Lecture Notes on PDE's: Separation of Variables and Orthogonality

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1 Three Problems

We will use the following three problems in steady state heat conduction to motivate our study of a variety of math methods:

Problem "A": Heat conduction in a cube

$$\nabla^2 u = 0 \ \text{ for } \ 0 < x < L, \ 0 < y < L, \ 0 < z < L \tag{1}$$

with the assumption that $u = u(x, z, \text{only})$ (that is, no y dependence)),
and with the boundary conditions:

$$u = 0 \quad \text{on} \quad x = 0, L \tag{2}$$

$$u = 0 \quad \text{on} \quad z = 0 \tag{3}$$

$$u = 1 \quad \text{on} \quad z = L \tag{4}$$

Problem "B": Heat conduction in a circular cylinder

$$\nabla^2 u = 0 \ \text{ for } \ 0 < r < a, \ 0 < z < L \tag{5}$$

with the assumption that $u = u(r, z, \text{only})$ (that is, no θ dependence),
and with the boundary conditions:

$$u = 0 \quad \text{on} \quad r = a \tag{6}$$

$$u = 0 \quad \text{on} \quad z = 0 \tag{7}$$

$$u = 1 \quad \text{on} \quad z = L \tag{8}$$

Problem "C": Heat conduction in a sphere

$$\nabla^2 u = 0 \ \text{ for } \ 0 < \rho < a \tag{9}$$

with the assumption that $u = u(\rho, \phi, \text{only})$ (that is, no θ dependence),
and with the boundary conditions:

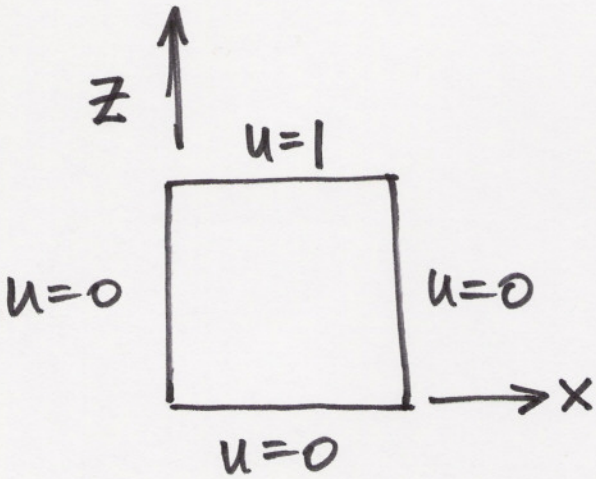
$$u = 0 \quad \text{on} \quad r = a, \quad \pi/2 \le \phi \le \pi \tag{10}$$

$$u = 1 \quad \text{on} \quad r = a, \quad 0 \le \phi < \pi/2 \tag{11}$$

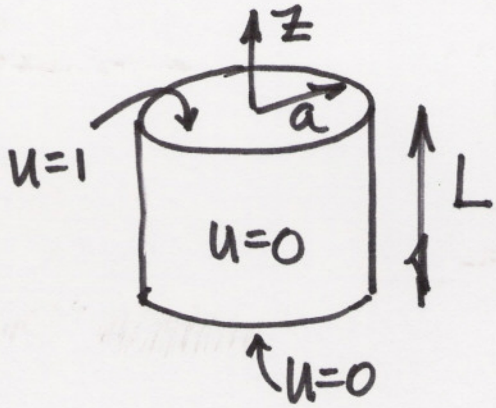
Here ϕ is the colatitude and θ is the longitude.

Three problems in heat conduction

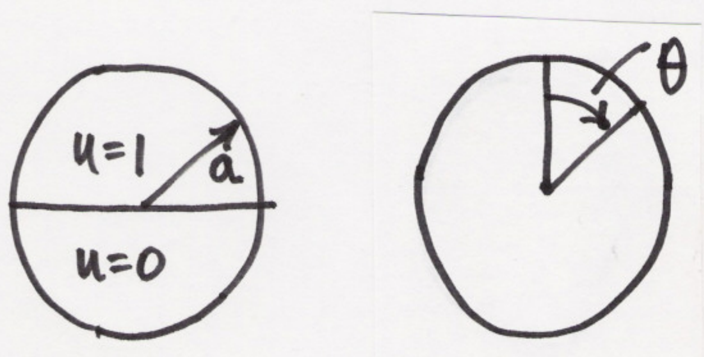
problem A:



problem B:



problem C:



2 The Laplacian ∇^2 in three coordinate systems

Rectangular coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{12}$$

Circular cylindrical coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \tag{13}$$

where

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{that is,} \quad r^2 = x^2 + y^2 \tag{14}$$

and where

$$0 \le \theta < 2\pi \tag{15}$$

Spherical coordinates

$$\nabla^2 u = \frac{1}{\rho^2}\left[\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\sin\phi\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right] \tag{16}$$

where

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\theta, \quad \text{that is,} \quad \rho^2 = x^2 + y^2 + z^2 \tag{17}$$

and where

$$0 \le \theta < 2\pi, \qquad 0 \le \phi \le \pi \tag{18}$$

3 Solution to Problem "A" by Separation of Variables

In this section we solve Problem "A" by separation of variables. This is intended as a review of work that you have studied in a previous course.

We seek a solution to the PDE (1) (see eq.(12)) in the form

$$u(x,z) = X(x)Z(z) \tag{19}$$

Substitution of (19) into (12) gives:

$$X''Z + XZ'' = 0 \tag{20}$$

where primes represent differentiation with respect to the argument, that is, X' means dX/dx whereas Z' means dZ/dz. Separating variables, we obtain

$$\frac{Z''}{Z} = -\frac{X''}{X} = \lambda \tag{21}$$

where the two expressions have been set equal to the constant λ because they are functions of the independent variables x and z, and the only way these can be equal is if they are both constants. This yields two ODE's:

$$X'' + \lambda X = 0 \quad \text{and} \quad Z'' - \lambda Z = 0 \tag{22}$$

Substituting the ansatz (19) into the boundary conditions (=B.C.) (2) and (3), we find:

$$X(0) = 0, \quad X(L) = 0 \quad \text{and} \quad Z(0) = 0 \tag{23}$$

The general solution of the X equation in (22) is

$$X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x \tag{24}$$

where c_1 and c_2 are arbitrary constants. The first B.C. of (23), $X(0) = 0$, gives c_1=0. The second B.C. of (23), $X(L) = 0$, gives $\sqrt{\lambda} = n\pi/L$, for $n = 1, 2, 3, \cdots$:

$$\lambda = \frac{n^2\pi^2}{L^2}, \qquad X(x) = c_2 \sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \cdots \tag{25}$$

The general solution of the Z equation in (22) can be written in either of the equivalent forms:

$$Z(z) = c_3 \cosh\sqrt{\lambda}z + c_4 \sinh\sqrt{\lambda}z \tag{26}$$

or

$$Z(z) = c_5\, e^{\sqrt{\lambda}z} + c_6\, e^{-\sqrt{\lambda}z} \tag{27}$$

where $\lambda = \frac{n^2\pi^2}{L^2}$ and where

$$\cosh v = \frac{e^v + e^{-v}}{2} \quad \text{and} \quad \sinh v = \frac{e^v - e^{-v}}{2} \tag{28}$$

Choosing the form (26), the third B.C. of (23), $Z(0) = 0$, gives c_3=0.

Substituting the derived results into the assumed form of the solution (19), we have

$$u(x,z) = X(x)Z(z) = a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi z}{L}, \qquad n = 1, 2, 3, \cdots \tag{29}$$

where a_n=$c_2\ c_4$ is an arbitrary constant. Since the PDE (1) is linear, we may superimpose solutions to obtain the form:

$$u(x,z) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi z}{L} \tag{30}$$

We still have to satisfy the B.C. (4), $u(x, L)$=1, which gives (from (30)):

$$1 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} \sinh n\pi \tag{31}$$

Eq.(31) is a Fourier series. We may obtain the values of the constants a_n by using the orthogonality of the eigenfunctions $\sin \frac{n\pi x}{L}$ on the interval $0 < x < L$:

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = 0 \qquad n \neq m \tag{32}$$

where n and m are integers. Eq.(32) follows from the trig identity:

$$\sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} = -\frac{1}{2} \cos \frac{(n+m)\pi x}{L} + \frac{1}{2} \cos \frac{(n-m)\pi x}{L} \tag{33}$$

Substituting (33) into (32), the cosines integrate to sines and vanish at the upper and lower limits if $n \neq m$. In the case that n=m, we have

$$\int_0^L \left(-\frac{1}{2} \cos \frac{(n+m)\pi x}{L} - \frac{1}{2} \cos \frac{(n-m)\pi x}{L} \right) dx = \int_0^L \left(-\frac{1}{2} \cos \frac{2m\pi x}{L} + \frac{1}{2} \right) dx = \frac{L}{2} \tag{34}$$

We return now to eq.(31) and use (32) and (34) to obtain the coefficients a_n. Multiplying (31) by $\sin \frac{m\pi x}{L}$ and integrating from 0 to L, we obtain:

$$\int_0^L \sin \frac{m\pi x}{L} dx = a_m \frac{L}{2} \sinh m\pi \tag{35}$$

Changing the index from m to n, this gives

$$a_n \frac{L}{2} \sinh n\pi = \int_0^L \sin \frac{n\pi x}{L} dx = \left[-\frac{L}{n\pi} \cos \frac{n\pi x}{L} \right]_0^L = \begin{cases} \frac{2L}{n\pi} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \tag{36}$$

whereupon we obtain the following solution $u(x, z)$ from eqs.(30) and (36):

$$u(x,z) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{4}{n\pi} \frac{\sinh \frac{n\pi z}{L}}{\sinh n\pi} \sin \frac{n\pi x}{L} \tag{37}$$

4 Solving Problem "B" by Separation of Variables

Problem "B" has the PDE (see (5) and (13)):

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{38}$$

Following the procedure we used on problem "A", we seek a solution to the PDE (38) in the form

$$u(r,z) = R(r)Z(z) \tag{39}$$

Substitution of (39) into (38) gives:

$$R''Z + \frac{1}{r}R'Z + RZ'' = 0 \tag{40}$$

where again primes represent differentiation with respect to the argument. Separating variables, we obtain

$$\frac{Z''}{Z} = -\frac{R'' + \frac{1}{r}R'}{R} = \lambda \tag{41}$$

This yields two ODE's:

$$R'' + \frac{1}{r}R' + \lambda R = 0 \quad \text{and} \quad Z'' - \lambda Z = 0 \tag{42}$$

Substituting the ansatz (39) into the B.C. (6) and (7), we find:

$$R(a) = 0 \quad \text{and} \quad Z(0) = 0 \tag{43}$$

Now if we were to continue to follow the procedure we used on problem "A", we would solve the R equation of (42) and use the first B.C. of (43) to find λ, and so on.

However, the R equation has a <u>variable coefficient</u>, namely in the $\frac{1}{r}R'$ term. Thus we must digress and find out to how to solve such ODE's before we can continue with the solution of problem "B".

5 Euler's Differential Equation

The simplest case of a linear variable coefficient second order ODE is Euler's equation:

$$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + c\, y = 0 \tag{44}$$

We look for a solution with the ansatz:

$$y = x^r \tag{45}$$

Substitution of (45) into (44) gives

$$ar(r-1) + br + c = 0 \qquad \text{that is,} \qquad ar^2 + (b-a)r + c = 0 \tag{46}$$

We may use the quadratic formula to obtain (in general) a pair of complex conjugate roots r_1 and r_2. Thus the general solution may be written

$$y = c_1\, x^{r_1} + c_2\, x^{r_2} \tag{47}$$

If the roots are both real, then eq.(47) suffices. However in the general case in which r_1 and r_2 are complex, say $r_1 = \mu + i\nu$, we obtain the form

$$y = c_1\, x^{\mu+i\nu} + c_2\, x^{\mu-i\nu} = x^\mu(c_1 x^{i\nu} + c_2 x^{-i\nu}) \tag{48}$$

Using Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, and the identity $x = e^{\log x}$, we obtain the real form

$$y = x^\mu(c_1 e^{i\nu \log x} + c_2 e^{-i\nu \log x}) = x^\mu(c_3 \cos(\nu \log x) + c_4 \sin(\nu \log x)) \tag{49}$$

where c_3 and c_4 are *real* arbitrary constants, and where $\log x$ stands for *natural* logarithms.

In the case that the roots are repeated, r_1=r_2=r, the general solution to (44) is

$$y = c_1\, x^r + c_2\, x^r \log x \tag{50}$$

6 Power Series Solutions

So now we know how to solve Euler's equation. What about linear differential equations with variable coefficients which are not in the form of Euler's equation? A natural approach would be to look for the solution in the form of a power series:

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots \tag{51}$$

where the coefficients c_i are to be found. The method is best illustrated with an example.
Example 1

$$\frac{d^2y}{dx^2} + x\ \frac{dy}{dx} + y = 0 \tag{52}$$

We substitute (51) into (52)

$$2c_2 + 6c_3 x + 12c_4 x^2 + \cdots + x(c_1 + 2c_2 x + 3c_3 x^2 + \cdots) + c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = 0 \tag{53}$$

and collect terms:

$$2c_2 + c_0 + x(6c_3 + 2c_1) + x^2(12c_4 + 3c_2) + x^3(20c_5 + 4c_3)\cdots \tag{54}$$

Next we require the coefficient of each power of x^n to vanish, giving:

$$2c_2 + c_0 = 0 \tag{55}$$
$$6c_3 + 2c_1 = 0 \tag{56}$$
$$12c_4 + 3c_2 = 0 \tag{57}$$
$$20c_5 + 4c_3 = 0 \tag{58}$$
$$\cdots$$

Note that eqs.(55)-(58) can be abbreviated by the single *recurrence relation:*

$$(n+2)(n+1)c_{n+2} + (n+1)c_n = 0, \qquad n = 0, 1, 2, 3, \cdots \tag{59}$$

which may be written in the form

$$c_{n+2} = -\frac{c_n}{n+2}, \qquad n = 0, 1, 2, 3, \cdots \tag{60}$$

The nature of the recurrence relation (60) is that c_0 and c_1 can be chosen arbitrarily, after which all the other c_i's will be determined in terms of c_0 and c_1. This leads to the following expression for the general solution of eq.(52):

$$y = c_0\ f(x) + c_1\ g(x) \tag{61}$$

where

$$f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 4} - \frac{x^6}{2\cdot 4\cdot 6} + \frac{x^8}{2\cdot 4\cdot 6\cdot 8} + \cdots \tag{62}$$

$$g(x) = x - \frac{x^3}{3} + \frac{x^5}{3\cdot 5} - \frac{x^7}{3\cdot 5\cdot 7} + \frac{x^9}{3\cdot 5\cdot 7\cdot 9} + \cdots \tag{63}$$

The method of power series has worked great on Example 1. Let's try it on Example 2:
<u>Example 2</u>

$$2x^2\frac{d^2y}{dx^2} + 3x\ \frac{dy}{dx} + (-1+x)y = 0 \tag{64}$$

We substitute (51) into (64)

$$2(2c_2x^2+6c_3x^3+12c_4x^4+\cdots)+3x(c_1+2c_2x+3c_3x^2+\cdots)+(-1+x)(c_0+2c_1x+2c_2x^2+2c_3x^3+\cdots) = 0 \tag{65}$$

and collect terms:

$$-c_0 = 0 \tag{66}$$

$$2c_1 + c_0 = 0 \tag{67}$$

$$9c_2 + c_1 = 0 \tag{68}$$

$$20c_3 + c_2 = 0 \tag{69}$$

$$\cdots$$

This time eqs.(67)-(69) have the recurrence relation:

$$c_{n+1} = -\frac{c_n}{(2n+1)(n+2)}, \qquad n = 1, 2, 3, \cdots \tag{70}$$

But since eq.(66) requires that c_0=0, we find from (70) that all the c_i's must vanish. In other words, the method of power has failed to produce a solution for Example 2, eq.(64).

This raises two questions:
1. How can we obtain a solution to eq.(64)?, and
2. For which class of equations will the method of power series work?

We answer the first question in the next section by replacing the method of power series by a more general method called the method of Frobenius. We put off answering the second question until later.

7 The Method of Frobenius

The method of Frobenius generalizes the method of power series by seeking a solution in the form of a "generalized power series":

$$y = x^r(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots) \tag{71}$$

Note that eq.(71) reduces to the power series (51) if r=0.

Let's try this method on Example 2:
Example 2, continued

$$2x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + (-1+x)y = 0 \tag{72}$$

We substitute (71) into (72) and collect terms:

$$(2r^2 + r - 1)c_0 x^r + [(2r^2 + 5r + 2)c_1 + c_0]x^{r+1} + [(2r^2 + 9r + 9)c_2 + c_1]x^{r+2} + \cdots = 0 \tag{73}$$

$$(2r^2 + r - 1)c_0 = 0 \tag{74}$$
$$(2r^2 + 5r + 2)c_1 + c_0 = 0 \tag{75}$$
$$(2r^2 + 9r + 9)c_2 + c_1 = 0 \tag{76}$$
$$\cdots$$
$$[2r^2 + (4n+1)r + (n^2 + n - 1)]c_{n+1} + c_n = 0 \tag{77}$$
$$\cdots$$

Eqs.(75)-(77) have the recurrence relation:

$$c_{n+1} = -\frac{c_n}{(2r + 2n - 1)(r + n + 1)}, \qquad n = 1, 2, 3, \cdots \tag{78}$$

In addition to (78), we also must satisfy, from (74), the *indicial equation:*

$$2r^2 + r - 1 = 0 \qquad \Rightarrow \qquad r = -1, \ \frac{1}{2} \tag{79}$$

We consider each of these r-values separately, taking c_0=1 for each:
For r=−1 we get:

$$y = f(x) = \frac{1}{x} + 1 - \frac{x}{2} + \frac{x^2}{18} - \frac{x^3}{360} + \frac{x^4}{12600} - \frac{x^5}{680400} + \cdots \tag{80}$$

For $r=\frac{1}{2}$ we get:

$$y = g(x) = \sqrt{x}\left(1 - \frac{x}{5} + \frac{x^2}{70} - \frac{x^3}{1890} + \frac{x^4}{83160} - \frac{x^5}{5405400} + \cdots\right) \tag{81}$$

The general solution of Example 2, eq.(72), is thus given by

$$y = k_1 \ f(x) + k_2 \ g(x) \tag{82}$$

where k_1 and k_2 are arbitrary constants.

Next, let's try the method of Frobenius on Example 3:
Example 3

$$x^3 \frac{d^2y}{dx^2} + y = 0 \tag{83}$$

We substitute (71) into (83) and collect terms:

$$c_0 x^r + [(r^2 - r)c_0 + c_1]x^{r+1} + [(r^2 + r)c_1 + c_2]x^{r+2} + [(r^2 + 3r + 2)c_2 + c_3]x^{r+3} + \cdots + = 0 \tag{84}$$

$$c_0 = 0 \tag{85}$$

$$(r^2 - r)c_0 + c_1 = 0 \tag{86}$$

$$(r^2 + r)c_1 + c_2 = 0 \tag{87}$$

$$(r^2 + 3r + 2)c_2 + c_3 = 0 \tag{88}$$

$$\cdots$$

$$(r + n)(r + n - 1)c_n + c_{n+1} = 0 \tag{89}$$

$$\cdots$$

Eqs.(86)-(89) have the recurrence relation:

$$c_{n+1} = -(r + n)(r + n - 1)c_n, \qquad n = 1, 2, 3, \cdots \tag{90}$$

But since eq.(85) requires that c_0=0, we find from (90) that all the c_i's must vanish. In other words, the method of Frobenius has failed to produce a solution for Example 3, eq.(83).

Summarizing, we have seen that for some equations the method of power series works (Example 1), whereas for other equations that method fails but the more general method of Frobenius works (Example 2). Now we have seen an example in which the method of Frobenius fails (Example 3).

We need a classification scheme which will tell us which equations we can be guaranteed to solve using these two methods.

8 Ordinary Points and Singular Points

Let's consider the general class of linear second order ODE's of the form:

$$A(x)\ y'' + B(x)\ y' + C(x)\ y = 0 \tag{91}$$

in which $A(x)$, $B(x)$ and $C(x)$ have power series expansions about x=0:

$$A(x) = A_0 + A_1 x + A_2 x^2 + \cdots \tag{92}$$

$$B(x) = B_0 + B_1 x + B_2 x^2 + \cdots \tag{93}$$

$$C(x) = C_0 + C_1 x + C_2 x^2 + \cdots \tag{94}$$

This includes the possibility that $A(x)$, $B(x)$ and $C(x)$ are polynomials.

Definition: If $A_0 \neq 0$ then x=0 is called an *ordinary point.* If A_0=0 and not both B_0 and C_0 are zero, then x=0 is called a *singular point.*

Now suppose that x=0 is a singular point. Then let $p(x)$ be defined by

$$p(x) = x\frac{B(x)}{A(x)} = \frac{B_0 + B_1 x + B_2 x^2 + \cdots}{A_1 + A_2 x + A_3 x^2 + \cdots} \tag{95}$$

and let $q(x)$ be defined by

$$q(x) = x^2\frac{C(x)}{A(x)} = x\frac{C_0 + C_1 x + C_2 x^2 + \cdots}{A_1 + A_2 x + A_3 x^2 + \cdots} \tag{96}$$

Definition: Let x=0 be a singular point. If $p(x)$ and $q(x)$ don't blow up as x approaches zero, then x=0 is called a *regular singular point.* A singular point which is not regular is called an *irregular singular point.*

These definitions will help us to determine whether the method of power series or the method of Frobenius will work to solve a given equation of the form (91).

Rule 1: If x=0 is an ordinary point, then two linearly independent solutions can be obtained by the method of power series expansions about x=0.

Rule 2: If x=0 is a regular singular point, then at least one solution can be obtained by the method of Frobenius expanded about x=0.

As an example, consider Example 1, eq.(52):

$$\frac{d^2y}{dx^2} + x\ \frac{dy}{dx} + y = 0 \tag{97}$$

In this case $A(x)$=1 and $x = 0$ is an ordinary point. According to Rule 1, two independent solutions can be obtained in the form of power series, see eqs.(62) and (63).

Consider now Example 2, eq.(72):

$$2x^2\frac{d^2y}{dx^2} + 3x\ \frac{dy}{dx} + (-1+x)y = 0 \tag{98}$$

In this case A_0=0 and C_0=−1 so that x=0 is a singular point. The quantities $p(x)$ and $q(x)$ are given by

$$p(x) = x\frac{B(x)}{A(x)} = \frac{3}{2} \qquad \text{and} \qquad q(x) = x^2\frac{C(x)}{A(x)} = \frac{-1+x}{2} \tag{99}$$

and since neither $p(x)$ nor $q(x)$ blows up as $x \to 0$, we see that x=0 is a regular singular point. Then by Rule 2, at least one solution can be obtained by the method of Frobenius. In fact we found two such solutions, see eqs.(80) and (81).

Next consider Example 3, eq.(83):

$$x^3\frac{d^2y}{dx^2} + y = 0 \tag{100}$$

In this case A_0=0 and C_0=1 so that x=0 is a singular point. The quantities $p(x)$ and $q(x)$ are given by

$$p(x) = x\frac{B(x)}{A(x)} = 0 \qquad \text{and} \qquad q(x) = x^2\frac{C(x)}{A(x)} = \frac{1}{x} \tag{101}$$

Since $q(x)$ blows up as $x \to 0$, we see that x=0 is an irregular singular point and neither Rule 1 nor Rule 2 applies. In fact we saw that the method of Frobenius did not work on Example 3.

You may have noticed that Rule 1 guarantees two linearly independent solutions in the case of an ordinary point, while Rule 2 only guarantees one solution in the case of a regular singular point. Nevertheless in the case of Example 2 (which has a regular singular point) we found two linearly independent solutions. The case where the method of Frobenius only yields one solution is illustrated by Example 4:
Example 4

$$x^2\frac{d^2y}{dx^2} + 4x\ \frac{dy}{dx} + (2+x)\ y = 0 \tag{102}$$

Here x=0 is a regular singular point. We apply the method of Frobenius by substituting the Frobenius series (71) into eq.(102) and collecting terms:

$$(r^2+3r+2)c_0x^r + [(r^2+5r+6)c_1+c_0]x^{r+1} + [(r^2+7r+12)c_2+c_1]x^{r+2} + \cdots = 0 \tag{103}$$

$$(r^2+3r+2)c_0 = 0 \tag{104}$$

$$(r^2+5r+6)c_1+c_0 = 0 \tag{105}$$

$$(r^2+7r+12)c_2+c_1 = 0 \tag{106}$$

$$\cdots$$

$$[r^2+(2n+5)r+(n^2+5n+6)]c_{n+1}+c_n = 0 \tag{107}$$

$$\cdots$$

Eqs.(105)-(107) have the recurrence relation:

$$c_{n+1} = -\frac{c_n}{(r+n+2)(r+n+3)}, \qquad n = 1, 2, 3, \cdots \tag{108}$$

In addition to (108), we also must satisfy, from (104), the indicial equation:

$$r^2 + 3r + 2 = 0 \qquad \Rightarrow \qquad r = -1, -2 \tag{109}$$

Let us consider each of these r-values separately:
For r=−1 we get:

$$y = c_0 \left(\frac{1}{x} - \frac{1}{2} + \frac{x}{12} - \frac{x^2}{144} + \frac{x^3}{2880} - \frac{x^4}{86400} + \frac{x^5}{3628800} + \cdots \right) \tag{110}$$

For r=−2, eq.(105) requires that we take c_0=0. Then we obtain the following solution:

$$y = c_1 \left(\frac{1}{x} - \frac{1}{2} + \frac{x}{12} - \frac{x^2}{144} + \frac{x^3}{2880} - \frac{x^4}{86400} + \frac{x^5}{3628800} + \cdots \right) \tag{111}$$

Eqs.(110) and (111) are obviously not linearly independent. The method of Frobenius has obtained only one linearly independent solution. This case can be characterized by Rule 3:

Rule 3: If x=0 is a regular singular point, and if the two indicial roots r_1 and r_2 are not identical and do not differ by an integer, then the method of Frobenius will yield two linearly independent solutions.

In the case of Example 2, eq.(72), the indicial equation (79) gave r_1=−1 and $r_2=\frac{1}{2}$. Since these r-values do not differ by an integer, Rule 3 guarantees that the method of Frobenius will generate two linearly independent solutions. See eqs.(80) and (81). However, in the case of Example 4, eq.(102), the indicial roots were r_1=−1 and r_2=−2, see eq.(109). In this case Rule 3 does not apply and we have no guarantee that the method of Frobenius will generate two linearly independent solutions.

A natural question to ask at this point is: How can we find the second linearly independent solution in cases like that of Example 4, where there is a regular singular point for which the indicial roots are repeated or differ by an integer? The answer is given by Rule 4:

Rule 4: If x=0 is a regular singular point, and if the two indicial roots r_1 and r_2 are either identical or differ by an integer, then one solution can be obtained by the method of Frobenius. Let us refer to that solution as $y = f(x)$. A second linearly independent solution can be obtained in the form

$$y = C f(x) \log x + g(x) \tag{112}$$

where $g(x)$ is a Frobenius series. The constant C may or may not be equal to zero. If C=0, then both linearly independent solutions can be obtained in the form of Frobenius series. In the repeated root case r_1=r_2, the constant C is not equal to zero.

In the case of repeated indicial roots, we should not be surprised to find the occurrence of $\log x$ in the solution, since we have seen that in Euler's equation, eq.(44), which has a regular singular point at x=0, the presence of a repeated root implies the presence of $\log x$ in the solution, see eq.(50).

9 Solving Problem "B" by Separation of Variables, continued

Now that we know how to solve linear variable coefficient ODE's, let us return to the problem which motivated our digression, namely the application of separation of variables to problem "B". We wrote $u(r,z) = R(r)Z(z)$, see eq.(39), and we found that the unkown function $R(r)$ had to satisfy the following boundary value problem (see eqs.(42),(43)):

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda R = 0 \qquad \text{with the B.C.} \quad R = 0 \;\; \text{when} \;\; r = a \tag{113}$$

To begin with, let's stretch the r coordinate so as to absorb the separation constant λ and thereby make it disappear from the ODE. Let

$$x = \sqrt{\lambda} r \tag{114}$$

and let us change notation from $R(r)$ to $y(x)$:

$$\frac{dR}{dr} = \frac{dy}{dx}\frac{dx}{dr} = \frac{dy}{dx}\frac{d}{dr}(\sqrt{\lambda} r) = \sqrt{\lambda}\frac{dy}{dx} \tag{115}$$

where we have used the chain rule. Similarly, $\frac{d^2R}{dr^2} = \lambda \frac{d^2y}{dx^2}$. The differential equation in (113) becomes

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0 \tag{116}$$

Next let us multiply eq.(116) by x so as to put it in the form of eq.(91):

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \tag{117}$$

Now we have to decide upon which method to use to solve eq.(117). Inspection of (117) shows that x=0 is not an ordinary point (since $A_0 \neq 0$, see eq.(92)). In fact x=0 is a regular singular point. To prove this we compute the quantities $p(x)$ and $q(x)$ defined in eqs.(95),(96):

$$p(x) = x\frac{B(x)}{A(x)} = 1 \qquad \text{and} \qquad q(x) = x^2\frac{C(x)}{A(x)} = x^2 \tag{118}$$

and we note that neither $p(x)$ nor $q(x)$ blows up as $x \to 0$, telling us that x=0 is a regular singular point. Then by Rule 2 we know that at least one solution of (117) can be obtained by the method of Frobenius.

We proceed with the method of Frobenius by substituting a Frobenius series (71) into eq.(117) and collecting terms. We find:

$$r^2 c_0 x^{r-1} + (r^2+2r+1)c_1 x^r + [(r^2+4r+4)c_2 + c_0]x^{r+1} + [(r^2+6r+9)c_3 + c_1]x^{r+2} + \cdots = 0 \tag{119}$$

$$r^2 c_0 = 0 \tag{120}$$

$$(r^2+2r+1)c_1 = 0 \tag{121}$$

$$(r^2 + 4r + 4)c_2 + c_0 = 0 \tag{122}$$
$$(r^2 + 6r + 9)c_3 + c_1 = 0 \tag{123}$$
$$\cdots$$
$$(r + n + 2)^2 c_{n+2} + c_n = 0 \tag{124}$$
$$\cdots$$

Eqs.(122)-(124) have the recurrence relation:

$$c_{n+2} = -\frac{c_n}{(r+n+2)^2}, \qquad n = 0, 1, 2, 3, \cdots \tag{125}$$

The indicial equation for this system is obtained from eq.(120):

$$r^2 = 0 \qquad \Rightarrow \qquad r = 0, 0 \quad \text{(repeated root)} \tag{126}$$

So we are in the case of a regular singular point with a repeated indicial root. Rule 4 applies to this situation and tells us that the method of Frobenius will be able to generate only one solution, and that a second linearly independent solution will involve a $\log x$ term, see eq.(112).

Note that once r=0 is chosen to satisfy eq.(120), we find from eq.(121) that c_1=0. This implies, from the recurrence relation (125) that all the c_{odd} coefficients must vanish. Thus we obtain the following solution:

$$y = c_0 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147456} - \frac{x^{10}}{14745600} + \cdots\right) \tag{127}$$

The function so generated is very famous and is known as a "Bessel function of order zero of the first kind". It is usually represented by the symbol J_0 and can be written in the following form:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \tag{128}$$

We present without derivation the following expression for $J_0(x)$ which is valid for large values of x:

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) \qquad \text{as} \quad x \to \infty \tag{129}$$

From Rule 4, a second linearly independent solution will take the form:

$$y = C J_0(x) \log x + g(x) \tag{130}$$

where $g(x)$ can be written as a power series. Let us abbreviate this second linearly independent solution by the symbol Y_0.

Thus we have obtained the following expression for the general solution to eq.(117):

$$y = k_1 J_0(x) + k_2 Y_0(x) \tag{131}$$

Recall that we defined x in order to simplify the form of the R differential equation in (113). Now we return to the original independent variable r by setting $x=\sqrt{\lambda}r$ (see eq.(114)). The general solution to the R differential equation in (113), $R'' + R'/r + \lambda R = 0$ can be written:

$$R(r) = k_1 J_0(\sqrt{\lambda}r) + k_2 Y_0(\sqrt{\lambda}r) \tag{132}$$

The next step is to apply the B.C. associated with this equation:

$$R = 0 \quad \text{when} \quad r = a \tag{133}$$

Before applying the B.C. (133), we note that the presence of the log term in Y_0 will produce infinite temperatures at r=0, and hence the Y_0 part of the solution must be removed by choosing k_2=0 in (132). The B.C. (133) then requires that

$$R(a) = k_1 J_0(\sqrt{\lambda}a) = 0 \tag{134}$$

There is an infinite sequence of λ values, $\{\lambda_i\}$, each of which satisfies eq.(134). These may be found by utilizing tables of the zeros of $J_0(x)$. We shall represent the n^{th} zero of $J_0(x)$ by Γ_n. The first five zeros of $J_0(x)$ are:

$$\Gamma_1 = 2.4048,\ \Gamma_2 = 5.5201,\ \Gamma_3 = 8.6537,\ \Gamma_4 = 11.7915,\ \Gamma_5 = 14.9309,\ \cdots \tag{135}$$

The corresponding values of λ are of the form λ_n=Γ_n^2/a^2:

$$\lambda_1 = \frac{5.7831}{a^2},\ \lambda_2 = \frac{30.4715}{a^2},\ \lambda_3 = \frac{74.8865}{a^2},\ \lambda_4 = \frac{139.039}{a^2},\ \lambda_5 = \frac{222.932}{a^2},\ \cdots \tag{136}$$

And the corresponding R-eigenfunctions are:

$$R_n(r) = J_0(\sqrt{\lambda_n}r) = J_0\left(\Gamma_n \frac{r}{a}\right) \tag{137}$$

Returning to the separation of variables solution of problem "B", we must also solve the Z-equation (see eqs.(42),(43)):

$$Z'' - \lambda Z = 0 \qquad \text{with the B.C.} \qquad Z = 0 \quad \text{when} \quad z = 0 \tag{138}$$

As in the separation of variables solution of problem "A", the general solution of the Z equation (138) can be written in the form:

$$Z(z) = c_3 \cosh\sqrt{\lambda}z + c_4 \sinh\sqrt{\lambda}z \tag{139}$$

The B.C. $Z(0) = 0$ gives c_3=0.

Substituting the derived results into the assumed form of the solution $u(r,z) = R(r)Z(z)$, we have

$$u(r,z) = R(r)Z(z) = a_n J_0\left(\Gamma_n \frac{r}{a}\right) \sinh \frac{\Gamma_n z}{a}, \qquad n = 1, 2, 3, \cdots \tag{140}$$

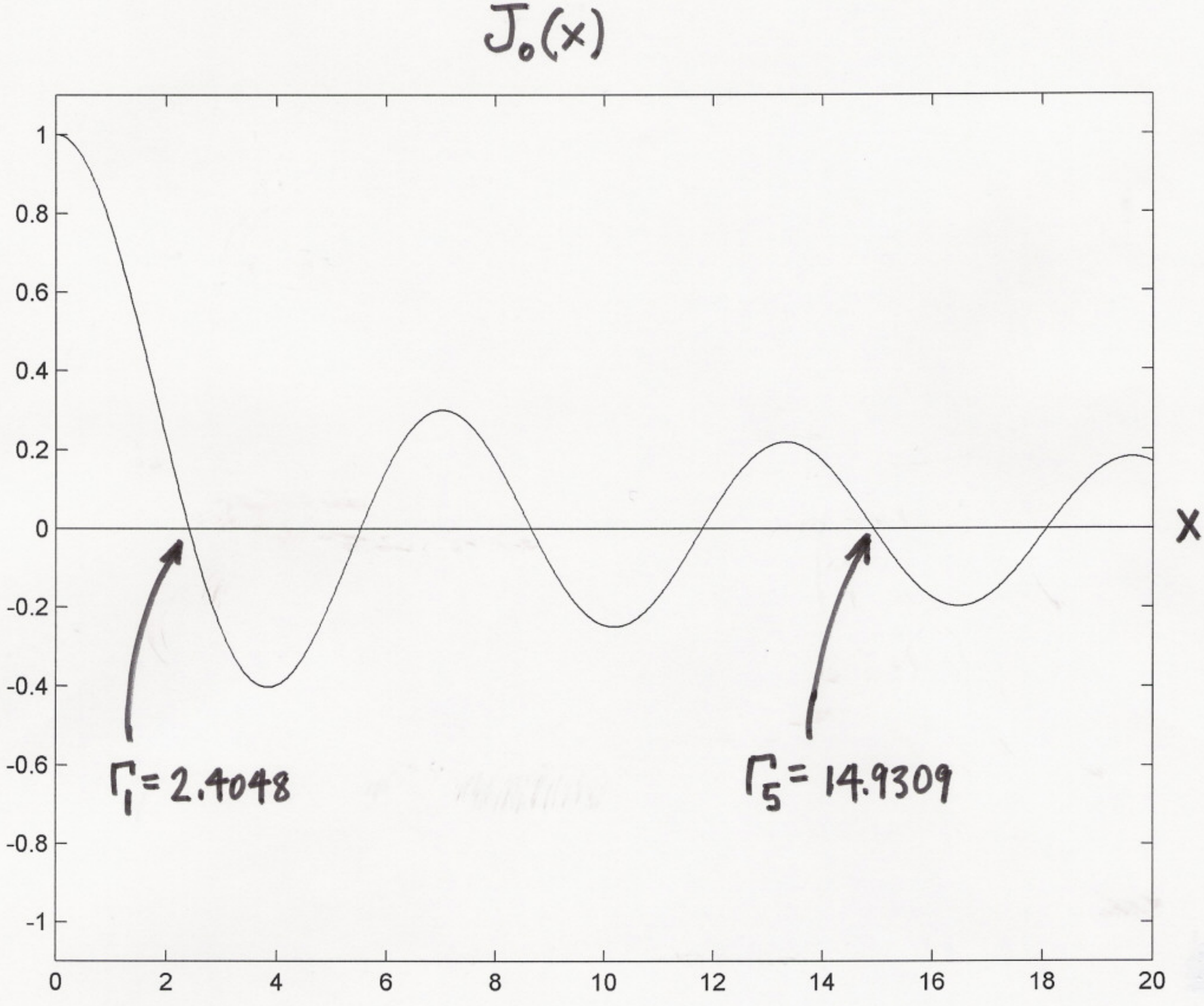
where a_n is an arbitrary constant. Since the PDE (38) is linear, we may superimpose solutions to obtain the form:

$$u(r,z) = \sum_{n=1}^{\infty} a_n J_0\left(\Gamma_n \frac{r}{a}\right) \sinh \frac{\Gamma_n z}{a} \tag{141}$$

We still have to satisfy the B.C. (8), $u(r,L)$=1, which gives (from (141)):

$$1 = \sum_{n=1}^{\infty} a_n J_0\left(\Gamma_n \frac{r}{a}\right) \sinh \frac{\Gamma_n L}{a} \tag{142}$$

Our task now is to find the constants a_n. When we were faced with this same task in solving problem "A", we had a Fourier series (see eq.(31)). We obtained the values of the constants a_n by using the orthogonality of the eigenfunctions $\sin \frac{n\pi x}{L}$. In the case of problem "B" we will similarly use the orthogonality of the eigenfunctions $J_0\left(\Gamma_n \frac{r}{a}\right)$.



J0(x)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
2
4
6
8
10
12
14
16
18
20
X
Γ1 = 2.4048
Γ5 = 14.9309

10 Orthogonality

In order to complete our treatment of problem "B", we need to use orthogonality of the associated eigenfunctions. In the case of problem "A", the orthogonal eigenfunctions $X(x)$ satisfied the following boundary value problem (see eqs.(22) and (23)):

$$X'' + \lambda X = 0 \quad \text{with the B.C.} \quad X(0) = 0, \quad X(L) = 0 \tag{143}$$

In order to prove orthogonality, we solved this problem to get

$$X_n(x) = \sin \lambda_n x = \sin \frac{n\pi x}{L} \tag{144}$$

and then we used the properties of trig functions to prove that

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = 0 \qquad n \neq m \tag{145}$$

See eqs.(32)-(34).

Unfortunately, we can't use a comparable method to prove the orthogonality of Bessel functions. Instead we will use a different method called Sturm-Liouville theory. We demonstrate the method by using it on eq.(143). Let $X_n(x)$ satisfy the eqs:

$$X_n'' + \lambda_n X_n = 0 \quad \text{with the B.C.} \quad X_n(0) = 0, \quad X_n(L) = 0 \tag{146}$$

and let $X_m(x)$ satisfy the eqs:

$$X_m'' + \lambda_m X_m = 0 \quad \text{with the B.C.} \quad X_m(0) = 0, \quad X_m(L) = 0 \tag{147}$$

We want to prove that

$$\int_0^L X_m X_n dx = 0 \quad \text{for} \quad n \neq m \tag{148}$$

We multiply (146) by X_m and (147) by X_n, and subtract, giving

$$X_m X_n'' - X_n X_m'' + (\lambda_n - \lambda_m) X_n X_m = 0 \tag{149}$$

Next we integrate (149) from 0 to L:

$$\int_0^L (X_m X_n'' - X_n X_m'')\, dx + (\lambda_n - \lambda_m) \int_0^L X_n X_m \, dx = 0 \tag{150}$$

The idea now is to get the first term to vanish by using integration by parts and the B.C. in (143). Consider the first part of the first term:

$$\int_0^L X_m X_n'' \, dx = [X_m X_n']_0^L - \int_0^L X_m' X_n' \, dx \tag{151}$$

The integrated terms vanish due to the B.C. $X_m(0)$=$X_m(L)$=0. Thus

$$\int_0^L X_m X_n'' \, dx = - \int_0^L X_m' X_n' \, dx \tag{152}$$

Similarly,

$$\int_0^L X_n X_m'' \, dx = -\int_0^L X_n' X_m' \, dx \tag{153}$$

Therefore eq.(150) becomes

$$(\lambda_n - \lambda_m) \int_0^L X_n X_m \, dx = 0 \quad \Rightarrow \quad \int_0^L X_n X_m \, dx = 0 \quad \text{if} \quad n \neq m \tag{154}$$

We have thus proven the orthogonality of the eigenfunctions $X_n(x)$ *without constructing* $X_n(x)$. This method is quite different from the method we used in eqs.(32)-(34), where we solved for $X_n(x) = \sin \frac{n\pi x}{L}$ and used various trig identities.

Now in the case of problem "B", the eigenfunctions $R_n(r)$ satisfy the boundary value problem (113):

$$\frac{d^2 R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} + \lambda_n R_n = 0 \qquad \text{with the B.C.} \quad R_n = 0 \ \text{ when } \ r = a \tag{155}$$

Based on the procedure given above in eqs.(146)-(154), it would seem like the right way to proceed with eq.(155) would be to multiply by R_m, subtract the same equation with m and n interchanged, and integrate by parts to eliminate the terms which are not part of the orthogonality condition. Try it: it doesn't work!

The correct procedure is to multiply eq.(155) by r and write it in the form:

$$\frac{d}{dr}\left(r\frac{dR_n}{dr}\right) + \lambda_n r R_n = 0 \tag{156}$$

Now we proceed as before, multiplying (156) by R_m and subtracting the same equation with m and n interchanged, giving:

$$R_m \frac{d}{dr}\left(r\frac{dR_n}{dr}\right) - R_n \frac{d}{dr}\left(r\frac{dR_m}{dr}\right) + (\lambda_n - \lambda_m) r R_n R_m = 0 \tag{157}$$

Now we integrate (157) from 0 to a:

$$\int_0^a \left(R_m \frac{d}{dr}\left(r\frac{dR_n}{dr}\right) - R_n \frac{d}{dr}\left(r\frac{dR_m}{dr}\right)\right) dr + (\lambda_n - \lambda_m) \int_0^a r R_n R_m dr = 0 \tag{158}$$

As before, the idea is to get the first term to vanish by using integration by parts and the B.C. in (155). Consider the first part of the first term:

$$\int_0^a R_m \frac{d}{dr}\left(r\frac{dR_n}{dr}\right) dr = \left[r R_m \frac{dR_n}{dr}\right]_0^a - \int_0^a r \frac{dR_n}{dr}\frac{dR_m}{dr} dr \tag{159}$$

The integrated terms vanish at the upper limit because R_m=0 at r=a. At the lower limit, the presence of the factor r kills the term, assuming that R and dR/dr remain bounded as $r \to 0$. So we have

$$\int_0^a R_m \frac{d}{dr}\left(r\frac{dR_n}{dr}\right) dr = -\int_0^a r \frac{dR_n}{dr}\frac{dR_m}{dr} dr \tag{160}$$

Similarly, we find that

$$\int_0^a R_n \frac{d}{dr}\left(r\frac{dR_m}{dr}\right)dr = -\int_0^a r\frac{dR_m}{dr}\frac{dR_n}{dr}dr \tag{161}$$

so that eq.(158) becomes

$$(\lambda_n - \lambda_m)\int_0^a rR_nR_m dr = 0 \quad \Rightarrow \quad \int_0^a rR_nR_m dr = 0 \quad \text{if} \quad n \neq m \tag{162}$$

Since we showed in eq.(137)

$$R_n(r) = J_0(\sqrt{\lambda_n} r) = J_0\left(\Gamma_n \frac{r}{a}\right) \tag{163}$$

we have the following orthogonality condition for Bessel functions:

$$\int_0^a rJ_0\left(\Gamma_n\frac{r}{a}\right)J_0\left(\Gamma_m\frac{r}{a}\right)dr = 0 \quad \text{if} \quad n \neq m \tag{164}$$

where as a reminder we note that Γ_n is the n^{th} zero of $J_0(x)$, see eq.(135).

11 Sturm-Liouville Theory

In this section we present some results which generalize the orthogonality results in the previous section. A linear second order ODE is said to be in Sturm-Liouville form if it can be written as follows:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + (q(x) + \lambda w(x))y = 0 \tag{165}$$

We suppose that this equation is defined on an interval $a \le x \le b$ and that B.C. on $y(x)$ are given at the endpoints $x{=}a$ and $x{=}b$.

[In the case of problem "B", eq.(156) is in Sturm-Liouville form with $R(r)$ corresponding to $y(x)$, and where $p(x){=}x$, $q(x){=}0$, and $w(x){=}x$. The interval $0 \le r \le a$ corresponds to $a \le x \le b$.]

Let $\{\lambda_n\}$ be a set of eigenvalues satisfying the ODE (165) and the B.C., and let $\{y_n(x)\}$ be the corresponding eigenfunctions. The orthogonality condition for this system is

$$\int_a^b w(x)y_n(x)y_m(x)dx = 0 \quad \text{if} \quad n \neq m \tag{166}$$

Here $w(x)$ is called the *weight function.*

In order for eq.(166) to hold, the following terms must vanish:

$$[p(x)(y_m(x)y_n'(x) - y_n(x)y_m'(x))]_a^b = 0 \tag{167}$$

A variety of B.C. will satisfy eq.(167), including $y(x)$ vanishing at both endpoints $x{=}a$ and $x{=}b$. As we have seen in the case of problem "B", another way to satisfy (167) is if $p(x){=}0$ at one endpoint and $y(x){=}0$ at the other endpoint.

A related question is given a second order ODE which is not in Sturm-Liouville form:

$$A(x)y'' + B(x)y' + C(x)y + \lambda D(x)y = 0 \tag{168}$$

How can it be put into Sturm-Liouville form?

Recall that this was the situation in problem "B", where the original R-equation, (155), had to be multiplied by r to get it into the Sturm-Liouville form (156).

The procedure is to first calculate the following expression for $p(x)$:

$$p(x) = e^{\int (B(x)/A(x))dx} \tag{169}$$

Then multiply eq.(168) by $p(x)/A(x)$. The result may then be written in the Sturm-Liouville form (165), where

$$q(x) = \frac{p(x)C(x)}{A(x)} \tag{170}$$

$$w(x) = \frac{p(x)D(x)}{A(x)} \tag{171}$$

For example, in the case of eq.(155), which we may write in the form:

$$y'' + \frac{1}{x}y' + \lambda y = 0 \tag{172}$$

we have

$$A(x) = 1, \quad B(x) = \frac{1}{x}, \quad C(x) = 0, \quad D(x) = 1 \tag{173}$$

Then eqs.(169)-(171) give

$$p(x) = e^{\int (B(x)/A(x))dx} = e^{\int (1/x)dx} = e^{\log x} = x \tag{174}$$

$$q(x) = \frac{p(x)C(x)}{A(x)} = 0 \tag{175}$$

$$w(x) = \frac{p(x)D(x)}{A(x)} = x \tag{176}$$

which is to say that if eq.(172) is multiplied by $p(x)/A(x)$=x, it can be written in the Sturm-Liouville form:

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \lambda xy = 0 \tag{177}$$

12 Solving Problem "B" by Separation of Variables, concluded

Now that we know about the orthogonality of Bessel functions, let's return to solving problem "B" by separation of variables. We previously obtained the following expression for $u(r,z)$ (see eq.(141):

$$u(r,z) = \sum_{n=1}^{\infty} a_n J_0\left(\Gamma_n \frac{r}{a}\right) \sinh \frac{\Gamma_n z}{a} \tag{178}$$

We still have to satisfy the B.C. (8), $u(r,L)$=1, which gives (from (178)):

$$1 = \sum_{n=1}^{\infty} a_n J_0\left(\Gamma_n \frac{r}{a}\right) \sinh \frac{\Gamma_n L}{a} \tag{179}$$

Now we can use the orthogonality result (164)

$$\int_0^a r J_0\left(\Gamma_n \frac{r}{a}\right) J_0\left(\Gamma_m \frac{r}{a}\right) dr = 0 \quad \text{if} \quad n \neq m \tag{180}$$

We multiply (179) by $rJ_0\left(\Gamma_m \frac{r}{a}\right)$ and integrate from 0 to a. The orthogonality condition (180) zaps everything on the RHS of the resulting equation except for the n=m term:

$$\int_0^a r J_0\left(\Gamma_m \frac{r}{a}\right) dr = a_m \sinh \frac{\Gamma_m L}{a} \int_0^a r \left[J_0\left(\Gamma_m \frac{r}{a}\right)\right]^2 dr \tag{181}$$

Solving (181) for a_m, we obtain:

$$a_m = \frac{I_1(m)}{I_2(m) \sinh \frac{\Gamma_m L}{a}} \tag{182}$$

where we have used the notation I_1 and I_2 to abbreviate the following integrals:

$$I_1(m) = \int_0^a r J_0\left(\Gamma_m \frac{r}{a}\right) dr \quad \text{and} \quad I_2(m) = \int_0^a r \left[J_0\left(\Gamma_m \frac{r}{a}\right)\right]^2 dr \tag{183}$$

Using this notation, the expression (178) for $u(r,z)$ becomes:

$$u(r,z) = \sum_{n=1}^{\infty} \frac{I_1(n)}{I_2(n)} J_0\left(\Gamma_n \frac{r}{a}\right) \frac{\sinh \frac{\Gamma_n z}{a}}{\sinh \frac{\Gamma_n L}{a}} \tag{184}$$

Now it turns out that the integrals I_1 and I_2 can be evaluated in terms of a tabulated function $J_1(x)$, which is the notation for a Bessel function of order 1 of the first kind. Skipping all the details, the result is:

$$\frac{I_1(n)}{I_2(n)} = \frac{2}{\Gamma_n} \frac{1}{J_1(\Gamma_n)} \tag{185}$$

Using (185) in (184), we obtain the final solution of problem "B":

$$u(r,z) = \sum_{n=1}^{\infty} \frac{2}{\Gamma_n} \frac{J_0\left(\Gamma_n \frac{r}{a}\right)}{J_1(\Gamma_n)} \frac{\sinh \frac{\Gamma_n z}{a}}{\sinh \frac{\Gamma_n L}{a}} \tag{186}$$

13 Solving Problem "C" by Separation of Variables

Problem "C" has the PDE (see (9) and (16)):

$$\nabla^2 u = \frac{1}{\rho^2}\left[\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\frac{\partial u}{\partial \phi}\sin\phi\right)\right] = 0 \tag{187}$$

Following the procedure we used on problems "A" and "B", we seek a solution to the PDE (187) in the form

$$u(\rho,\phi) = R(\rho)\Phi(\phi) \tag{188}$$

Substitution of (188) into (187) gives:

$$\frac{1}{\rho^2}\left[(\rho^2 R')'\Phi + \frac{1}{\sin\phi}(\Phi'\sin\phi)'R\right] = 0 \tag{189}$$

where primes represent differentiation with respect to the argument. Separating variables, we obtain

$$\frac{\rho^2 R'' + 2\rho R'}{R} = -\frac{1}{\sin\phi}\frac{\Phi''\sin\phi + \Phi'\cos\phi}{\Phi} = \lambda \tag{190}$$

This yields two ODE's:

$$\rho^2 R'' + 2\rho R' - \lambda R = 0 \quad \text{and} \quad \Phi'' + \Phi'\cot\phi + \lambda\Phi = 0 \tag{191}$$

Let's begin with the Φ-equation. This equation can be simplified by changing the independent variable from the colatitude ϕ to $x{=}\cos\phi$. In order to accomplish this step, we will use the chain rule, and we will write $\Phi(\phi)$ as $y(x)$ instead of $\Phi(x)$ to avoid confusion. We have

$$\frac{d\Phi}{d\phi} = \frac{dy}{dx}\,\frac{dx}{d\phi} = -\sin\phi\,\frac{dy}{dx} \tag{192}$$

$$\frac{d^2\Phi}{d\phi^2} = \frac{d}{d\phi}\left(-\sin\phi\,\frac{dy}{dx}\right) = -\cos\phi\,\frac{dy}{dx} + \sin^2\phi\,\frac{d^2y}{dx^2} = -x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} \tag{193}$$

Using eqs.(192) and (193), the Φ-equation of (191) becomes:

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} + \cot\phi\left(-\sin\phi\,\frac{dy}{dx}\right) + \lambda y = 0 \tag{194}$$

Eq.(194) can be simplified by using $\cot\phi\sin\phi{=}\cos\phi{=}x$, giving:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0 \tag{195}$$

Note that $x{=}0$ is an ordinary point of eq.(195). By Rule 1, we can use the method of power series expansions about $x{=}0$ to obtain two linearly independent solutions. (See the section on "Ordinary Points and Singular Points".)1 We look for a solution in the form of a power series:

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots \tag{196}$$

Substituting (196) into (195) and collecting terms gives

$$c_0\,\lambda + 2\,c_2 + (c_1\,(\lambda-2) + 6\,c_3)\;x + (c_2\,(\lambda-6) + 12\,c_4)\;x^2 + (c_3\,(\lambda-12) + 20\,c_5)\;x^3 + \cdots \tag{197}$$

Requiring the coefficient of each power of x^n to vanish gives:

$$c_0\,\lambda + 2\,c_2 = 0 \tag{198}$$
$$c_1\,(\lambda-2) + 6\,c_3 = 0 \tag{199}$$
$$c_2\,(\lambda-6) + 12\,c_4 = 0 \tag{200}$$
$$c_3\,(\lambda-12) + 20\,c_5 = 0 \tag{201}$$
$$\cdots$$

which gives the following recurrence relation:

$$c_{n+2} = \left(\frac{n(n+1)-\lambda}{(n+2)(n+1)}\right) c_n \tag{202}$$

Here c_0 and c_1 can be chosen arbitrarily, giving rise to two linearly independent solutions $f(x)$ and $g(x)$:

$$y = c_0\;f(x) + c_1\;g(x) \tag{203}$$

where

$$f(x) = 1 - \frac{\lambda\,x^2}{2} + \frac{\lambda(\lambda-6)\,x^4}{24} + \cdots \tag{204}$$

$$g(x) = x - \frac{(\lambda-2)\;x^3}{6} + \frac{(\lambda-12)\;(\lambda-2)\,x^5}{120} + \cdots \tag{205}$$

It turns out that these two series diverge at $x{=}\pm 1$, that is, at $\cos\phi{=}\pm 1$, that is, at $\phi{=}0$ and $\phi{=}\pi$. Physically this represents unbounded temperatures along the polar axis of the sphere. No choice of the arbitrary constants c_0 and c_1 will eliminate this problem, as it did in the case of the cylinder, compare with eqs.(132),(134). However, if λ is taken equal to $n(n+1)$, for n an integer, then one of the two infinite series will terminate, that is, will become a polynomial, and the convergence problem will disappear.

We therefore take $\lambda{=}n(n+1)$, $n{=}0,1,2,3,\cdots$ and we obtain a sequence of polynomial solutions called *Legendre polynomials*, $y = P_n(x)$:

$$P_0(x) = 1 \tag{206}$$
$$P_1(x) = x = \cos\phi \tag{207}$$
$$P_2(x) = \frac{1}{2}(3x^2-1) = \frac{1}{2}(3\cos^2\phi - 1) \tag{208}$$
$$P_3(x) = \frac{1}{2}(5x^3-3x) = \frac{1}{2}(5\cos^3\phi - 3\cos\phi) \tag{209}$$
$$\cdots$$

Rodrigues's formula gives an expression for $P_n(x)$:

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n \tag{210}$$

Let us turn now to the $R(\rho)$-equation in (191):

$$\rho^2 R'' + 2\rho R' - \lambda R = 0, \quad \lambda = n(n+1) \tag{211}$$

This is an Euler equation, so we look for a solution in the form $R(\rho) = \rho^r$ (see eq.(45)). Substitution into (211) gives:

$$r(r-1) + 2r - n(n+1) = 0 \quad \Rightarrow \quad r = n \text{ and } r = -(n+1) \tag{212}$$

Thus eq.(211) has the general solution:

$$R(\rho) = k_1 \rho^n + k_2 \frac{1}{\rho^{n+1}} \tag{213}$$

For bounded temperatures at the sphere's center, ρ=0, we require that k_2=0:

$$R(\rho) = k_1 \rho^n \tag{214}$$

Substituting the derived results into the assumed form of the solution (188), we have

$$u(\rho, \phi) = R(\rho)\Phi(\phi) = b_n P_n(\cos\phi)\rho^n, \qquad n = 0, 1, 2, 3, \cdots \tag{215}$$

where b_n is an arbitrary constant. Since the PDE (187) is linear, we may superimpose solutions to obtain the form:

$$u(\rho, \phi) = \sum_{n=0}^{\infty} b_n P_n(\cos\phi)\rho^n \tag{216}$$

Our task now is to choose the constants b_n so as to satisfy the B.C. on the outside of the sphere, $\rho = a$, see eqs.(10),(11):

$$u(a, \phi) = \sum_{n=0}^{\infty} b_n a^n P_n(\cos\phi) = \begin{cases} 1 & \text{for } 0 \le \phi < \pi/2 \\ 0 & \text{for } \pi/2 \le \phi \le \pi \end{cases} \tag{217}$$

or, switching to x=$\cos\phi$,

$$u(a, x) = \sum_{n=0}^{\infty} b_n a^n P_n(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{for } -1 \le x < 0 \end{cases} \tag{218}$$

As in the case of problems "A" and "B", we need to use orthogonality of the eigenfunctions to find the b_n coefficients.

We can write Legendre's equation (195) in Sturm-Liouville form (165) as follows:

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \lambda y = 0 \tag{219}$$

Comparison of (219) with the Sturm-Liouville form (165) gives the weight function $w(x)$=1, from which we obtain the following orthogonality condition:

$$\int_{-1}^{1} P_n(x) P_m(x) dx = 0 \quad \text{if} \quad n \ne m \tag{220}$$

We multiply eq.(217) by $P_m(x)$ and integrate from $x{=}-1$ to $x{=}1$. The orthogonality condition (220) eliminates all terms in the infinite series except for the $n{=}m$ term:

$$b_m a^m \int_{-1}^{1} [P_m(x)]^2 dx = \int_0^1 P_m(x) dx \tag{221}$$

Solving (221) for b_m, we obtain:

$$b_m = \frac{I_3(m)}{a^m I_4(m)} \tag{222}$$

where we have used the notation I_3 and I_4 to abbreviate the following integrals:

$$I_3(m) = \int_0^1 P_m(x) dx \quad \text{and} \quad I_4(m) = \int_{-1}^{1} [P_m(x)]^2 dx \tag{223}$$

Using this notation, the expression (216) for $u(\rho, \phi)$ becomes:

$$u(\rho, \phi) = \sum_{n=0}^{\infty} \frac{I_3(n)}{I_4(n)} P_n(\cos\phi) \left(\frac{\rho}{a}\right)^n \tag{224}$$

Now it turns out that the integrals I_3 and I_4 can be evaluated in closed form. Here are the results, without derivation:

$$I_4(n) = \int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1} \tag{225}$$

$$I_3(n) = \int_0^1 P_n(x) dx = \begin{cases} 1 & \text{if } n{=}0 \\ 1/2 & \text{if } n{=}1 \\ 0 & \text{if } n{=}2,4,6,\cdots \\ (-1)^{\frac{n-1}{2}} \frac{(n-2)!!}{(n+1)!!} & \text{if } n{=}3,5,7,\cdots \end{cases} \tag{226}$$

where

$$k!! = \begin{cases} k(k-2)(k-4)\cdots 6 \cdot 4 \cdot 2 & \text{for } k \text{ even} \\ k(k-2)(k-4)\cdots 5 \cdot 3 \cdot 1 & \text{for } k \text{ odd} \end{cases} \tag{227}$$