We begin with a review of <u>integrating factor</u>

We showed that we can solve the linear ODE

$$\frac{dy}{dx} + \frac{1}{x} y = x$$

by first computing the integrating factor

$$e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

and then multiplying the ODE by the integrating factor;

$$x\left(\frac{dy}{dx} + \frac{1}{x} y\right) = x^2$$

$$\underbrace{x\frac{dy}{dx} + y}_{\frac{d}{dx}(xy)} = x^2$$

After multiplying by the integrating factor, these terms can be written as $\frac{d}{dx}(\dots)$

Integrating both sides, we get

$$\int \frac{d}{dx}(xy)\, dx = \int x^2 dx + C$$

arbitrary constant

$$xy = \frac{x^3}{3} + C$$

$$\boxed{y = \frac{x^2}{3} + \frac{C}{x}}$$

The general solution (contains the arbitrary constant)

In this lecture we will reexamine the foregoing process.

After multiplying by the integrating factor, the ODE became:

$$x \frac{dy}{dx} + y = x^2$$

Let's multiply by dx and move everything to one side:

$$(y - x^2)\, dx + x\, dy = 0 \qquad (*)$$

Now we know that this equation has the following integral:

$$xy - \frac{x^3}{3} = C$$

To show this directly (without deriving it), take the differential of it:

$$d\left(xy - \frac{x^3}{3}\right) = x\, dy + dx \cdot y - \frac{3x^2}{3}$$

$$= (y - x^2)\, dx + x\, dy = 0$$

which is eq. $(*)$ above.

We say that $(y-x^2)dx + x\,dy$ is an <u>exact differential</u>

More generally, $M(x,y)\, dx + N(x,y)\, dy = 0$ is an exact differential if it can be written in the form $\quad d\,\psi(x,y) = 0$

2 Questions:

1) How can you tell if a d.e. of the form

$$M\,dx + N\,dy = 0$$ is <u>exact</u> ?

2) If it is exact, how can you find the integral

$$\psi(x,y) = \text{constant} \quad ?$$

Work backwards:

$$\psi = c$$
$$d\psi = 0$$
$$\psi_x\,dx + \psi_y\,dy = 0$$

But $$M\,dx + N\,dy = 0$$

So identify $\left.\begin{array}{l}\psi_x = M \\ \psi_y = N\end{array}\right\} \begin{array}{l}\psi_{yx} = M_y \\ \psi_{xy} = N_x\end{array}$

$\therefore$ $\boxed{M_y = N_x = \psi_{xy}}$ is the test condition

<u>Example</u> Let's use the previous example, but let's make believe we don't know the solution.

$$\text{Equation } (*) \Rightarrow \underbrace{(y-x^2)}_{M}\,dx + \underset{\downarrow \atop N}{x}\,dy = 0$$

Test to see if it is <u>exact</u>:

$$M_y \stackrel{?}{=} N_x$$

$$\left.\begin{array}{l} M_y = \dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}(y-x^2) = 1 \\ N_x = \dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}(x) = 1 \end{array}\right\}$$ $M_y = N_x \Rightarrow$ equation is EXACT (we knew this)

Now, continuing with making believe we don't know the solution, how to solve this <u>exact</u> equation:

$$\underbrace{(y - x^2)}_{M = y - x^2} dx + \underbrace{x}_{N = x} dy = 0$$

We have shown that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

We seek an integral, $\psi(x,y) = \text{constant}$

$$\therefore \; d\psi = 0$$

$$\therefore \; \psi_x \, dx + \psi_y \, dy = 0$$

We are given $\dfrac{\partial \psi}{\partial x} = M = y - x^2$; $\dfrac{\partial \psi}{\partial y} = N = x$

In order for ψ to exist,

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x} \quad \text{or} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

We have shown this to be the case. But what is ψ?

Integrate this eq., $\dfrac{\partial \psi}{\partial x} = y - x^2$, with respect to x:

$$\psi = xy - \frac{x^3}{3} + f(y)$$

an arbitrary function of y

this function plays the role of an arbitrary constant, but for partial integration it is a function of y instead of a constant.

Now ψ must also satisfy $\dfrac{\partial \psi}{\partial y} = N = x$.

So differentiate (with respect to y), $\psi = xy - \dfrac{x^3}{3} + f(y)$

$$\underbrace{\frac{\partial}{\partial y}\left[xy - \frac{x^3}{3} + f(y)\right]}_{x + \frac{df}{dy}} = x$$

$$\Rightarrow \quad \frac{df}{dy} = 0 \Rightarrow f(y) = \text{constant}$$

<u>Conclusion</u>: $\psi = xy - \frac{x^3}{3} + \text{constant}$

(which agrees with what we already knew.)

<u>Another example</u>

p.75, problem 3: $(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$

1) Is it <u>exact</u>? $M = 3x^2 - 2xy + 2$, $N = 6y^2 - x^2 + 3$

Compute $M_y = -2x$ and $N_x = -2x$

Since they are equal, the answer is <u>yes</u> the eq. is exact.

2) Find $\psi(x,y)$.

$$\psi(x,y) = \text{constant}$$

$$d\psi = \underset{\overset{\shortparallel}{M}}{\psi_x}\,dx + \underset{\overset{\shortparallel}{N}}{\psi_y}\,dy = 0$$

$$\frac{\partial \psi}{\partial x} = 3x^2 - 2xy + 2 \Rightarrow \psi = x^3 - x^2 y + 2x + f(y)$$

We require $\frac{\partial \psi}{\partial y} = N = 6y^2 - x^2 + 3$.

We have so far $\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(\;\;) = -x^2 + \frac{df}{dy}$

comparing these, we find

$$\frac{df}{dy} = 6y^2 + 3 \Rightarrow f = 2y^3 + 3y + \text{constant}$$

$$\therefore \boxed{\psi(x,y) = x^3 - x^2 y + 2x + 2y^3 + 3y = \text{constant}}$$