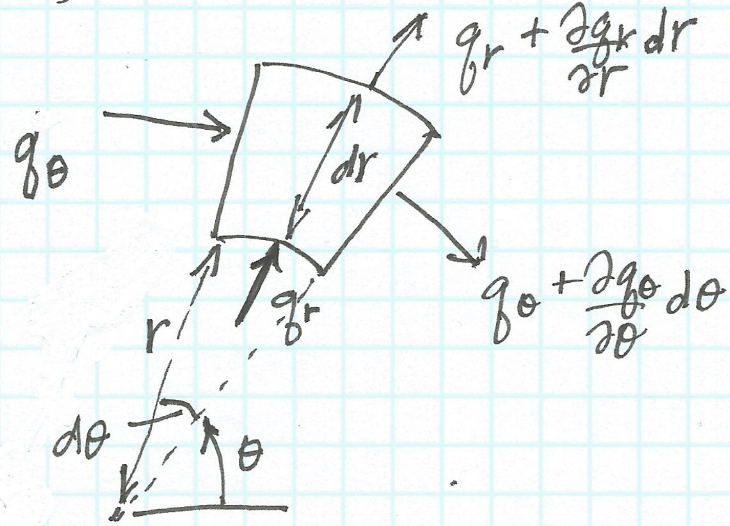
deriving the expression for

$\nabla^2 u$ in polar coordinates:

Examine a differential element of area:

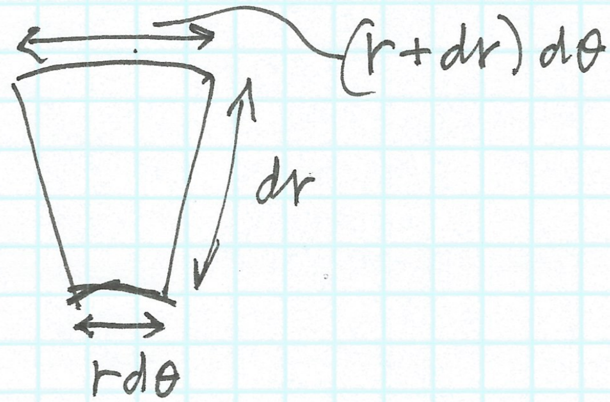


q = heat flux = [calories/area-second]

q_θ = heat flux in θ direction

q_r = " " " r "

Multiply by the area (= length of side × 1 unit depth) to get calories/sec



In time Δt,

$$\text{heat in} = \left[q_\theta\, dr + q_r\,(r\,d\theta)\right]\Delta t$$

$$\text{heat out} = \left[\left(q_\theta + \frac{\partial q_\theta}{\partial \theta}d\theta\right)dr + \left(q_r + \frac{\partial q_r}{\partial r}dr\right)(r+dr)\,d\theta\right]\Delta t$$

rate of increase in
$$\text{heat in} - \text{heat out} = -\frac{\partial q_\theta}{\partial \theta}d\theta\, dr - \frac{\partial q_r}{\partial r} r\, dr\, d\theta - q_r\, dr\, d\theta$$

Fourier's Law of heat conduction:

$$\underset{\substack{\parallel \\ q_r\hat{e}_r + q_\theta\hat{e}_\theta}}{\bar{q}} = -k\nabla u = -k\left(\frac{\partial u}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{e}_\theta\right)$$

So $q_r = -k\dfrac{\partial u}{\partial r}$, $q_\theta = -k\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

So rate of increase in heat $= k\left[\dfrac{\partial}{\partial \theta}\left(\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}\right) + r\dfrac{\partial^2 u}{\partial r^2} + \dfrac{\partial u}{\partial r}\right]dr\,d\theta$

$= c\rho\dfrac{\partial u}{\partial t}\,dV$ equate

$= c\rho\dfrac{\partial u}{\partial t}\,(r\,d\theta \cdot dr)$

$$\boxed{\frac{c\rho}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

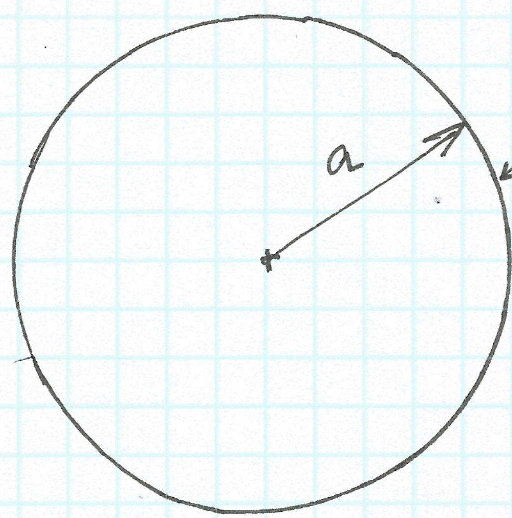
Laplace's Equation in Polar Coordinates

the heat equation in polar coordinates:

$$\frac{c\rho}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \nabla^2 u$$

In steady state, $\frac{\partial u}{\partial t} = 0$, and we get Laplace's equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$



temperature is given on $r = a$

Boundary Condition: $u = f(\theta)$ = given function when $r = a$

That is, $u(a,\theta) = f(\theta)$

Separation of Variables

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$u = R(r)\,\Theta(\theta)$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}\Theta'' = 0$$

Multiply by $\frac{r^2}{R\Theta} \Rightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$

Separate variables

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$r^2R'' + rR' - R\lambda = 0, \qquad \Theta'' + \lambda\Theta = 0$$

$\Theta(\theta)$ must be 2π-periodic $\Rightarrow$

$$\Theta(\theta) = c_1 \sin\sqrt{\lambda}\,\theta + c_2 \cos\sqrt{\lambda}\,\theta$$

$$\therefore \frac{2\pi n}{\sqrt{\lambda}} = \text{period of } \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\sqrt{\lambda}\,\theta = 2\pi, \quad n = 1, 2, 3, \ldots$$

(In words, if you go around the disk in a circle, n times, you end up with the same temperature as when you started.)

$$\therefore \quad \lambda = n^2$$

$$r^2R'' + rR' - n^2R = 0$$

Note: $\lambda = 0$ also works, and gives $\Theta(\theta) = 1$ (which may be thought of as being periodic.)

$$r^2R'' + rR' - n^2R = 0 \qquad (1)$$

This ODE is called a Cauchy-Euler eq. and has a solution of the form

$R = r^k$ where k is to be found.

$R' = k\,r^{k-1}$

$R'' = k(k-1)r^{k-2}$

Substituting into eq. (1), we get

$$k(k-1) + k - n^2 = 0$$

$$k^2 = n^2, \qquad k = \pm n$$

general solution $R = c_1 r^n + c_2 r^{-n}$

Note: r^{-n} blows up at $r=0$.
For bounded solution, take $c_2 = 0$

Note: $n=0$ is a separate case:

$r^2R'' + rR' = 0$ for $n=0$

$r\frac{d}{dr}(rR') = 0$, $rR' = \text{Constant} = c_3$

$R' = \frac{c_3}{r}$, $R = c_3 \ln r + c_4$ (general solution)

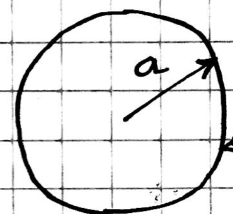
We must take $c_3 = 0$ for a bounded solution since $\ln r$ blows up at $r=0$.

Thus we obtain

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n(c_n \cos n\theta + d_n \sin n\theta)$$

Boundary Condition

$$u = u(a, \theta) = f(\theta) = \text{given}$$



temperature given at $r=a$

Then

$$f(\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos n\theta + d_n \sin n\theta)$$

Fourier series with $L = \pi$

$$a^n c_n = \frac{1}{L}\int_{-L}^{L} f(\theta) \cos n\theta \, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta$$

$$a^n d_n = \frac{1}{L}\int_{-L}^{L} f(\theta) \sin n\theta \, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta$$

Example

$$f(\theta) = \begin{cases} 0, & -\pi < \theta < 0 \\ 1, & 0 < \theta < \pi \end{cases}$$

$$a^n c_n = \frac{1}{\pi}\int_0^{\pi} \cos n\theta \, d\theta = \frac{1}{\pi} \left. \frac{\sin n\theta}{n} \right|_0^{\pi} = 0, \quad n = 1, 2, 3, \ldots$$

But $c_0 = \frac{1}{\pi}\int_0^{\pi} 1 \, d\theta = \frac{\pi}{\pi} = 1$

$$a^n d_n = \frac{1}{\pi}\int_0^{\pi} \sin n\theta \, d\theta = \frac{1}{\pi} \left. \frac{\cos n\theta}{-n} \right|_0^{\pi}$$

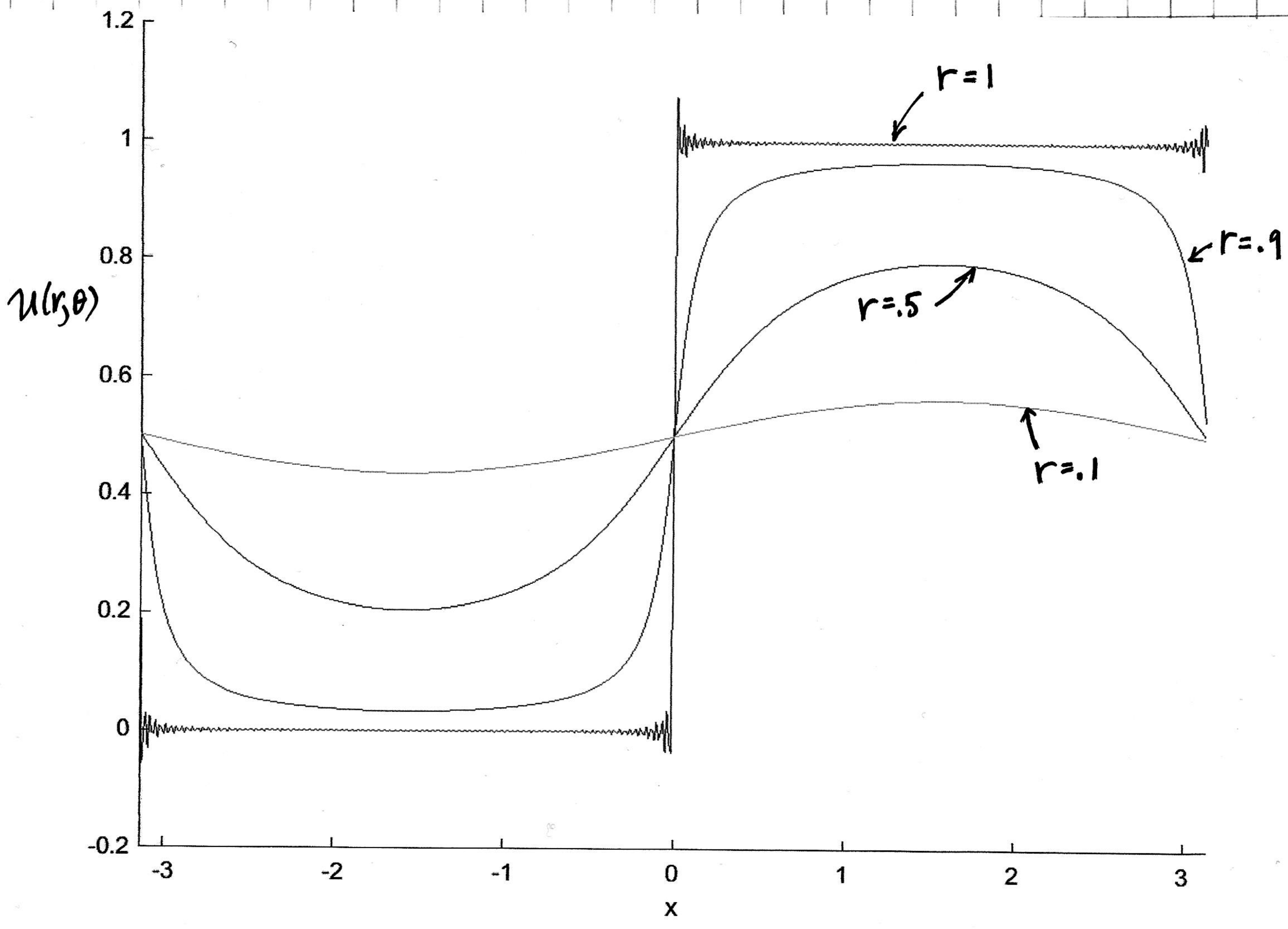
$$= \frac{1}{n\pi}\Big(1 - \underbrace{\cos n\pi}_{(-1)^n}\Big) = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Choose outer radius $a = 1$. Then

$$d_n = \begin{cases} \frac{2}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$f(\theta) = \frac{1}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{2}{n\pi} \sin n\theta$$

$$u(r,\theta) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} r^{2n-1} \frac{\sin(2n-1)\theta}{2n-1}$$



$a=1$, 100 terms

```
syms x
n=1:100;
hold on
f=(1/2)+(2/pi)*(.1.^(2.*n-1).*(sin((2.*n-1)*x)./(2.*n-1)))*(ones(100,1));
ezplot(f,[-pi,pi,-.2,1.2]);
```