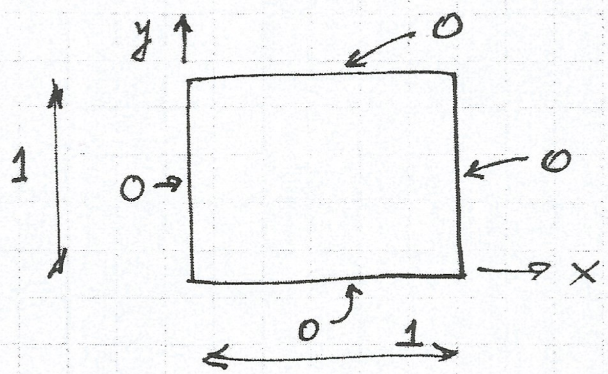
Heat conduction in 2 dimensions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$




IC
$t=0,\ u=f(x,y)$

$$u = X(x)\,Y(y)\,T(t)$$

$$T'XY = X''YT + XY''T$$

divide by $XYT$

$$\frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

$$\frac{X''}{X} = -\left(\frac{Y''}{Y}\right) - \lambda = -\mu$$

$$X'' + \mu X = 0 \qquad X(0) = X(1) = 0$$

$$Y'' + (\lambda - \mu) Y = 0 \qquad Y(0) = Y(1) = 0$$

$$T' + \lambda T = 0$$

$$\left.\begin{array}{ll} X = \sin n\pi x, & \mu = (n\pi)^2 \\ Y = \sin m\pi y, & (\lambda - \mu) = (m\pi)^2 \end{array}\right\} \lambda = (n^2 + m^2)\pi^2$$

$$u(x,y,t) = \sum_m \sum_n b_{n,m} \sin n\pi x \, \sin m\pi y \, e^{-(n^2+m^2)\pi^2 t}$$

to find $b_{n,m}$, use Fourier's theorem twice

at $t=0$, $f(x,y) = \sum \sum b_{n,m} \sin n\pi x \sin m\pi y$

Write it as

$$f(x,y) = \sum A_n \sin n\pi x$$

depends on $y$

$$A_n = \underset{1''}{\frac{2}{L}} \int_0^L f(x,y) \sin n\pi x \, dx = A_n(y)$$

then write

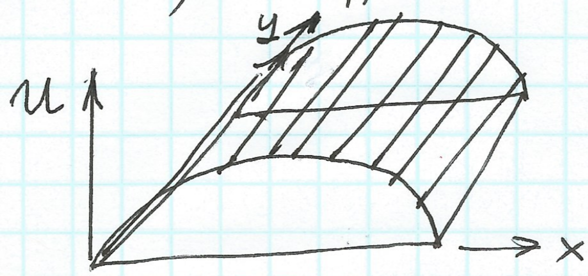
$$A_n(y) = \sum b_{n,m} \sin m\pi y$$

$$b_{n,m} = 2 \int_0^1 A_n(y) \sin m\pi y \, dy$$

$$= 4 \int_0^1 dx \int_0^1 dy \sin n\pi x \sin m\pi y \, f(x,y)$$

Initial Conditions

$t=0, \quad u(x,y) = \sin \pi x$




(no y dependence)

$$b_{n,m} = 4\int_0^1 dx \int_0^1 dy \, \sin n\pi x \, \sin m\pi y \, \sin \pi x$$

$$= 4 \underbrace{\int_0^1 \sin n\pi x \, \sin \pi x \, dx}_{\begin{cases} \int_0^1 (\sin \pi x)^2 dx, & n=1 \\ 0, & n\neq 1 \end{cases}} \int_0^1 \sin m\pi y \, dy$$

$$b_{n,m} = \begin{cases} 4\left(\frac{1}{2}\right), & n=1 \\ 0, & n\neq 1 \end{cases} \underbrace{\int_0^1 \sin m\pi y \, dy}$$

$$-\frac{\cos m\pi y}{m\pi}\Big|_0^1 = \begin{cases} \frac{2}{m\pi}, & m=1,3,5,\ldots \\ 0, & m=2,4,\ldots \end{cases}$$

$$u(x,y,t) = 2\sin\pi x \left(\frac{2}{\pi}\right)\left(\sin\pi y \, e^{-2\pi^2 t} + \frac{\sin 3\pi y}{3} e^{-10\pi^2 t} + \frac{\sin 5\pi y}{5} e^{-26\pi^2 t} + \cdots\right)$$

$$= \sum\sum b_{n,m} \sin n\pi x \, \sin m\pi y \, e^{-(n^2+m^2)\pi^2 t}$$