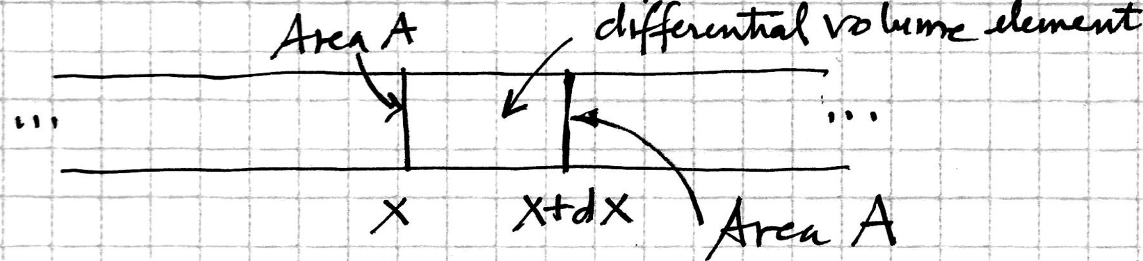
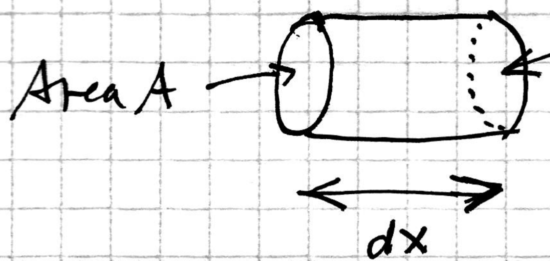
Heat flow along a rod (in 1 dimension)





Volume $= dV = A\,dx$

Let $u(x,t)$ = temperature at pt x at time t

The total heat energy in the volume element is

$$c \rho u \, dV$$

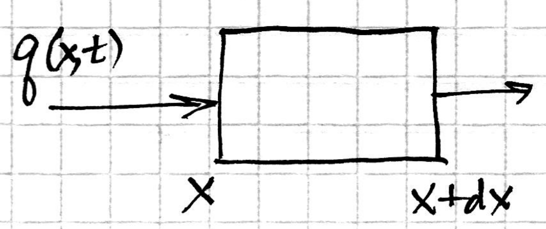
c = specific heat [calories/(mass °C)]

ρ = density [mass/volume]

In time Δt, the quantity of energy that has increased in the volume element is

$$c \rho \frac{\partial u}{\partial t} dV \Delta t$$

This increase in energy must balance the increase due to heat flow into the volume element at position x, minus the heat flow that is going out of the volume element at $x+dx$



$$q(x+dx,t) = q(x,t) + q_x(x,t)\,dx + \cdots$$

Taylor series

where $q(x,t)$ = heat flux

q has units of [calories/(area − sec)]

$$q = -k \frac{\partial u}{\partial x}$$ Fourier's Law of heat conduction

k = thermal conductivity [calories/(cm−sec °c)]

In time Δt, the increase in heat energy in the volume element = heat in − heat out

$$= q(x,t)\cdot A\cdot \Delta t - \underbrace{q(x+dx,t)\cdot A\cdot \Delta t}_{q(x,t) + \frac{\partial q}{\partial x}(x,t)\,dx + \cdots}$$

$$= -\frac{\partial q}{\partial x}\, dx\, A\, \Delta t$$

$$= +k\frac{\partial^2 u}{\partial x^2}\, dx\, A\, \Delta t$$

And this increase must equal the previously computed increase in energy, $c\rho \frac{\partial u}{\partial t}\, dV\, \Delta t$

$$k\frac{\partial^2 u}{\partial x^2}\, dx\, A\, \Delta t = c\rho \frac{\partial u}{\partial t}\, \underset{\substack{\parallel \\ A\,dx}}{dV}\, \Delta t$$

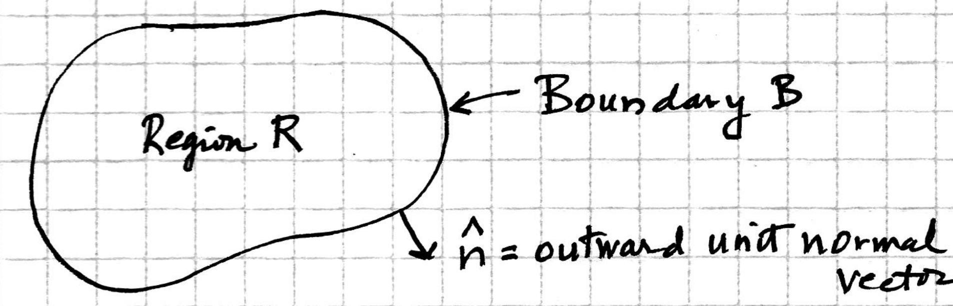
$$\Rightarrow \quad k\frac{\partial^2 u}{\partial x^2} = c\rho \frac{\partial u}{\partial t}$$

or $$\boxed{\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}}$$

where $\alpha^2 = \frac{k}{c\rho}$, α = thermal diffusivity

Heat flow in 3 dimensions

We will need the divergence theorem:



Let $\bar{q}(x,y,z,t)$ be a vector field (heat flux)

$$\iint_B \bar{q}\cdot\hat{n}\,dA = \iiint_R \nabla\cdot\bar{q}\,dV$$

The overall argument is the same as in the previous case.

In time Δt, the increase in heat energy stored in R is

$$\Delta t \iiint_R c\rho \frac{\partial u}{\partial t}\,dV$$

This increase in energy must be balanced by the increase due to heat flow into region R via its boundary B:

$$-\Delta t \iint_B \bar{q}\cdot\hat{n}\,dA$$

(minus sign because without it we have the flow out of R, since $\hat{n}$ is the outward normal)

Equating the two expressions, we obtain

$$\Delta t \iiint_R c\rho \frac{\partial u}{\partial t}\, dV = -\Delta t \iint_B \bar{q}\cdot\hat{n}\, dA$$

$$= -\Delta t \iiint_R \nabla\cdot\bar{q}\, dV \quad \left(\begin{matrix}\text{divergence}\\ \text{theorem}\end{matrix}\right)$$

And since this equation must be valid for every region R,

$$c\rho \frac{\partial u}{\partial t} = -\nabla\cdot\bar{q}$$

The vector form of Fourier's Law is

$$\bar{q} = -k\nabla u \quad \left(\text{compare with } q = -k\frac{\partial u}{\partial x} \text{ from previous case}\right)$$

which gives

$$c\rho\frac{\partial u}{\partial t} = k\,\nabla\cdot\nabla u = k\nabla^2 u$$

which may be written

$$\boxed{\alpha^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial u}{\partial t}}$$

where $\alpha^2 = \dfrac{k}{c\rho}$ as before.

In the case of steady state heat conduction in two dimensions, we have Laplace's equation:

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0}$$