2L$.

On the other hand, we have seen Fourier series come out of solving a PDE by separation of variables



BC at $x=0$ for example $u=0$ or $\frac{\partial u}{\partial x}=0$

BC at $x=L$ Typically $u=0$ or $\frac{\partial u}{\partial x}=0$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Separation of Variables:

$$u = X(x)\,T(t)$$

$$X\,T' = X''\,T \quad \text{or}$$

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda$$

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + \lambda T = 0$$

with typical BC:

at $x=0$, $X=0$ or $X'=0$

at $x=L$, $X=0$ or $X'=0$

We looked at 3 cases:

i) $X(0)=0$, $X(L)=0$ gave $X_n(x) = \sin\frac{n\pi x}{L}$

and
$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

If the PDE has initial condition $t=0$, $u=f(x)$,

we get the Fourier series problem: ($f(x)$ defined on $[0,L)$.)

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

ii) $X'(0)=0$, $X'(L)=0$ gave $X_n(x) = \cos\frac{n\pi x}{L}$, $X_0(x)=1$

and $$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}\, e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

And the IC $t=0$, $u=f(x)$ gives the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

iii) $X(0)=0$, $X'(L)=0$ gave

$$u(x,t) = \sum_{n=1,3,5,\dots}^{\infty} a_n \sin\frac{n\pi x}{2L}\, e^{-\left(\frac{n\pi}{2L}\right)^2 t}$$

and $$f(x) = \sum_{n=1,3,5,\dots}^{\infty} a_n \sin\frac{n\pi}{2L}x$$

<u>SUMMARY</u>

The general Fourier series has <u>both sine & cosine terms</u>, and is defined on $-L \le x < L$.

But the Fourier series which come out of PDE's have <u>only sine terms</u>, or <u>only cosine terms</u>, and are defined on $0 \le x < L$.

So a natural question is: what is the relationship between Fourier series which have only sine, or only cosine terms, and the general F.S. (Fourier series) which have both sine and cosine terms?

<u>Fourier sine series</u> are defined as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

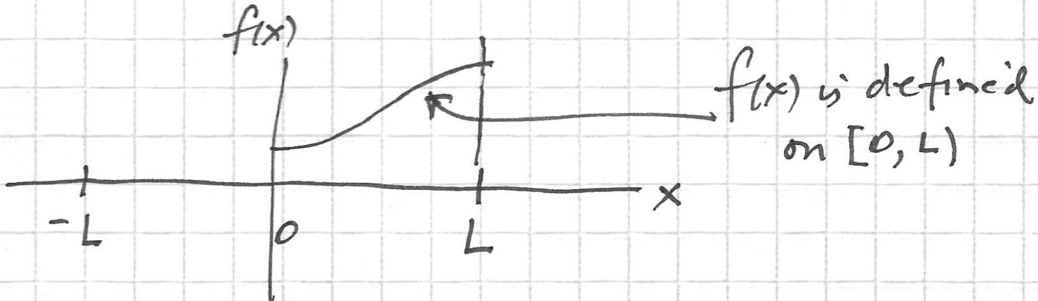
where $f(x)$ is defined on $0 \leq x < L$

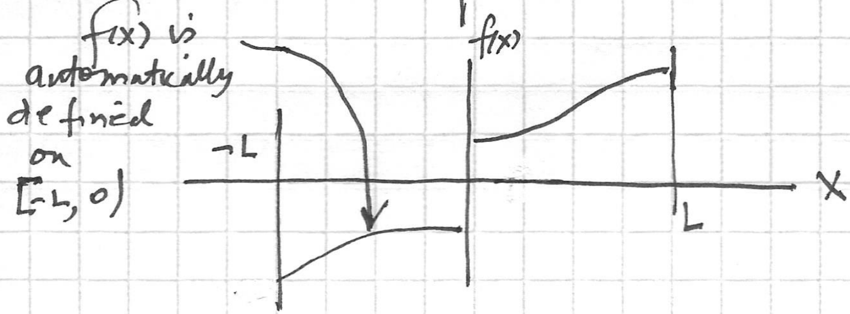
and where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

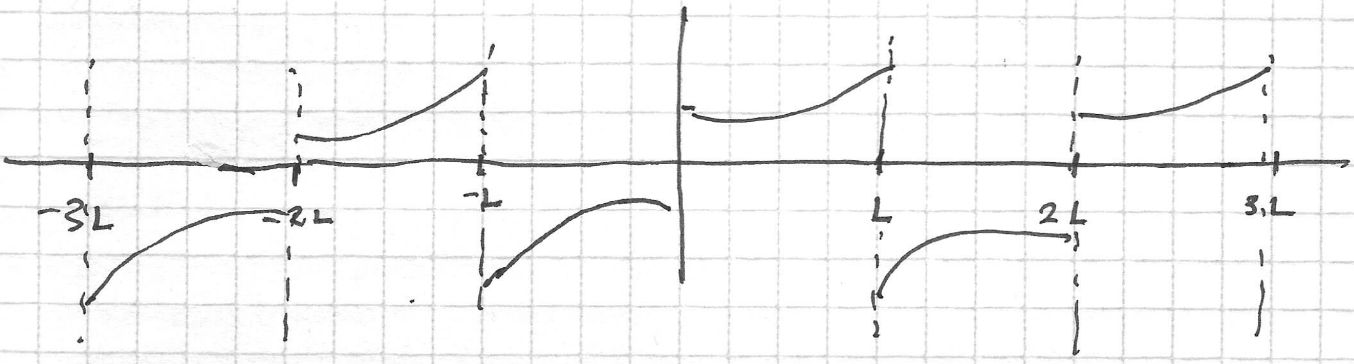
Note that $f(x)$ so defined is an "odd" function. That is $f(-x) = -f(x)$ (because $\sin \frac{n\pi x}{L}$ is odd)

So whatever $f(x)$ looks like on $[0, L)$, it AUTOMATICALLY is defined on $[-L, 0)$





And of course $f(x)$, once defined on $[-L, L)$,
is $2L$ periodic (because the general F.S. is $2L$ periodic)



That is, the sine series is a special case of the general F.S.

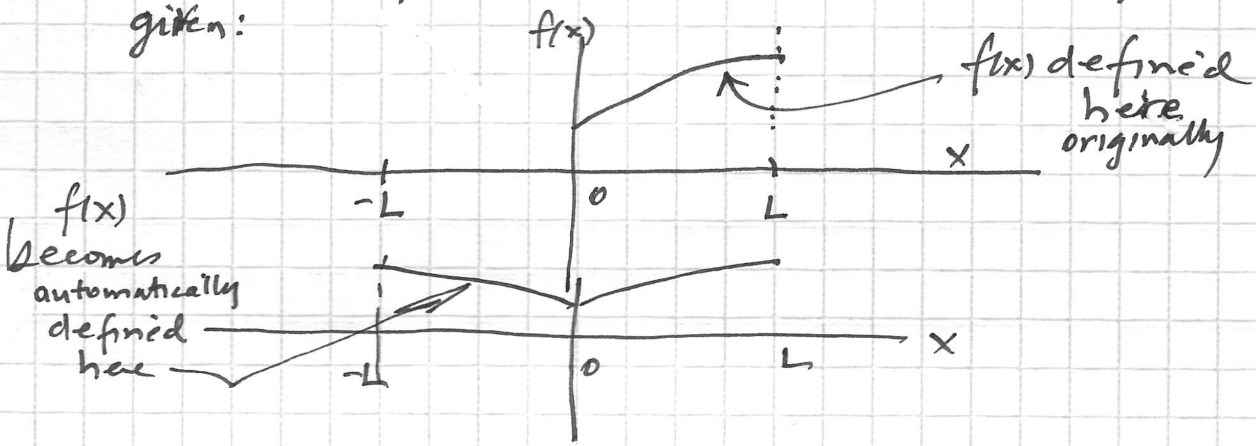
Similarly, a Fourier cosine series, defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

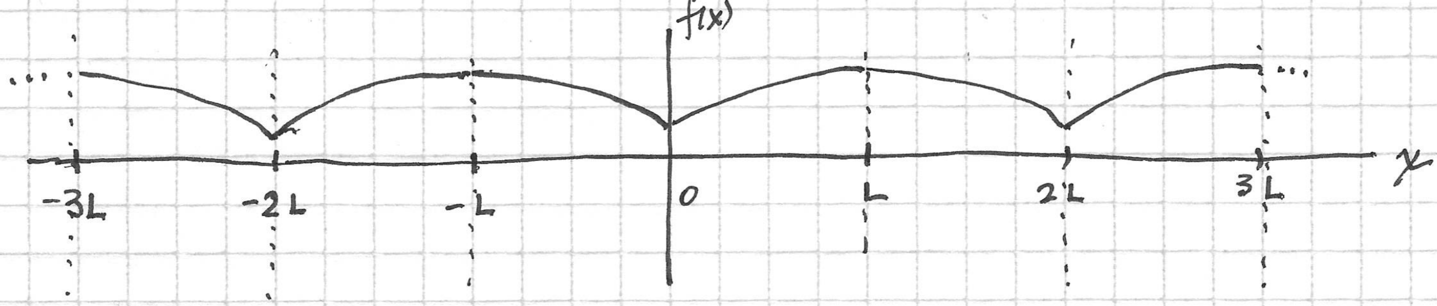
where $$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

is a special case of the general F.S.

Here $f(x)$ is again defined on $[0, L)$ but, since $\cos \frac{n\pi x}{L}$ is "even", i.e. $f(-x) = f(x)$, the value of $f(x)$ on $[-L, 0)$ is automatically given:



And just as in the case of the Fourier sine series, the Fourier cosine series is $2L$ periodic:



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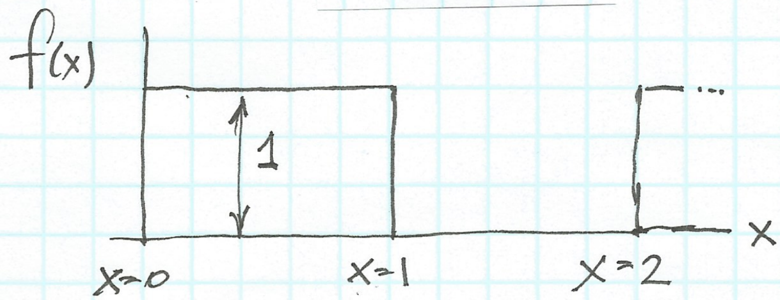

$$\left.\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \end{aligned}\right\} \text{ with } L=2$$

$$a_n = \int_0^2 f(x)\cos\frac{n\pi x}{L}\,dx = \int_0^1 \cos\frac{n\pi x}{L}\,dx \qquad (n>0)$$

$$= \left.\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right|_0^1 = \frac{\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} = \frac{2}{\pi}\,\frac{\sin\frac{n\pi}{2}}{n}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{2}\int_0^1 dx = 1$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \cdots\right)$$