First Order Linear ODE's

We have so far looked at separable ODE's:

$$\frac{dy}{dt} = f(y)\, g(t)$$

Method of solution: separate the variables

$$\frac{dy}{f(y)} = g(t)\, dt$$

Then integrate both sides

$$\int \frac{dy}{f(y)} = \int g(t)\, dt + C$$

arbitrary constant

Now we consider a class of ODE's called First order linear:

$$\frac{dy}{dt} + p(t)\, y = g(t)$$

Example:

$$\frac{dy}{dt} + y = e^{5t}$$

Note that this ODE is NOT SEPARABLE.

Watch what happens if I multiply both sides by e^t:

$$e^t \frac{dy}{dt} + e^t y = e^t \cdot e^{5t} = e^{6t}$$

It still doesn't appear to be separable. BUT:

$$e^t \frac{dy}{dt} + e^t y = \frac{d}{dt}(e^t y)$$

So now we have

$$\frac{d}{dt}(e^t y) = e^{6t}$$

or

$$d(e^t y) = e^{6t}\,dt$$

Integrate both sides

$$\int d(e^t y) = \int e^{6t}\,dt + C$$

$$e^t y = \frac{e^{6t}}{6} + C$$

or

$$\boxed{y = \frac{e^{5t}}{6} + Ce^{-t}}$$

The general solution ("general" meaning it contains an arbitrary constant and can fit an arbitrary IC (initial condition).)

Review of what we did:

Started with

$$\frac{dy}{dt} + y = e^{5t} \quad \text{(not separable)}$$

Multiplied by e^t to get

$$e^t \frac{dy}{dt} + e^t y = e^{6t}$$

Noticed that

$$e^t \frac{dy}{dt} + e^t y = \frac{d}{dt}\left(e^t y\right)$$

So that

$$\frac{d}{dt}\left(e^t y\right) = e^{6t}$$

which is effectively separable:

$$d\left(e^t y\right) = e^{6t}\, dt$$

The key move was multiplying by e^t

which is called "an integrating factor"

If we generalize this example to

$$\frac{dy}{dt} + p(t) y = g(t)$$

what can we multiply it by, to make it behave like the above example?

Answer Multiply by $e^{\int p(t)dt}$

$\uparrow$ integrating factor

Then we get

$$\underbrace{e^{\int p\,dt}\frac{dy}{dt} + e^{\int p\,dt}\,py}_{\frac{d}{dt}\left(e^{\int p\,dt}\,y\right)} = e^{\int p\,dt}\,g$$

because $\frac{d}{dt}e^{\int p\,dt} = e^{\int p\,dt}\cdot p$

Then proceeding as in the foregoing example,

$$\frac{d}{dt}\left(e^{\int p\,dt}\,y\right) = e^{\int p\,dt}\,g$$

$$d\left(e^{\int p\,dt}\,y\right) = e^{\int p\,dt}\,g\,dt$$

Integrate both sides

$$e^{\int p\,dt}\,y = \int e^{\int p\,dt}\,g(t)\,dt + C$$

$$y = e^{-\int p\,dt}\left[\int^{t} e^{\int p\,dt_1}\,g(t_1)\,dt_1\right] + Ce^{-\int p\,dt}$$

Here t_1 is used as the integration variable so that you don't get mixed up and try to push $e^{-\int p\,dt}$ into the inside of the integral.

Another example

$$\frac{dy}{dt} + \frac{y}{t} = t^2 \longleftarrow \text{Here } p(t) = \frac{1}{t}, \quad q(t) = t^2$$

Step 1 Compute $\int p\,dt = \int \frac{1}{t}\,dt = \ln t$

Step 2 Form the integrating factor

$$e^{\int p\,dt} = e^{\ln t} = t$$

Step 3 Multiply by $e^{\int p\,dt} = t$ to get

$$t\left(\frac{dy}{dt} + \frac{y}{t}\right) = t^3$$

or

$$t\frac{dy}{dt} + y = t^3$$

Step 4 Note that $t\frac{dy}{dt} + y = \frac{d}{dt}(ty)$

So that the ODE becomes

$$\frac{d}{dt}(ty) = t^3$$

or

$$d(ty) = t^3\,dt$$

or

$$ty = \frac{t^4}{4} + C$$

or

$$\boxed{y = \frac{t^3}{4} + \frac{C}{t}}$$

Check it

$$y = \frac{t^3}{4} + \frac{c}{t}$$

$$y' = \frac{3t^2}{4} - \frac{c}{t^2}$$

$$y' + \frac{y}{t} = \left(\frac{3}{4}t^2 - \frac{c}{t^2}\right) + \left(\frac{t^2}{4} + \frac{c}{t^2}\right)$$

$$= t^2$$ ✓ok