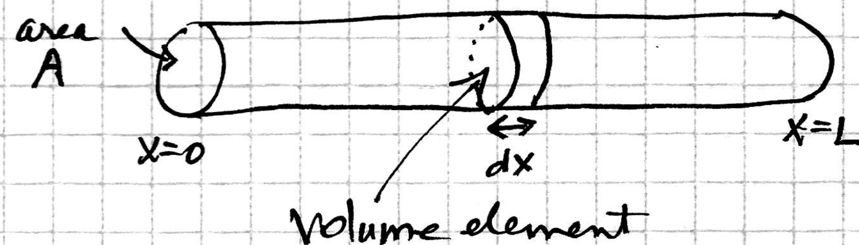
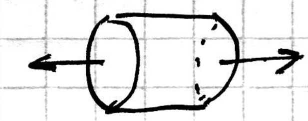
The Wave Equation

We begin with longitudinal waves in a rod



Free body diagram of volume element:

$A \cdot \sigma(x,t)$  $\sigma(x+dx,t)\cdot A$

$\sigma(x,t)$ = stress [force/area]

$F = ma \Rightarrow$

$$\Big[\underbrace{\sigma(x+dx,t)}_{\sigma(x,t)+\frac{\partial\sigma}{\partial x}dx+\cdots} - \sigma(x,t)\Big]A = \underbrace{\rho\, dV}_{\substack{\text{mass of}\\ \text{volume}\\ \text{element}}}\ \underbrace{u_{tt}}_{\text{acceleration}}$$

$u(x,t)$ = displacement

ρ = density [mass/volume]

$dV = A\,dx$

$$\Rightarrow \quad \frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}$$

Hooke's Law $\quad \sigma = E \dfrac{\partial u}{\partial x}$ ← strain

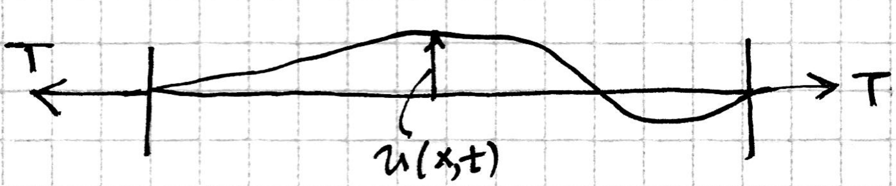
Young's modulus [force/area]

giving

$$E \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}$$

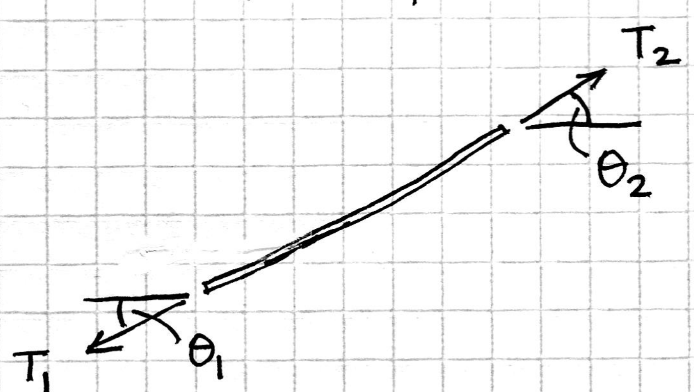
or $\quad a^2 u_{xx} = u_{tt}, \quad a^2 = \dfrac{E}{\rho}$ [length^2/sec^2]

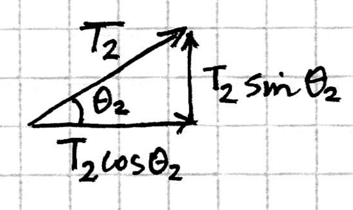
Transverse waves in a stretched string



a string is stretched tight by a force T as shown.
We assume the motions are small and vertical.
Note that the tension T tends to pull the deformed string back to the straight line position.
(This is versus the string being caused to return to its original position due to its microstructure, i.e. its elastic stiffness. Such a device is called a "beam". The string is assumed to have zero stiffness.)

Free body of a portion of the string:





Equilibrium in horizontal direction $\Rightarrow$

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T = \text{constant}$$

F = ma in the vertical direction:

$$T_2 \sin\theta_2 - T_1 \sin\theta_1 = m \frac{\partial^2 u}{\partial t^2}$$

where now we take the portion of the string to have length dx so that u in $\frac{\partial^2 u}{\partial t^2}$ is evaluated at (x,t).

Then $m = \rho\, dx$ where ρ = mass / length

Divide both sides by T $(= T_1 \cos\theta_1 = T_2 \cos\theta_2)$:

$$\frac{T_2 \sin\theta_2}{T} - \frac{T_1 \sin\theta_1}{T} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2} dx$$

$$\frac{T_2 \sin\theta_2}{T_2 \cos\theta_2} - \frac{T_1 \sin\theta_1}{T_1 \cos\theta_1} =$$

$$\tan\theta_2 - \tan\theta_1 =$$

$$\underbrace{\left.\frac{\partial u}{\partial x}\right|_{x+dx}}_{\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx + \cdots} - \left.\frac{\partial u}{\partial x}\right|_{x} =$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$$a^2 u_{xx} = u_{tt}, \quad a^2 = \frac{T}{\rho}$$

So far we have concluded that waves exist by observing numerical results. But we can show this analytically. For example, we derived the expression

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} \cos\frac{an\pi}{L}t$$

From the identity

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

we get (adding the 2 identities)

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

$\therefore$ we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{C_n}{2}\left\{ \sin\frac{n\pi}{L}(x+at) + \sin\frac{n\pi}{L}(x-at) \right\}$$

Eq.*

$$= \sum_{n=1}^{\infty} \frac{C_n}{2}\sin\frac{n\pi}{L}(x+at) + \sum_{n=1}^{\infty} \frac{C_n}{2}\sin\frac{n\pi}{L}(x-at)$$

Note that at $t=0$, we have

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = f(x)$$

Therefore Eq.* may be written

$$u(x,t) = \frac{1}{2}f(x+at) + \frac{1}{2}f(x-at)$$

In general $f(x-at)$ represents a wave moving to the right with speed a.

Thus the motion of a string released from rest consists of two waves, one moving to the right and one moving to the left, each with shape $= \frac{1}{2}$ the initial shape of the string.