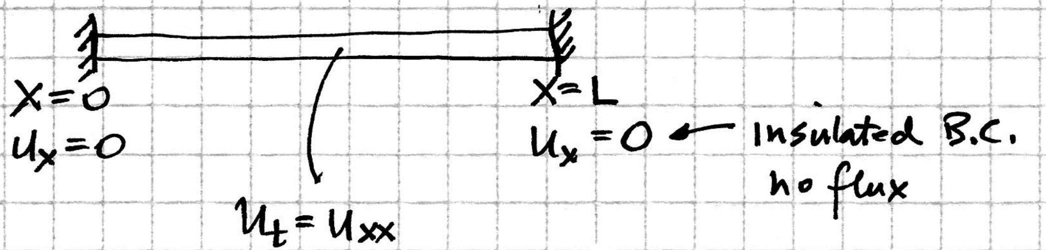
The Heat Equation (continued)



IC $t=0$, $u(x,0)=f(x)$

As before we use separation of variables to obtain

$$u(x,t) = X(x)\,T(t) \Rightarrow$$

$$X'' + \lambda X = 0, \qquad T'' + \lambda T = 0$$

$$X'(0) = X'(L) = 0$$

$$X(x) = c_1 \sin\sqrt{\lambda}\,x + c_2 \cos\sqrt{\lambda}\,x$$

$$X'(x) = c_1\sqrt{\lambda}\cos\sqrt{\lambda}\,x - c_2\sqrt{\lambda}\sin\sqrt{\lambda}\,x$$

$$X'(0) = c_1\sqrt{\lambda} = 0 \Rightarrow c_1 = 0$$

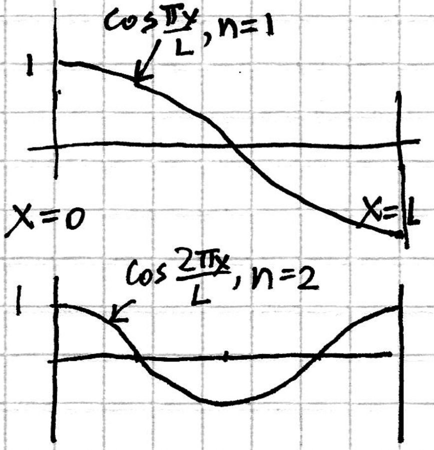
$$X'(L) = -c_2\sqrt{\lambda}\sin\sqrt{\lambda}\,L = 0$$

$$\Rightarrow \text{ either } \lambda = 0 \text{ or } \sqrt{\lambda}\,L = n\pi,\ n=1,2,3,\ldots$$

eigenfunctions: $X(x) = \begin{cases} 1 & \text{if } \lambda = 0 \\ \cos\frac{n\pi x}{L} & \text{if } \lambda = \left(\frac{n\pi}{L}\right)^2 \end{cases}$

$$T' + \lambda T = 0 \Rightarrow T = c_3 e^{-\lambda t}$$

$$= c_3 e^{-\frac{n^2\pi^2}{L^2}t}$$



general solution:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}\, e^{-\frac{n^2\pi^2}{L^2}t} + \frac{a_0}{2}$$

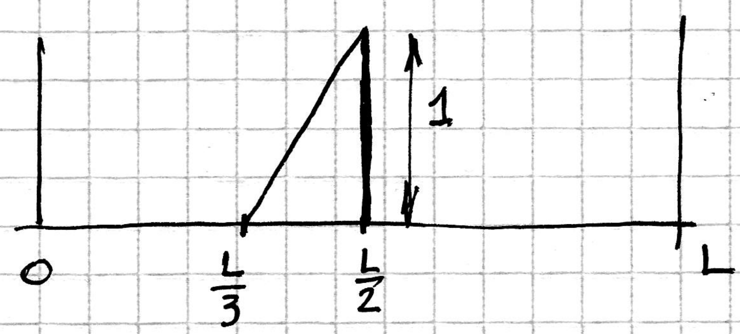
Initial Condition : $t=0$, $u(x,0) = f(x) = \text{given}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Fourier cosine series.

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \quad , \; n=0,1,2,\ldots$$

Example

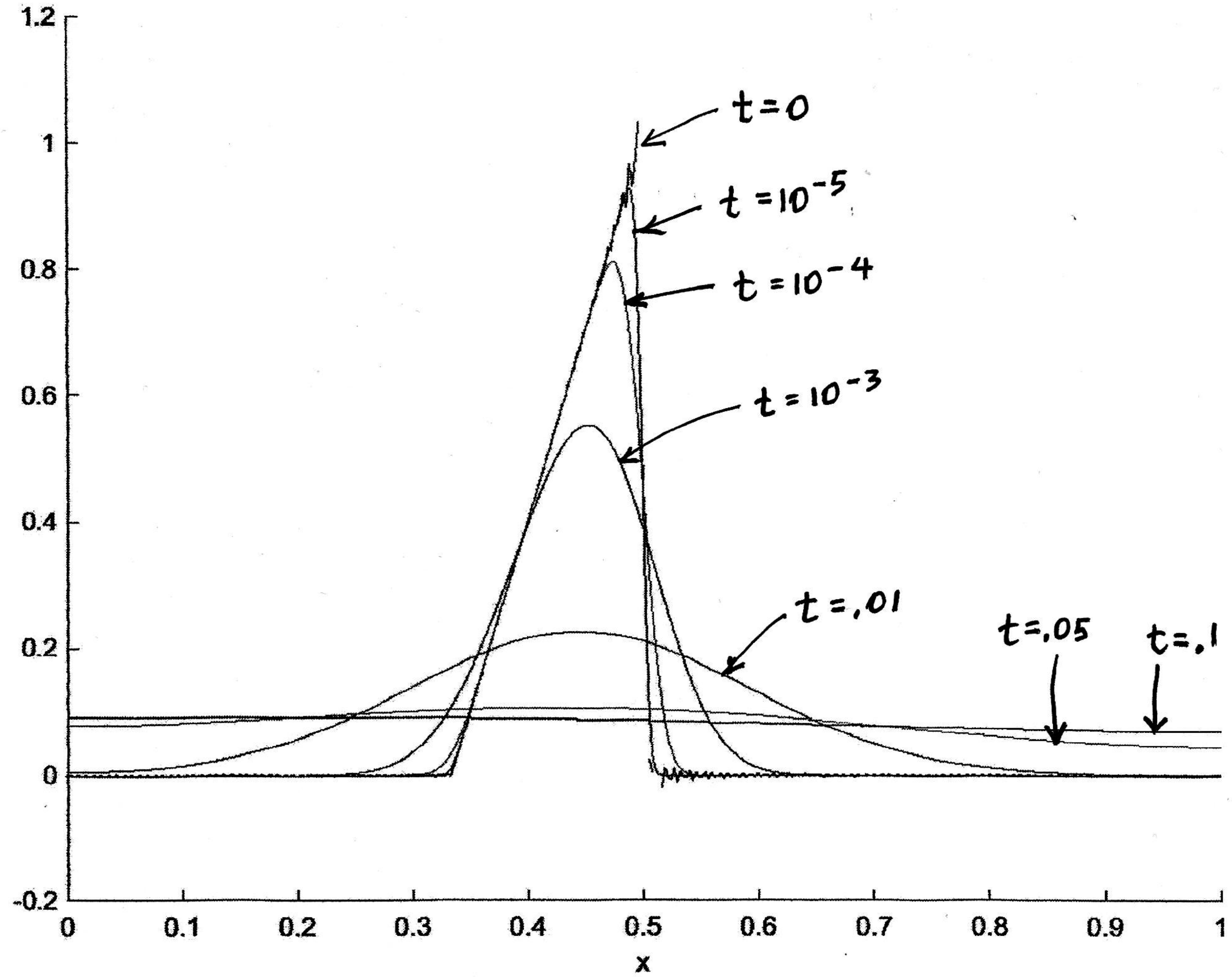


$$f(x) = u(x,0) = \begin{cases} 0, & 0 < x < L/3 \\ \frac{6}{L}\left(x-\frac{L}{3}\right), & \frac{L}{3} < x < \frac{L}{2} \\ 0, & \frac{L}{2} < x < L \end{cases}$$

$$a_n = \frac{2}{L}\int_{\frac{L}{3}}^{L/2} \frac{6}{L}\left(x-\frac{L}{3}\right)\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{12}{\pi^2 n^2}\left(\cos\frac{n\pi}{2} - \cos\frac{n\pi}{3}\right) + \frac{2}{\pi n}\sin\frac{n\pi}{2} \quad , \; n>0$$

$$a_0 = \frac{1}{6}$$


t=0
t = 10^{-5}
t = 10^{-4}
t = 10^{-3}
t = ,01
t=,05
t=,1
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

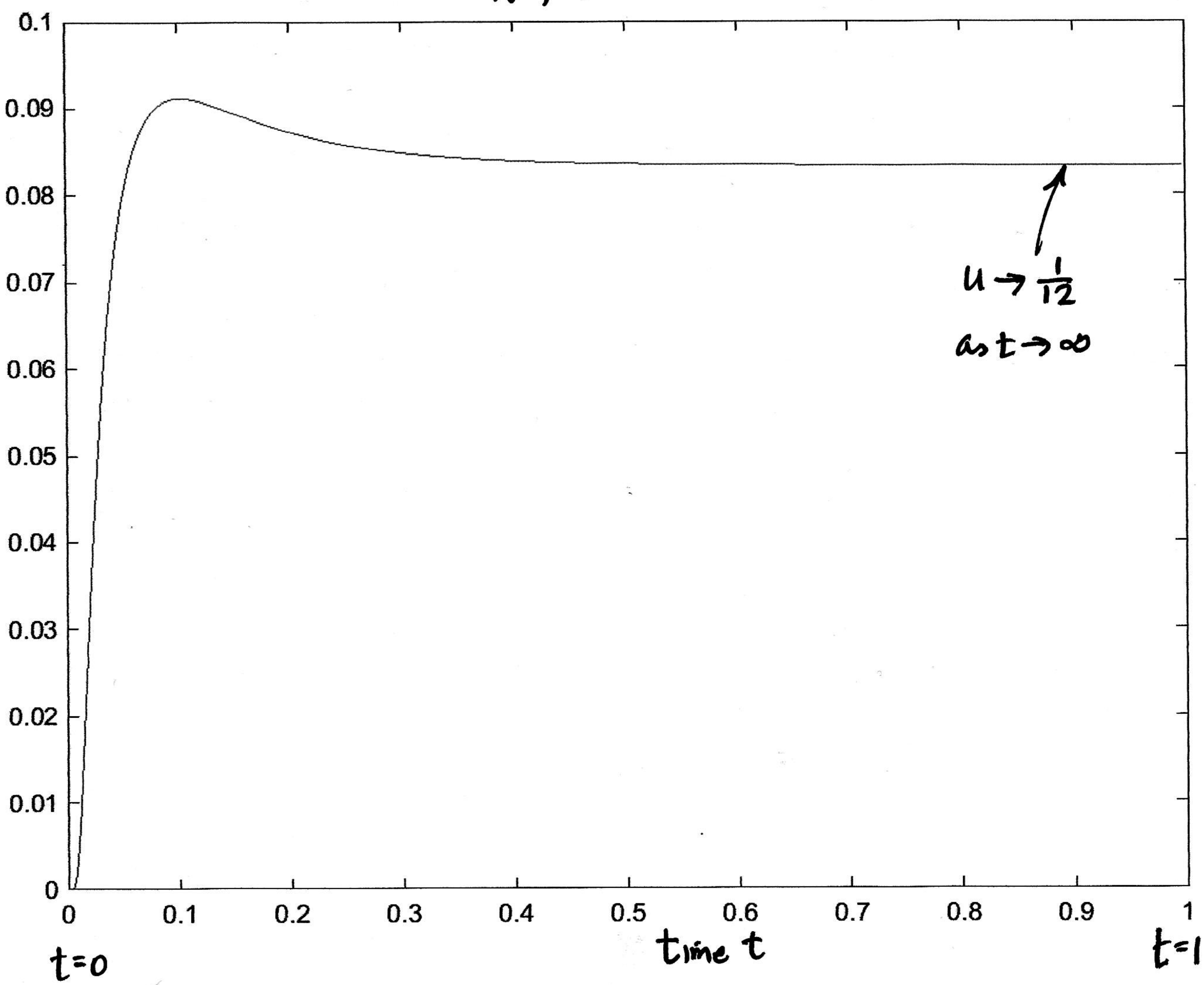
L=1 , 300 terms

Behavior at $x=0$:

$$u(0,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{L^2}t} \quad \left(\text{since } \cos\frac{n\pi x}{L} = 1 \text{ at } x=0\right)$$

The long-time behavior is $u(0,t) \sim \frac{a_0}{2} = \frac{1}{12} = .0833\ldots$

$u(0,t) = \{u \text{ at } x=0\}$



L=1, 300 terms

Notice the interesting overshoot at $x=0$.
It peaks at around $t=0.1$, then it starts coming down.