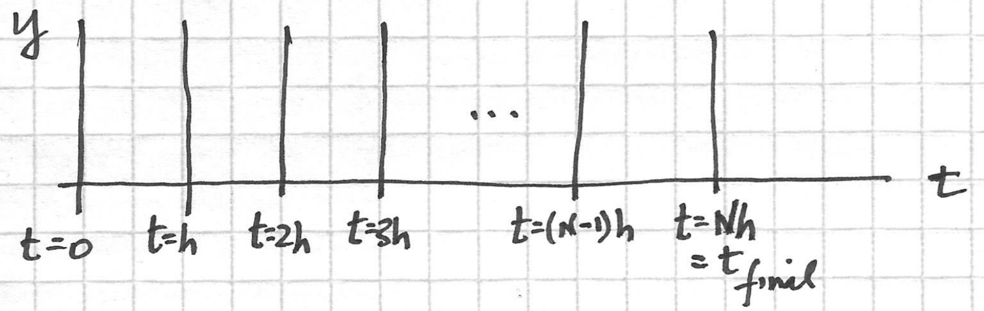
Numerical Integration : Euler's Method

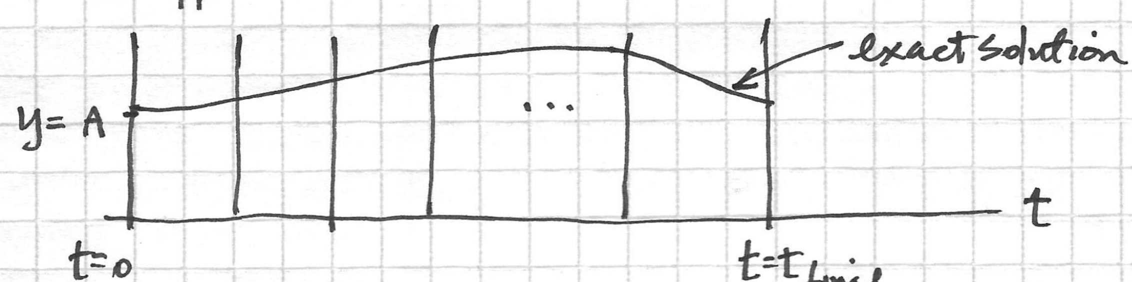
Suppose we want to solve a 1^{st} order ODE:

$$\frac{dy}{dt} = f(y,t) \quad \text{with I.C.} \quad t=0, y=A$$

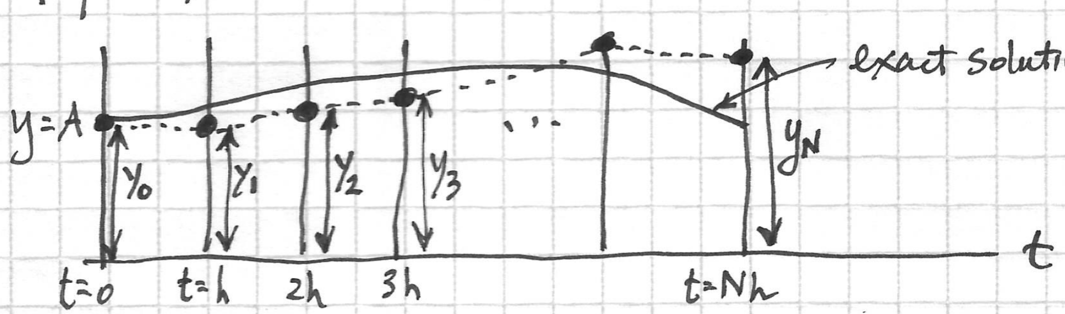
We may obtain a "numerical approximation" to the solution by breaking the interval $0 \leq t \leq t_{final}$ up into N subintervals (where t_{final} is the final time t that you are interested in):



Now suppose the exact solution to the ODE looks like:



We will approximate the exact solution by a series of points, one at each of the times $\{0, h, 2h, \ldots, Nh\}$:



The N+1 vector $\{y_0, y_1, y_2, \ldots, y_N\}$ will be an approximation of the exact solution $y(t)$.

We obtain it be replacing the ODE by a <u>difference</u> eq. based on the formula

$$\left.\frac{dy}{dt}\right|_{t=kh} \rightarrow \frac{y_{k+1}-y_k}{h}$$

So the ODE

$$\frac{dy}{dt} = f(y,t)$$

becomes

$$\frac{y_{k+1}-y_k}{h} = f(y_k,t)$$

The procedure is to solve for y_{k+1}:

$$y_{k+1} = y_k + h\, f(y_k,t)$$

and then use this formula <u>recursively</u>,

i.e., starting with $y_0 = A$ (the given initial condition (IC))

we compute y_1, then knowing y_1, we compute y_2, etc.

<u>Example</u> p. 83, problem 1.

$$\frac{dy}{dt} = 3 + t - y, \quad y(0) = 1$$

Say we choose $h = 0.1$ (the smaller h is, the better the approximation – in general)

Then we have $y_0 = 1$, y_1 is the computed approximate value of y at $t = h$,

y_2 is the approximation at $t = 2h$, etc.

We find:

$$y_1 = y_0 + (3 + t_0 - y_0)h$$
$$= 1 + (3 + 0 - 1)\,0.1$$
$$= 1.2$$
$$y_2 = y_1 + (3 + t_1 - y_1)h$$
$$= 1.2 + (3 + 0.1 - 1.2)\,0.1$$
$$= 1.39$$

and so on.

How does the approximate solution compare with the exact solution?

$$\frac{dy}{dt} = 3 + t - y \quad \text{or} \quad \frac{dy}{dt} + y = 3 + t$$

The integrating factor is $\exp \int 1\, dt = e^t$.

So multiply by e^t, giving

$$\underbrace{e^t \frac{dy}{dt} + e^t y}_{\frac{d}{dt}(e^t y)} = (3+t)e^t$$

and $e^t y = \int (3+t)e^t\, dt = 3e^t + (t-1)e^t + C$

$$y = 2 + t + Ce^{-t} \quad \text{(general solution)}$$

Use IC to find the arbitrary constant C:

$$y(0) = 1 = 2 + C \Rightarrow C = -1$$

$$\boxed{y = 2 + t - e^{-t}}$$

Comparison between numerical solution and Exact:

t	Numerical	Exact $y = 2 + t - e^{-t}$
0	$y_0 = 1$	$y(0) = 1$
$h = 0.1$	$y_1 = 1.2$	$y(.1) = 2 + 0.1 - e^{-.1}$ $= 1.1952$
$2h = 0.2$	$y_2 = 1.39$	$y(.2) = 2 + 0.2 - e^{-.2}$ $= 1.3813$