Separation of Variables Solution of Wave Equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

BC $\quad x=0,\ u=0$

$\quad\quad x=L,\ u=0$

IC $\quad t=0,\ u=f(x)$

$\quad\quad t=0,\ \frac{\partial u}{\partial t}=0$ (released from rest, as it would be if a string were plucked)

Ansatz: $u(x,t) = X(x)\,T(t)$

$a^2 X''T = XT''$ where $'$ = differentiation with respect to the argument of the function

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

$$X'' + \lambda X = 0,$$

$$T'' + a^2 \lambda T = 0$$

BC $\quad x=0,\ u=0 \Rightarrow X(0)=0$

$\quad\quad x=L,\ u=0 \Rightarrow X(L)=0$

IC $\quad t=0,\ u_t=0 \Rightarrow T'(0)=0$

(Note that the 2nd IC, $t=0,\ u=f(x)$, is not separable.)

$$X''+\lambda X,\ X(0)=0,\ X(L)=0$$

is the same as we encountered in the heat equation.

We saw in the heat eq. that

$$X(x) = \sin\frac{n\pi x}{L}, \quad \lambda = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \ldots$$

The T equation is $T'' + a^2\lambda T = 0$, or

$$T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0$$

giving $\quad T = c_1 \sin\frac{an\pi}{L}t + c_2 \cos\frac{an\pi}{L}t$

The IC $t=0$, $\frac{\partial u}{\partial t} = 0$ gives $c_1 = 0$

$$T = c_2 \cos\frac{an\pi}{L}t$$

We have $\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \cos\frac{an\pi}{L}t}$

Now for the last IC: $t=0$, $u(x,0) = f(x)$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$$

The same sort of Fourier sine series we saw before.

$$\boxed{c_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{an\pi}{L} t$$

where

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

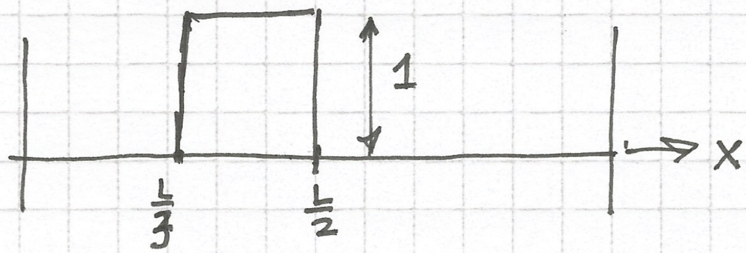
Note that for fixed x, $u(x,t)$ is a Fourier series in t which means it is periodic in t. What is its period? $n=1$ has the longest period $\left(= \frac{2\pi}{\omega} = \frac{2\pi}{\frac{a\pi}{L}} = \frac{2L}{a} \right)$

and thus determines the period of the sum.

So, for example, if $L=1$ and $a=1$, the period is 2

<u>Example</u> Let us take $a=1$ $\left(a^2 = \frac{T}{\rho}\right)$

and same IC as we chose for heat equation:



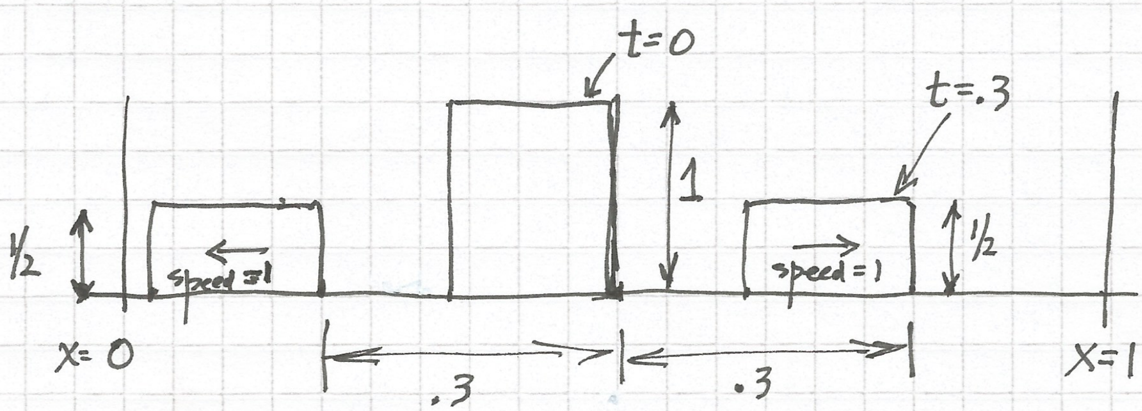
$$f(x) = \begin{cases} 0 & 0 < x < \frac{L}{3} \\ 1 & \frac{L}{3} \le x \le \frac{L}{2} \\ 0 & \frac{L}{2} < x < L \end{cases}$$

Then,

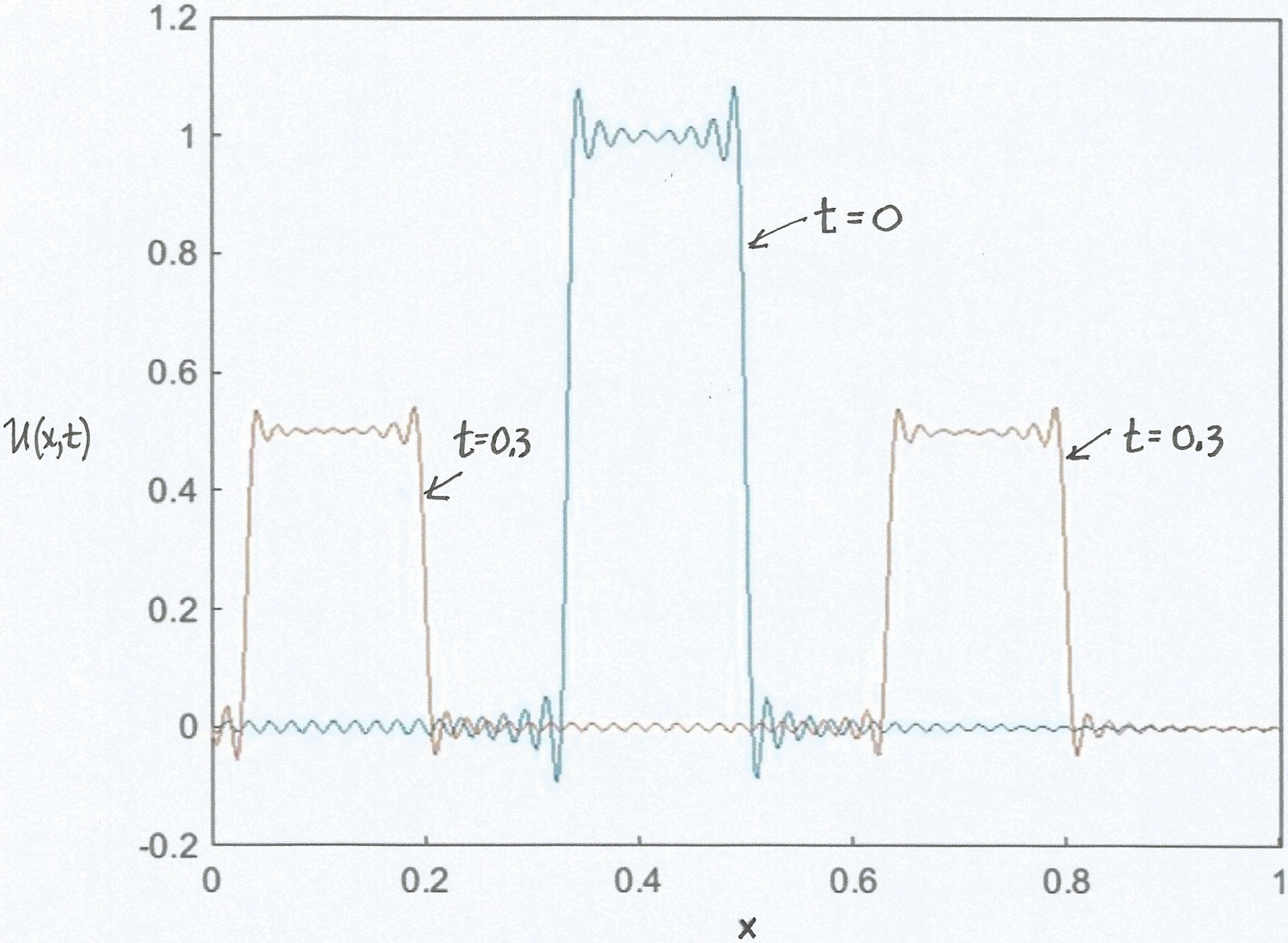
$$c_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{L/3}^{L/2} 1\cdot \sin\frac{n\pi x}{L}\,dx$$

$$= -\frac{2}{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Bigg|_{L/3}^{L/2} = -\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos\frac{n\pi}{3}\right)$$

Numerical evaluation (see next page)



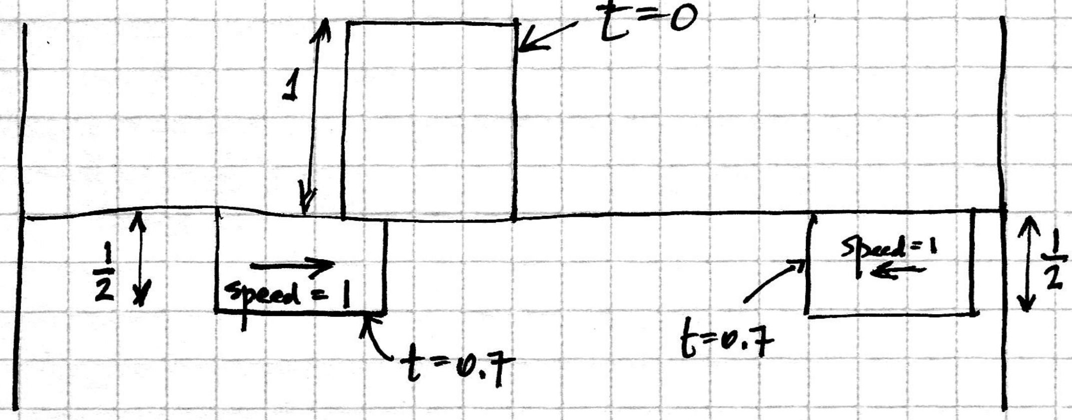
Note that 1) the original IC gives rise to two half-size copies
2) each moving in opposite directions with speed $a=1$



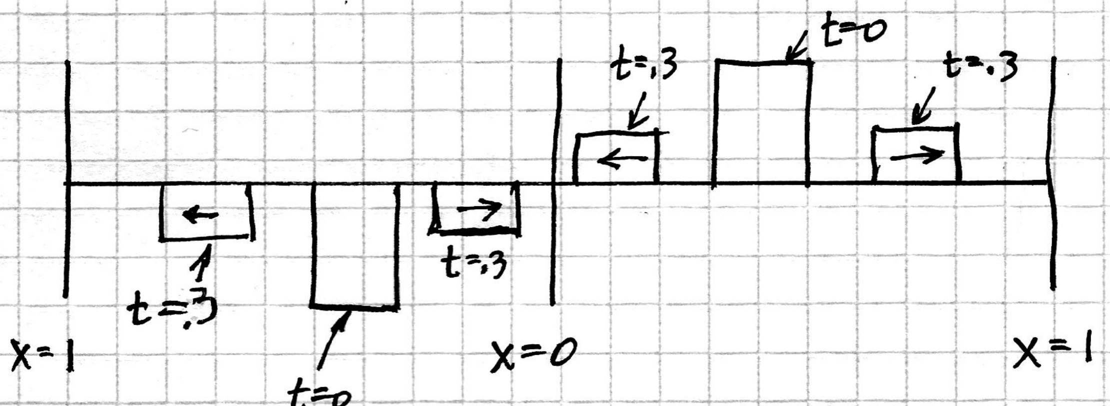
L=1 , 100 terms
a=1

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left(\cos\frac{n\pi}{2} - \cos\frac{n\pi}{3}\right) \sin\frac{n\pi x}{L} \cos\frac{an\pi t}{L}$$

Further numerical evaluations (see next page) reveals <u>reflections</u> at the boundaries $x=0, L$



Explanation: Since the sine series is odd, we have these solutions in the region $[-1,0)$:

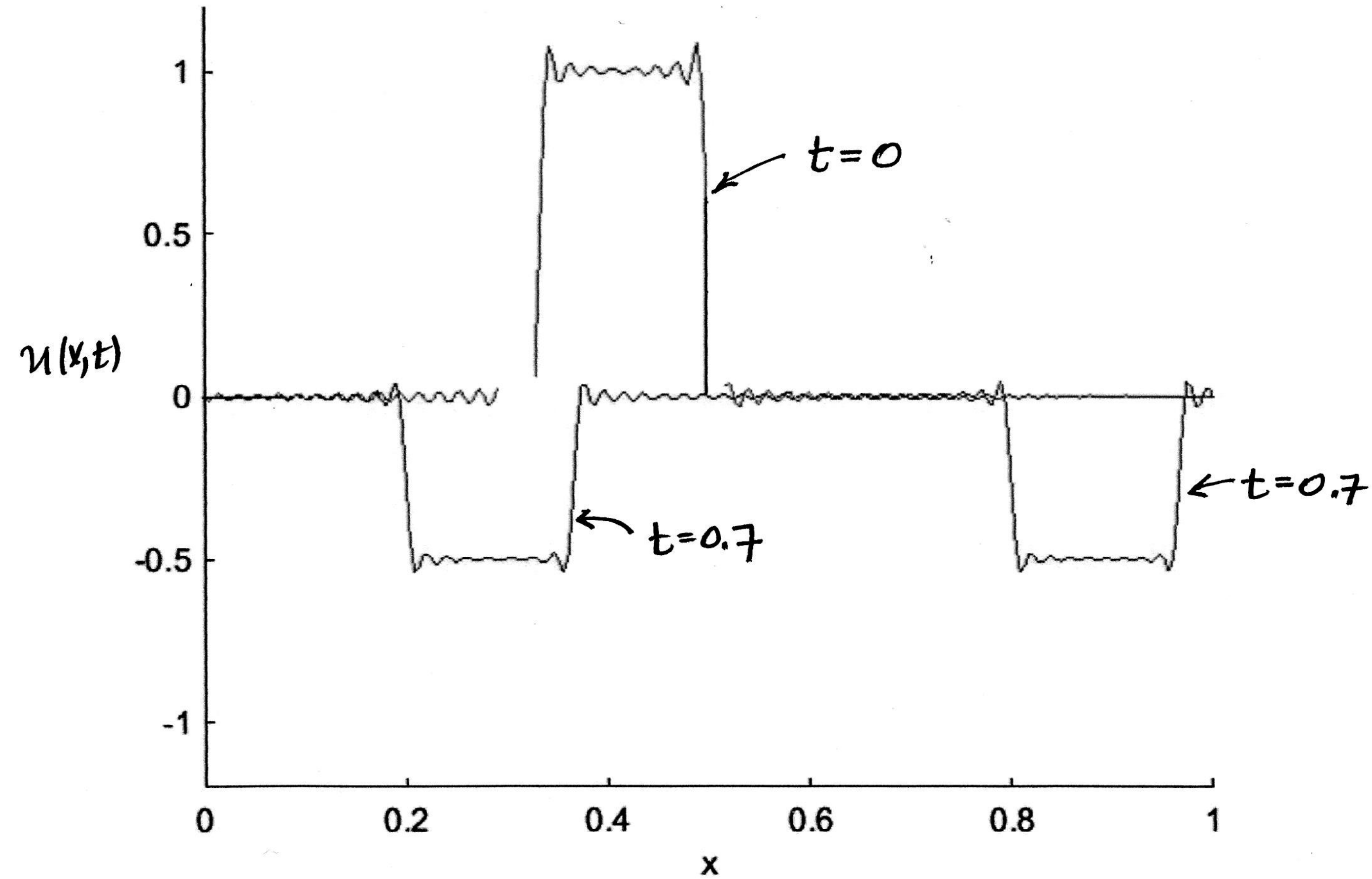


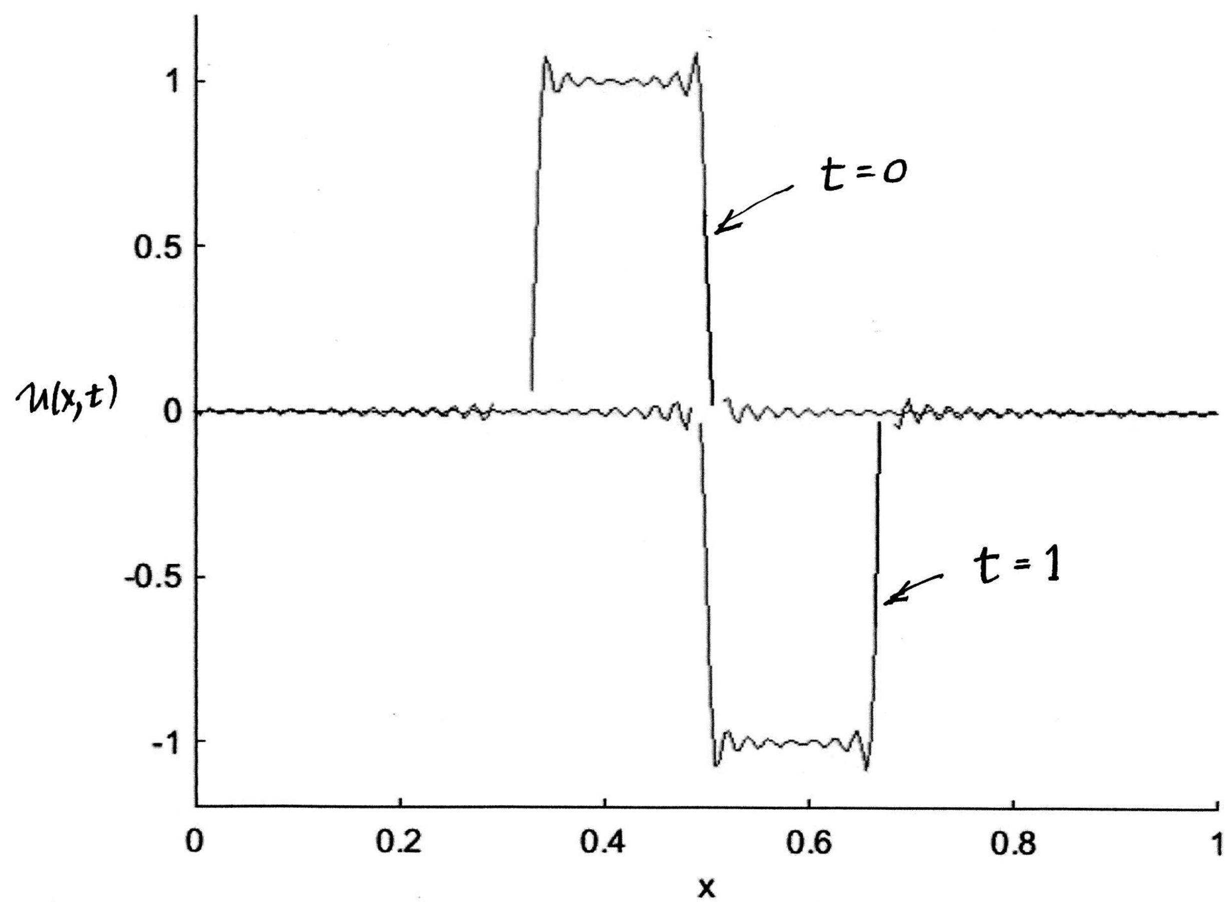
At $t=0.7$, the half-size copies in $[-1,0)$ for $t=0.3$ have moved into neighboring regions.

As they pass over the boundaries $x=0,1$, they cancel out with the half-size copies moving in the opposite direction, satisfying the BC that $u=0$ at $x=0,1$.

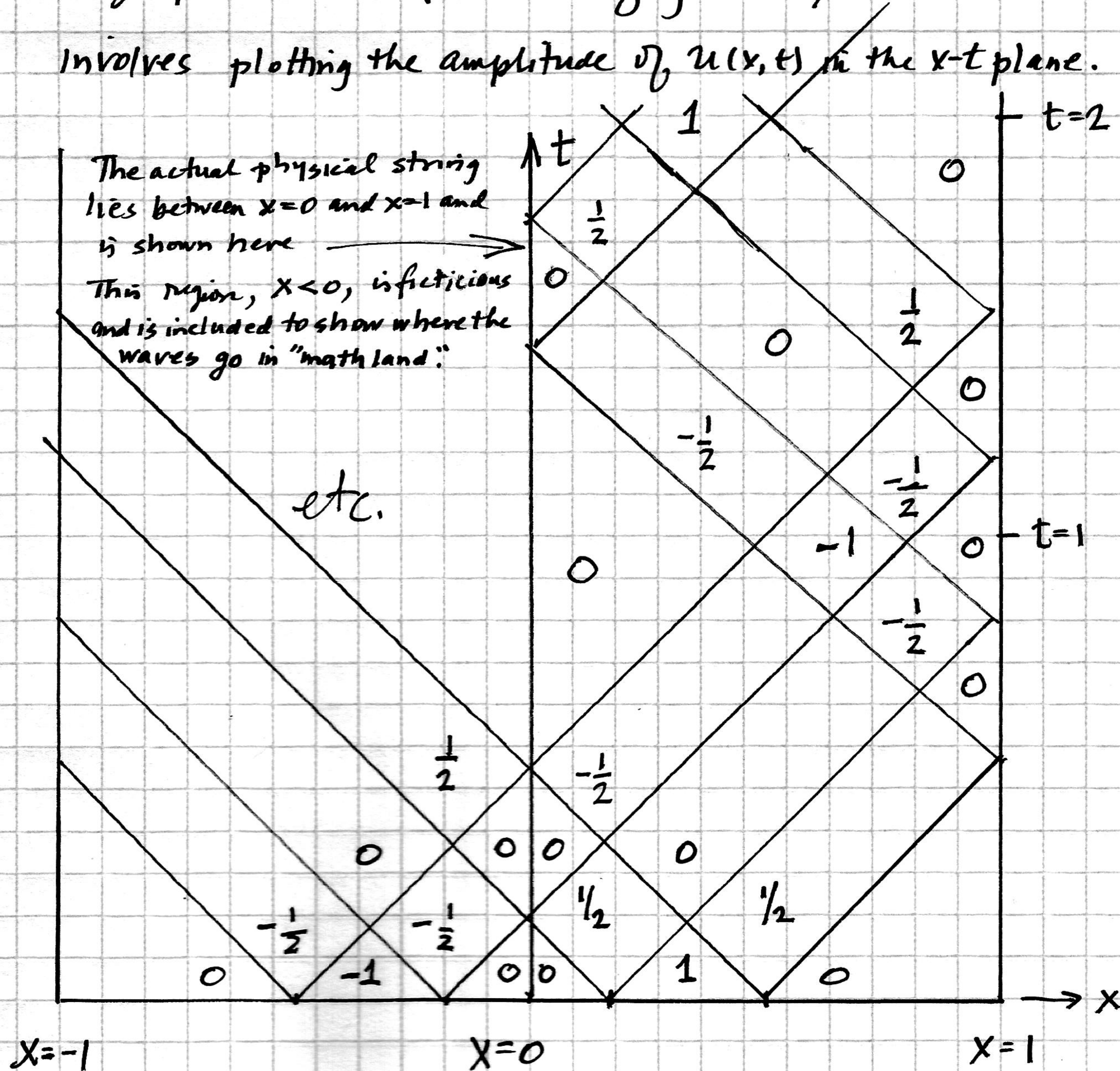
At $t=1$ (see next page), the two half-size copies seen at $t=0.7$, superimpose to create a full-size copy of the IC, upside-down and reflected.

At $t=2$ (not shown) this process repeats itself, and the original IC are regenerated, showing the period $=2$.


t=0
t=0.7
t=0.7
u(x,t)
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
x


t=0
t = 1
u(x,t)
1
0.5
0
-0.5
-1
0
0.2
0.4
0.6
0.8
1
x

A graphic scheme for visualizing these results involves plotting the amplitude of $u(x,t)$ in the x-t plane.



A motion that moves with speed 1 shows up as a straight line $x = t + \text{constant}$, i.e. a line with slope 1.

So far we have concluded that waves exist by observing numerical results. But we can show this analytically. For example, we derived the expression

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \cos \frac{an\pi}{L} t$$

From the identity

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$

we get (adding the 2 identities)

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2 \sin\alpha \cos\beta$$

$\therefore$ we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{C_n}{2} \left\{ \sin \frac{n\pi}{L}(x+at) + \sin \frac{n\pi}{L}(x-at) \right\}$$

Eq. *

$$= \sum_{n=1}^{\infty} \frac{C_n}{2} \sin \frac{n\pi}{L}(x+at) + \sum_{n=1}^{\infty} \frac{C_n}{2} \sin \frac{n\pi}{L}(x-at)$$

Note that at $t=0$, we have

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} = f(x)$$

Therefore Eq. * may be written

$$u(x,t) = \tfrac{1}{2} f(x+at) + \tfrac{1}{2} f(x-at)$$

In general $f(x-at)$ represents a wave moving to the right with speed a.

Thus the motion of a string released from rest consists of two waves, one moving to the right and one moving to the left, each with shape $= \frac{1}{2}$ the initial shape of the string.