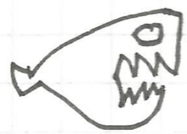
An example based on 2 competing populations

<u>Predator-Prey System</u>

We model a lake in which there are two distinct populations:

predators  population $= x(t)$

and prey  population $= y(t)$

The prey eat plankton (or whatever), which is assumed to exist in unlimited quantities, so that without interactions with the predators, the prey grow without bound, based on:

$$\frac{dy}{dt} = a\,y \qquad \left(\Rightarrow y = c_1 e^{at}\right)$$

The predators on the other hand eat only the prey, so that without interactions, they starve:

$$\frac{dx}{dt} = -b x \qquad \left(\Rightarrow x = c_2 e^{-bt}\right)$$

Every time there is a meeting between a predator and a prey, $\dot{x}$ increases and $\dot{y}$ decreases.

Suppose we model the magnitude of the interaction term by the number of ways that a meeting between x and y could occur, namely $x \cdot y$:

$$\frac{dx}{dt} = -bx + cxy$$

$$\frac{dy}{dt} = ay - dxy$$

(of course we have assumed that $a, b, c, d > 0$)

What will be the nature of the population dynamics in the lake?

To find out, we must solve the ODE's.

Note that we have two 1st order ODE's on $x(t)$ and $y(t)$, and that <u>the right hand sides of the ODE's do not depend on time t.</u>

In this case we may reduce the system of 2 ODE's on $x(t), y(t)$, to a single ODE on $y(x)$. To do so we use the <u>chain rule</u>:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{ay - dxy}{-bx + cxy}$$

For convenience in solving, we take

$$a = b = c = d = 1$$

giving

$$\frac{dy}{dx} = \frac{y - xy}{-x + xy} = \frac{y(1-x)}{x(-1+y)}$$

We may solve this by using <u>separation of variables</u>:

Move the terms with y (including dy) to one side,

and the terms with x (including dx) to the other side:

$$\frac{(-1+y)}{y}\,dy = \frac{(1-x)}{x}\,dx$$

Then perform an indefinite integral of both sides:

$$\int \frac{(-1+y)}{y}\,dy = \int \frac{(1-x)}{x}\,dx$$

$$\int \left(-\frac{1}{y} + 1\right) dy = \int \left(\frac{1}{x} - 1\right) dx$$

$$-\ln y + y = \ln x - x + C$$

arbitrary constant

As usual, we determine C using IC.

Suppose we pick $x(0) = y(0) = 0.5$

Then we find $C = 2(.5 - \ln(.5)) = 2.3863$

We can visualize the solution by using MATLAB:

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>> ezplot('-log(y)+y-log(x)+x-2.3863',[0 3 0 3]);
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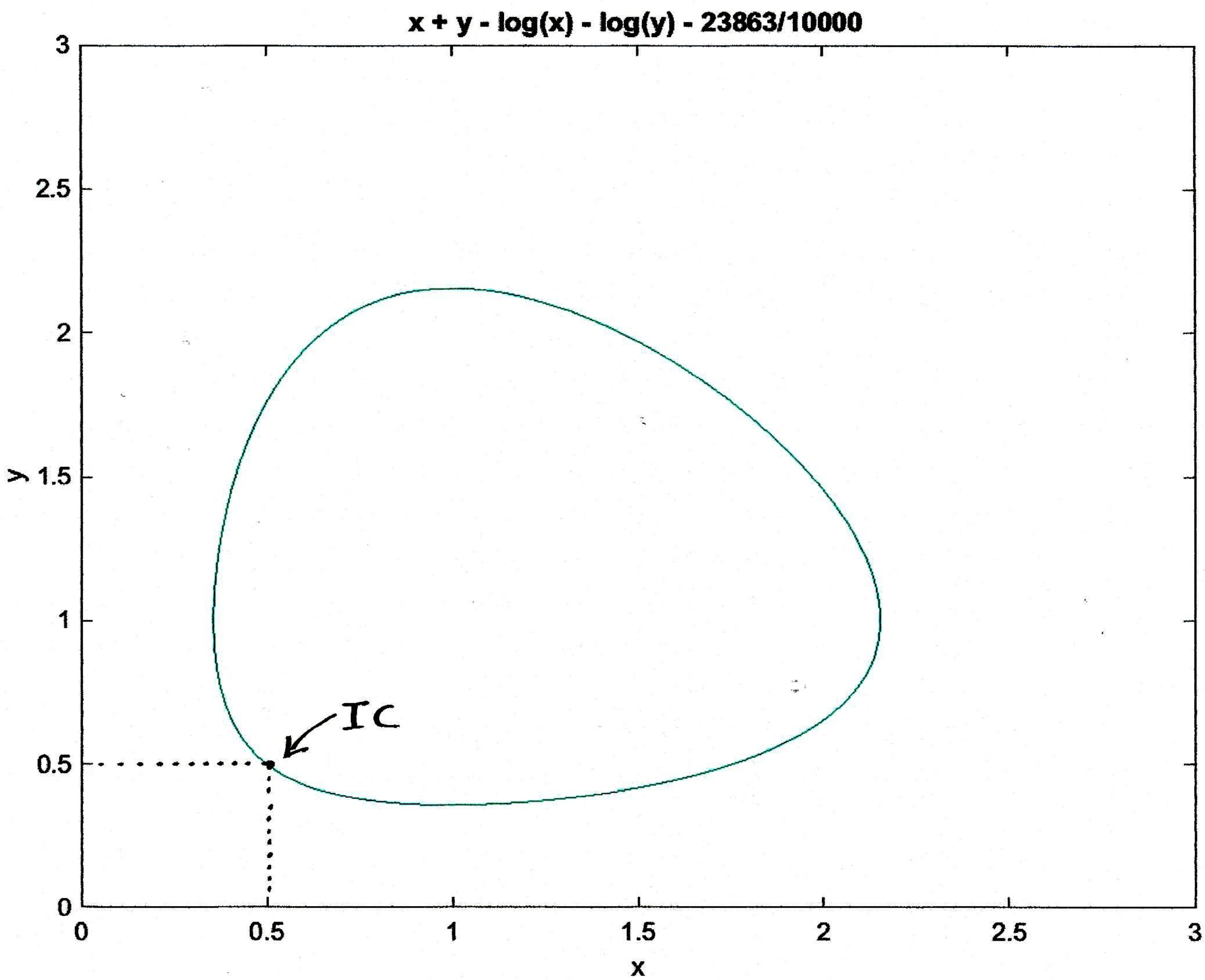

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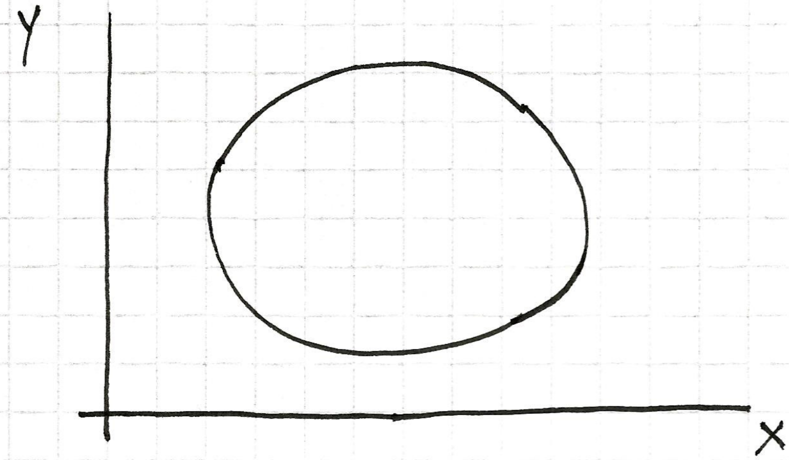


The motion is periodic in time, but which direction does the rotation occur in?

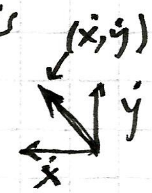
To find out, go back to the $\dot{x}$, $\dot{y}$ equations:

$$\dot{x} = -x + xy$$
$$\dot{y} = y - xy$$

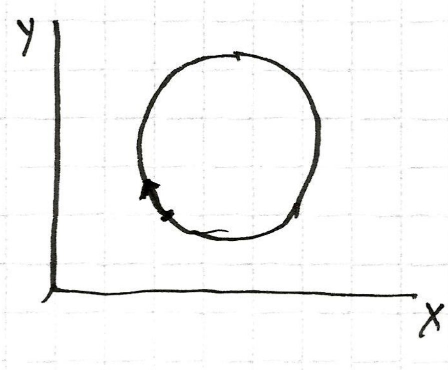
Choose any point on the solution curve and observe which direction the $(\dot{x}, \dot{y})$ vector points in.

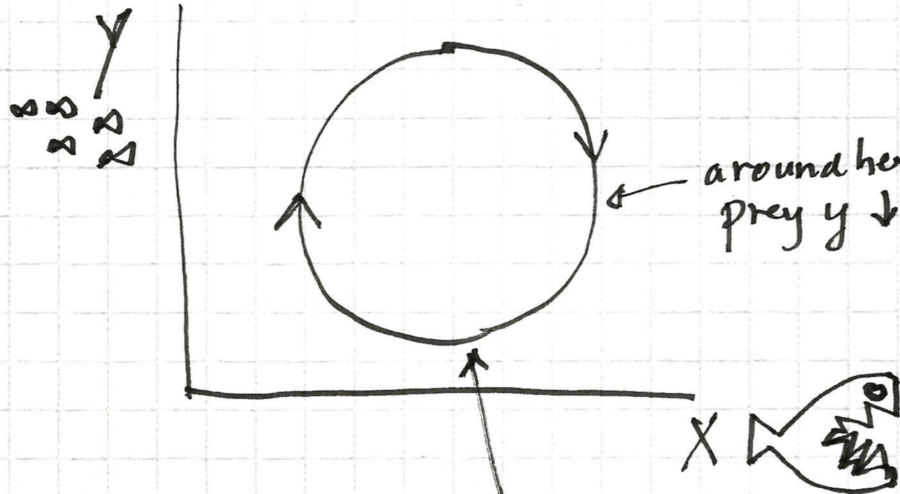
Take the IC, $x = .5$, $y = .5$ which gives

$$\left.\begin{aligned} \dot{x} &= -.5 + .25 = -.25 \\ \dot{y} &= +.5 - .25 = +.25 \end{aligned}\right\}$$



So the motion is clockwise:





around here the prey $y \downarrow$, being eaten

around here the predators $x \downarrow$, because their food (=prey) are relatively scarce.

Use pplane.jar to visualize the whole plane (google "pplane" and choose "dfield and pplane (Java versions)")

<u>Equilibrium Points</u>

Set $\dot{x}=0, \dot{y}=0$

$$\left.\begin{aligned} \dot{x} &= x(-1+y) \\ \dot{y} &= y(1-x) \end{aligned}\right\} \Rightarrow \begin{aligned} &x=y=0 \\ &\text{and } x=y=1 \end{aligned}$$

The point $(x=1, y=1)$ is the center of all the rotations.

pplane.jar