The Wave Equation (continued)

Note: Some of the material covered in this lecture is the subject of HW problems 13, 16 in § 10.7.

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

Let us define two new variables ξ, η defined by

$$\xi = x - ct$$
$$\eta = x + ct$$

Use the chain rule:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \underbrace{\frac{\partial \xi}{\partial x}}_{=1} + \frac{\partial u}{\partial \eta} \underbrace{\frac{\partial \eta}{\partial x}}_{=1}$$

$$= \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) \underbrace{\frac{\partial \xi}{\partial x}}_{=1} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) \underbrace{\frac{\partial \eta}{\partial x}}_{=1}$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\underset{-c}{\underset{\shortparallel}{\frac{\partial \xi}{\partial t}}} + \frac{\partial u}{\partial \eta}\underset{c}{\underset{\shortparallel}{\frac{\partial \eta}{\partial t}}}$$

$$= -c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(-c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}\right)$$

$$= -c\frac{\partial}{\partial \xi}\left(-c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}\right) + c\frac{\partial}{\partial \eta}\left(-c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}\right)$$

$$= c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2}$$

$$\therefore\ c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \Rightarrow c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$

$$\Rightarrow \quad 4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \Rightarrow \quad \frac{\partial u}{\partial \eta} = f(\eta)$$

an arbitrary function of η

$$\therefore\ u(\xi,\eta) = \underbrace{\int f(\eta)\,d\eta}_{\substack{\text{call it} \\ F(\eta)}} + g(\xi)$$

an arbitrary function of ξ

(still an arbitrary function of η)

Thus we have shown that

$$u(x,t) = F(x+ct) + g(x-ct)$$

a wave moving to the left

a wave moving to the right

Initial Conditions

$$u(x,0) = Q(x) = \text{given function}$$

$$u_t(x,0) = 0 \quad (\text{released from rest})$$

The general solution has been shown to be of the form:

$$u(x,t) = F(x+ct) + g(x-ct) \qquad (*)$$

Note that $u_t(x,t) = F'(x+ct)\frac{\partial}{\partial t}(x+ct) + g'(x-ct)\frac{\partial}{\partial t}(x-ct)$

$$= c\,F'(x+ct) - c\,g'(x-ct)$$

where $'$ denotes differentiation with respect to the argument.

Plug these expressions into the initial conds:

$$u(x,0) = Q(x) = F(x) + g(x) \qquad (1)$$

$$u_t(x,0) = 0 = c\,F'(x) - c\,g'(x) \qquad (2)$$

Integrate this last equation with respect to x:

$$c\,(F(x) - g(x)) = \text{Const} = K \qquad (3)$$

Solve eqs. (1), (3) for $F(x)$ and $g(x)$:

$$F(x) + g(x) = Q(x)$$

$$F(x) - g(x) = \frac{K}{c}$$

$$2F(x) = Q(x) + \frac{K}{c}$$

$$2g(x) = Q(x) - \frac{K}{c}$$

Plugging into equation $(*)$ above, we find

$$u(x,t) = \frac{1}{2}Q(x+ct) + \frac{1}{2}Q(x-ct)$$

Boundary Conditions

So far we have satisfied the PDE and the IC, and we have

$$u(x,t) = \frac{1}{2} Q(x+ct) + \frac{1}{2} Q(x-ct) \qquad (6)$$

Now suppose we are given the following BC:

$$x=0, \quad u(0,t)=0 \qquad (4)$$

$$x=L, \quad u(L,t)=0 \qquad (5)$$

Eq.(4) gives

$$u(0,t) = 0 = \frac{1}{2} Q(ct) + \frac{1}{2} Q(-ct)$$

$$\therefore \quad Q(ct) = -Q(-ct)$$

or, changing the argument from ct to s,

$$Q(s) = -Q(-s) \qquad (7)$$

That is, the function Q must be an odd function.

Eq.(6) may be written, using $Q(x+ct) = -Q(-x-ct)$, which is just eq.(7) with $s=x+ct$,

$$u(x,t) = \frac{1}{2} Q(x-ct) - \frac{1}{2} Q(-x-ct)$$

Now we apply the second BC, eq.(5)

$$u(L,t) = \frac{1}{2} Q(L-ct) - \frac{1}{2} Q(-L-ct) = 0$$

or, if we set $s=-L-ct$

$$Q(s) = Q(2L+s)$$

This proves that the function $Q(s)$ has period $2L$

Summary

We have considered the following problem

$$c^2 u_{xx} = u_{tt}$$

$$\text{IC} \quad u(x,0) = Q(x)$$
$$u_t(x,0) = 0$$

$$\text{BC} \quad u(0,t) = 0$$
$$u(L,t) = 0$$

In the previous lecture we solved this problem using separation of variables and Fourier series. We saw that the resulting infinite series gave an odd function (because it was a Fourier sine series), and it gave a function which was periodic, with period $2L$ in x, and period $\frac{2L}{c}$ in t.

In today's lecture we saw that these properties of the solution could be obtained without actually constructing the infinite series.