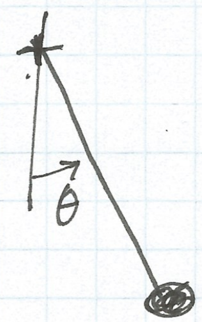
EQUILIBRIUM and STABILITY

Let's start off with an example:

A damped pendulum



The ODE is second order:

$$\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} + \sin\theta = 0$$

(where the various constants have been taken equal to 1 for simplicity)

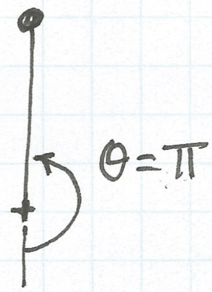
We can write this as a first order system by setting $V = \frac{d\theta}{dt}$ (= angular velocity)

$$\frac{d\theta}{dt} = V$$

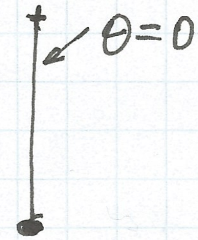
$$\frac{dV}{dt} = -V - \sin\theta$$

Although we can't solve this ODE right now, we understand how a pendulum works, so we can understand certain features of the solution of the ODE's.

There are 2 equilibrium solutions for a physical pendulum:



and



How can we confirm that these physical states are solutions of the ODE's:

$$\frac{d\theta}{dt} = V, \quad \frac{dV}{dt} = -V - \sin\theta$$

<u>Answer:</u> The essence of an equilibrium is that it doesn't change in time.

So we set $\dot{\theta} = 0$ and $\dot{V} = 0$

This gives 2 algebraic equations:

$$0 = V \text{ and } 0 = -V - \sin\theta$$

Since the first eq. tells you that $V=0$, the second eq. tells you that $\sin\theta = 0$

OR $\boxed{\theta = 0 \text{ and } \theta = \pi}$

These are the two physical equilibria, straight down and straight up.

Now we know that if you try to balance a pendulum upside down, it is not going to stay there: We say the upside-down equilibrium

$\theta = \pi$ is <u>UNSTABLE</u>

meaning that if you displace it slightly, it will start to fall and will move away from the equilibrium point.

On the other hand, the straight down equilibrium, $\theta = 0$ is <u>STABLE</u> because a small displacement will end up moving back towards the equilibrium, and will remain close to the equilibrium for all time.

<u>COMMENT</u>: In a real situation, say an engineering system with say 5 or 10 first order ODE's, all coupled, you cannot determine stability of a solution by inspection, as we did in the case of the pendulum.

You need a mathematical approach to determine stability.

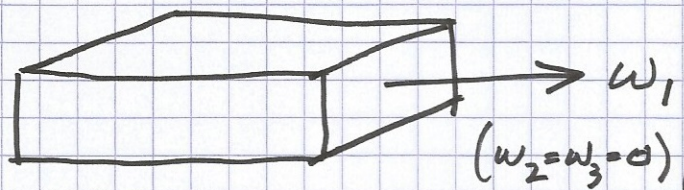
A solution without a stability analysis is worthless, because you will never see it in the real world if it is unstable.

Another example of STABILITY

Imagine tossing a RECTANGULAR SOLID-shaped object (like a board eraser, or a book (with a rubber band around it))

in the air so that it spins. The general motion is very complicated. But there are 3 special motions which are simple: The rotations about the symmetry axes:

1



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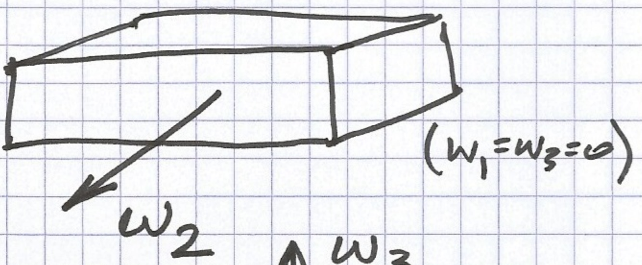

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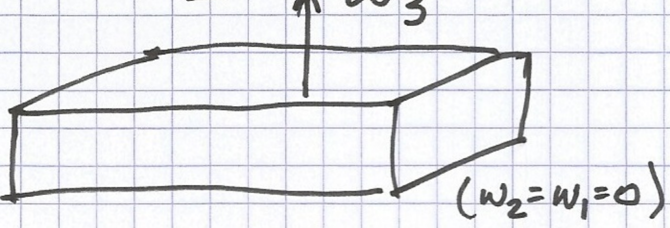


$\bar{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ is the angular velocity vector

The equations of motion are

$$\dot{\omega}_1 = -\omega_2 \omega_3$$
$$\dot{\omega}_2 = \omega_1 \omega_3$$
$$\dot{\omega}_3 = -\frac{1}{3} \omega_1 \omega_2$$

It turns out that 1 and 3 are stable, but 2 is unstable! TRY IT!

In this lecture we will be concerned with determining the equilibria and their stability for a first order ODE of the form:

$$\frac{dy}{dt} = f(y)$$

For example $\quad \frac{dy}{dt} = y(1-y)(2-y)$

We can think of the procedure consisting of two steps:

<u>Step 1</u>: Find the equilibria.

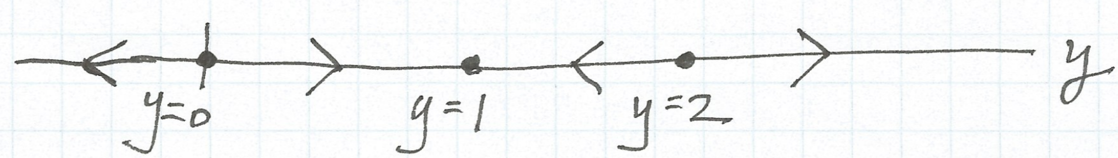
Set $\dot{y} = 0$, that is, $f(y) = 0$

In the example this becomes

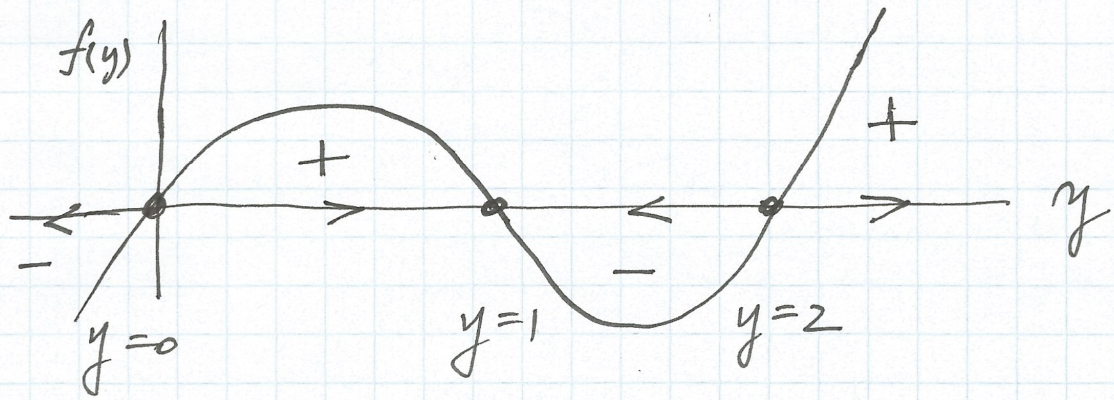
$$y(1-y)(2-y) = 0 \Rightarrow y = 0, 1, 2$$

So there are 3 equilibria.

<u>Step 2</u>: Draw the y-axis (called the phase line) and show the direction of the vector field (called the phase fluid's velocity) on the y-axis



How can you tell which way the arrows are pointing? Draw $f(y)$ over the y-line



$+$ means arrow points to the right
$-$ " " " " " left

Now once you have the arrows in place, you can determine the stability of each equilibrium by inspection: If you displace the system off of an equilibrium, does the phase fluid move it back, or away from the equilibrium?

Thus in the above example,

$y=0$ and $y=2$ are unstable

$y=1$ is stable.