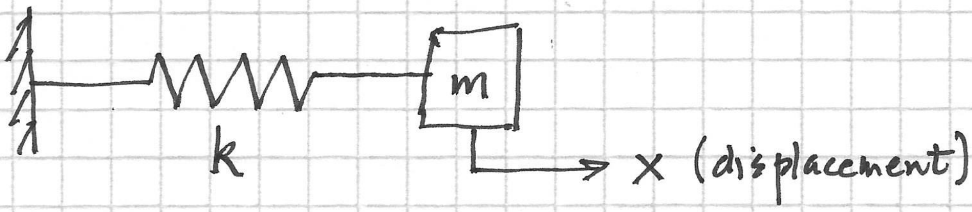
a mechanical spirng-mass system:



Here k is the spring stiffness [units of force/length] and m is the mass of the particle.

Newton's 2^{nd} Law gives

(free body)

$$F = -kx = ma = m\ddot{x}$$

or

$$\boxed{m\ddot{x} + kx = 0}$$

This equation can be written in the simplified form (by taking $m = k = 1$):

$$\boxed{\ddot{x} + x = 0}$$

This equation is one of the most famous ODE's, and is called the <u>simple harmonic oscillator</u>.

We will use a 2-step strategy to solve $\ddot{x}+x=0$:

1) Define $v=\dot{x}$, a new variable

This will turn $\ddot{x}+x=0$, a single second order ODE, into 2 first order ODE's

2) Convert the two 1st order ODE's

$$\dot{x} = \cdots$$
$$\dot{v} = \cdots$$

to a single 1st order ODE by using the chain rule:

$$\frac{dv}{dx} = \frac{\dot{v}}{\dot{x}} \qquad \left(\text{that is, } \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}\right)$$

<u>step 1</u>: Define $v=\dot{x}$. Then $\ddot{x} = \dot{v} = -x$ (from $\ddot{x}+x=0$)

So the two first order ODE's are:

$$\dot{x} = v$$
$$\dot{v} = -x$$

<u>step 2</u>: Convert to a single 1st order ODE:

$$\frac{dv}{dx} = \frac{\dot{v}}{\dot{x}} = \frac{-x}{v}$$ which is <u>separable</u>:

$$\frac{dv}{dx} = \frac{-x}{v} \Rightarrow \quad v\,dv = -x\,dx$$

Integrate both sides $\Rightarrow \int v\,dv = \int -x\,dx$

$$\frac{v^2}{2} = -\frac{x^2}{2} + C$$ arbitrary constant

Solve this for v:

$$v = \sqrt{2C - x^2}$$

Now substitute this result back into $\dot{x} = v$:

$$\dot{x} = \frac{dx}{dt} = v = \sqrt{2c - x^2}$$

This ODE is separable:

$$\frac{dx}{\sqrt{2c - x^2}} = dt$$

$$\int \frac{dx}{\sqrt{2c - x^2}} = t + K$$

a new arbitrary constant

to evaluate this integral, set $x = \sqrt{2c}\,\sin\theta$

then $dx = \sqrt{2c}\,\cos\theta\, d\theta$

$$\int \frac{dx}{\sqrt{2c - x^2}} = \int \frac{\sqrt{2c}\,\cos\theta\, d\theta}{\sqrt{2c - 2c\sin^2\theta}} = \int \frac{\sqrt{2c}\,\cos\theta\, d\theta}{\sqrt{2c}\sqrt{1 - \sin^2\theta}}$$

$$= \int d\theta = \theta = \arcsin\frac{x}{\sqrt{2c}}$$

So we have

$$\arcsin\frac{x}{\sqrt{2c}} = t + K$$

or

$$\frac{x}{\sqrt{2c}} = \sin(t + K)$$

or finally, $\boxed{x = \sqrt{2c}\,\sin(t + K)}$

which is the general solution of $\ddot{x} + x = 0$

(Check it!)

$$x = \sqrt{2c}\ \sin(t+k)$$

Note that there are 2 arbitrary constants.

There will be 2 initial conditions, typically $x(0)$, $\dot{x}(0)$

For example, suppose

$$x(0) = 1 \quad \text{and} \quad \dot{x}(0) = 2$$

We have $\quad x(0) = \sqrt{2c}\ \sin k = 1$

$$\dot{x}(0) = \sqrt{2c}\ \cos k = 2$$

Square and add $\Rightarrow \quad 2c = 5 \quad (\sin^2 k + \cos^2 k = 1)$

To find k, divide the 2 equations $\Rightarrow$

$$\tan k = \frac{1}{2} \Rightarrow k = .4636$$

So

$$x = \sqrt{5}\ \sin(t + .4636)$$

Use the identity $\sin(t+k) = \sin t \cos k + \cos t \sin k$

$$\cos k = \cos .4636 = .8944 \qquad \sin k = \sin .4636 = .4472$$

$$x = (\sqrt{5})(.8944) \sin t + (\sqrt{5})(.4472) \cos t$$

$$= 2 \sin t + \cos t$$

problem 25, page 39

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \text{EQUATION (1)}$$

For example (problem 26) $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$

NOT SEPARABLE, NOT LINEAR

So how can we obtain its solution?

ANSWER: Replace y by $z = \frac{y}{x}$, i.e. set $y = zx$

Then $\frac{dy}{dx} = \frac{d}{dx}(zx) = \frac{dz}{dx}x + z$

The original ODE (1) becomes

$$\frac{dz}{dx}x + z = f(z)$$

$$\frac{dz}{dx} = \frac{f(z) - z}{x}$$ which is SEPARABLE!

Example (problem 26, page 39)

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

Set $y = zx \Rightarrow \frac{dz}{dx} = \frac{(1 + z + z^2) - z}{x} = \frac{1 + z^2}{x}$

Separate variables: $\frac{dz}{1+z^2} = \frac{dx}{x}$, $\arctan z = \ln x + c$

or $z = \tan(\ln x + c) \Rightarrow$

$$y = zx = x\tan(\ln x + c)$$