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ON THE PART OF THE

MOTION OF THE LUNAR PERIGEE

WHICH IS A FUNCTION OF THE

MEAN MOTIONS OF THE SUN AND MOON[1]

BY

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in WASHINGTON.

For more than sixty years after the publication of the *Principia*, astronomers were puzzled to account for the motion of the lunar perigee, simply because they could not conceive that terms of the second and higher orders, with respect to the disturbing force, produced more than half of it. For a similar reason, the great inequalities of Jupiter and Saturn remained a long time unexplained.

The rate of motion of the lunar perigee is capable of being determined from observation with about a thirteenth of the precision of the rate of mean motion in longitude. Hence, if we suppose that the mean motion of the moon, in the century and a quarter which has elapsed since BRADLEY began to observe, is known within 3″, it follows that the motion of the perigee can be got to within about the 500,000th of the whole. None of the values hitherto computed from theory agrees as closely as this with the value derived from observation. The question then arises whether the discrepancy should be attributed to the fault of not having carried the approximation far enough, or is indicative of forces acting on the moon which have not yet been considered.

[1] Reprinted, with som additions, from a paper published at Cambridge U. S. A., 1877.

This question cannot be decisively answered until some method of computing the quantity considered is employed, which enables us to say, with tolerable security, that the neglected terms do not exceed a certain limit. If other forces besides gravity have a part in determining the positions of the heavenly bodies, the moon is unquestionably that one which will earliest exhibit traces of these actions; and the motion of the perigee is one of the things most likely to give us advice of them. Hence I propose, in this memoir, to compute the value of this quantity, so far as it depends on the mean motions of the sun and moon, with a degree of accuracy that shall leave nothing further to be desired.

I.

Denoting the potential function by Ω, the differential equations of motion of the moon, in rectangular coördinates, are

$$(1) \qquad \frac{d^2x}{dt^2} = \frac{d\Omega}{dx}, \qquad \frac{d^2y}{dt^2} = \frac{d\Omega}{dy}.$$

When terms, involving the solar eccentricity, are neglected, as is done here, it is known that these equations admit an integral,[1] the Eulerian multipliers for which are, respectively,

$$F = \frac{dx}{dt} + n'y, \qquad G = \frac{dy}{dt} - n'x,$$

n' being the mean angular motion of the sun. When the equations are multiplied by these factors and the products added, it is seen that, not only is the resulting first member an exact derivative with respect to t, but that the second is also the exact derivative of Ω. Hence the integral is

$$(2) \qquad \frac{dx^2 + dy^2}{2dt^2} - n'\frac{xdy - ydx}{dt} = \Omega + C.$$

[1] As JACOBI was the first to announce this integral (Comptes rendus de l'académie des sciences de Paris, Tome III, p. 59), we shall take the liberty of calling it the Jacobian integral.

Let us now suppose that the lunar inequalities independent of the eccentricity, that is, those having the argument of the variation, have already been obtained, and that it is desired to get those which are multiplied by the simple power of this quantity. Denoting the latter by δx and δy, and, for convenience, putting

$$\frac{d^2\Omega}{dx^2} = H, \qquad \frac{d^2\Omega}{dxdy} = J, \qquad \frac{d^2\Omega}{dy^2} = K,$$

which will be all known functions of t, we shall have the linear differential equations

$$(3) \qquad \frac{d^2\delta x}{dt^2} = H\delta x + J\delta y, \qquad \frac{d^2\delta y}{dt^2} = K\delta y + J\delta x.$$

The Jacobian integral also, being subjected to the operation δ, furnishes another equation. Here we notice that when the arbitrary constant C is developed in ascending powers of e, only even powers present themselves, hence we have $\delta C = 0$. In the equation, moreover, the partial derivatives of Ω may be replaced by their equivalents, the second differential quotients of the coördinates. Then, it is evident, the resulting equation may be written

$$(4) \qquad F\frac{d\delta x}{dt} + G\frac{d\delta y}{dt} - \frac{dF}{dt}\delta x - \frac{dG}{dt}\delta y = 0.$$

This is plainly an integral of equations (3) with the special value 0 attributed to the arbitrary constant. For taking the derivative of it with respect to t,

$$(5) \qquad F\frac{d^2\delta x}{dt^2} + G\frac{d^2\delta y}{dt^2} - \frac{d^2F}{dt^2}\delta x - \frac{d^2G}{dt^2}\delta y = 0.$$

Hence the Eulerian multipliers, for obtaining (4) from (3), are, for the first equation, F, and for the second, G. Making the multiplication and comparing the result with (5), we get the conditions

$$(6) \qquad \frac{d^2F}{dt^2} = HF + JG, \qquad \frac{d^2G}{dt^2} = KG + JF.$$

On comparing these with (3), we gather at once that the system of equations

$$\delta x = F, \qquad \delta y = G,$$

is a particular solution of equations (3); and it also satisfies (4). This solution, being composed of terms having the same argument as the variation, is foreign to the solution we seek, and, in consequence, the arbitrary constant, multiplying it in the complete integrals of (3), must, for our problem, be supposed to vanish. But advantage may be taken of it to depress the order of the final equation obtained by elimination. For this purpose we adopt new variables ρ and σ, such that

$$\delta x = F\rho, \qquad \delta y = G\sigma.$$

Relations (6) being considered, (3) and (4) then become

$$F\frac{d^2\rho}{dt^2} + 2\frac{dF}{dt}\frac{d\rho}{dt} + JG(\rho - \sigma) = 0,$$

$$G\frac{d^2\sigma}{dt^2} + 2\frac{dG}{dt}\frac{d\sigma}{dt} + JF(\sigma - \rho) = 0,$$

$$F^2\frac{d\rho}{dt} + G^2\frac{d\sigma}{dt} = 0.$$

Write the first equation

$$\frac{d}{dt}\left(F^2\frac{d\rho}{dt}\right) + JFG(\rho - \sigma) = 0,$$

and adopt a new variable λ such that

$$\frac{d\rho}{dt} = F^{-2}\lambda, \qquad \frac{d\sigma}{dt} = -G^{-2}\lambda.$$

If these values are substituted in the equation after dividing by JFG and differentiating it, we get

(7) $$\frac{d}{dt}\left[\frac{1}{JFG}\frac{d\lambda}{dt}\right] + \left[\frac{1}{F^2} + \frac{1}{G^2}\right]\lambda = 0.$$

Let us now assume a variable $\mathbf{w}$ such that

$$\frac{d\rho}{dt} = \sqrt{JFG}.$$

The second term of (7) is removed by this transformation, and the equation takes the form of the reduced linear equation of the second order,

$$(8) \qquad \frac{d^2\mathbf{w}}{dt^2} + \theta\mathbf{w} = 0,$$

in which, after some reductions,

$$(9) \qquad \theta = \frac{J(F^2 + G^2)}{FG} + \frac{d^2 \log(JFG)}{2dt^2} - \left[\frac{d\log(JFG)}{2dt}\right]^2.$$

It will be perceived that interchanging F and G produces no change in θ: hence had we eliminated ρ instead of σ, the equation obtained would have been the same; and this is true in general, — we arrive always at the same value for θ, no matter what variables may have been used to express the original differential equations. From this we may conclude that θ depends only on the relative position of the moon with reference to the sun, and that it can be developed in a periodic series of the form

$$\theta_0 + \theta_1 \cos 2\tau + \theta_2 \cos 4\tau + \ldots,$$

in which τ denotes the mean angular distance of the two bodies.

It may be noted also that θ, as expressed above, does not involve the quantities H and K. It is obvious that, by means of the original differential equations, all second and higher derivatives may be eliminated from this expression, and that the Jacobian integral suffices for eliminating the first derivative of one of the variables. But it is not possible to express θ as a function of the coördinates only without their derivatives.

II.

As the reduction of θ, in the form just given, presents some difficulties, we will derive another from differential equations in terms of coördinates expressing the relative position of the moon to the sun.

Let the axes of rectangular coördinates have a constant velocity of rotation, so that the axis of x constantly passes through the centre of the sun, and adopt the imaginary variables

$$u = x + y\sqrt{-1}, \qquad s = x - y\sqrt{-1},$$

and put $\varepsilon^{\tau\sqrt{-1}} = \zeta$. In addition, let D denote the operation $-\frac{d}{d\tau}\sqrt{-1}$, so that

$$D(\alpha\zeta^{\nu}) = \nu\alpha\zeta^{\nu},$$

and m denote the ratio of the synodic month to the sidereal year, or

$$\mathsf{m} = \frac{n'}{n - n'},$$

and μ being the sum of the masses of the earth and moon,

$$\varkappa = \frac{\mu}{(n - n')^2}.$$

Lastly, putting

$$\Omega = \frac{\varkappa}{\sqrt{us}} + \frac{3}{8}\mathsf{m}^2(u + s)^2, \tag{10}$$

the differential equations of motion are

$$(11) \quad \left| \begin{aligned} &D^2u + 2\mathsf{m}Du + 2\frac{d\Omega}{ds} = 0, \\ &D^2s - 2\mathsf{m}Ds + 2\frac{d\Omega}{du} = 0. \end{aligned} \right.$$

Multiplying the first of these by Ds, the second by Du, adding the products and integrating the resulting equation, we have the Jacobian integral

$$DuDs + 2\Omega = 2C.$$

When the last three equations are subjected to the operation δ, the results are

$$(12)\quad \begin{cases} D^2\delta u + 2\mathbf{m}D\delta u + 2\dfrac{d^2\Omega}{duds}\delta u + 2\dfrac{d^2\Omega}{ds^2}\delta s = 0, \\ D^2\delta s - 2\mathbf{m}D\delta s + 2\dfrac{d^2\Omega}{duds}\delta s + 2\dfrac{d^2\Omega}{du^2}\delta u = 0, \\ DuD\delta s + DsD\delta u + 2\dfrac{d\Omega}{du}\delta u + 2\dfrac{d\Omega}{ds}\delta s = 0. \end{cases}$$

If, in these equations, the symbol δ is changed into D, they evidently still hold, since they then become the derivatives of the preceding equations. Hence the system of equations

$$\delta u = Du, \qquad \delta s = Ds,$$

forms a particular solution of them. For a like purpose as before, let us adopt new variables v and w, such that

$$\delta u = Du \,.\, v, \qquad \delta s = Ds \,.\, w.$$

In terms of these, equations (12) become

$$Du\,.\,D^2v + 2\left[D^2u + \mathbf{m}Du\right]Dv + \left[D^3u + 2\mathbf{m}D^2u + 2\frac{d^2\Omega}{duds}Du\right]v + 2\frac{d^2\Omega}{ds^2}Ds\,.\,w = 0,$$

$$Ds\,.\,D^2w + 2\left[D^2s - \mathbf{m}Ds\right]Dw + \left[D^3s - 2\mathbf{m}D^2s + 2\frac{d^2\Omega}{duds}Ds\right]w + 2\frac{d^2\Omega}{du^2}Du\,.\,v = 0,$$

$$DuDs\,.\,D(v + w) + \left[DsD^2u + 2\frac{d\Omega}{du}Du\right]v + \left[DuD^2s + 2\frac{d\Omega}{ds}Ds\right]w = 0.$$

If the second and third derivatives of u and s are eliminated from these equations by means of equations (11), we get

$$(13)\quad \begin{cases} Du\,.\,D^2v - 2\left[2\dfrac{d\Omega}{ds} + \mathbf{m}Du\right]Dv - 2\dfrac{d^2\Omega}{ds^2}Ds\,.\,(v - w) = 0, \\ Ds\,.\,D^2w - 2\left[2\dfrac{d\Omega}{du} - \mathbf{m}Ds\right]Dw - 2\dfrac{d^2\Omega}{du^2}Du\,.\,(w - v) = 0, \\ DuDs\,.\,D(v + w) - 2\left[\dfrac{d\Omega}{ds}Ds - \dfrac{d\Omega}{du}Du + \mathbf{m}DuDs\right](v - w) = 0. \end{cases}$$

If the first of these equations is multiplied by Ds, the second by Du, and the products added, the resulting equation will evidently be the derivative of the third; but if the products are subtracted, the second from the first, we get

$$DuDs\,.\,D^2(v-w) - 2D\Omega\,.\,D(v-w)$$
$$-2\left[\frac{d\Omega}{ds}Ds - \frac{d\Omega}{du}Du + \mathbf{m}DuDs\right]D(v+w) - 2\left[\frac{d^2\Omega}{du^2}Du^2 + \frac{d^2\Omega}{ds^2}Ds^2\right](v-w) = 0.$$

For brevity we will write

$$\Delta = \frac{d\Omega}{ds}Ds - \frac{d\Omega}{du}Du + \mathbf{m}DuDs,$$

and put

$$\rho = v + w, \qquad \sigma = v - w,$$

then the last two equations, which will be those employed for the solution of the problem, become

$$(14)\quad \left|\begin{aligned} &DuDs\,.\,D\rho - 2\Delta\,.\,\sigma = 0, \\ &D[DuDs\,.\,D\sigma] - 2\Delta\,.\,D\rho - 2\left[\frac{d^2\Omega}{du^2}Du^2 + \frac{d^2\Omega}{ds^2}Ds^2\right]\sigma = 0. \end{aligned}\right.$$

Eliminating $D\rho$ between these equations, a single equation, involving only the unknown σ, is obtained,

$$(15)\qquad D[DuDs\,.\,D\sigma] - 2\left[\frac{d^2\Omega}{du^2}Du^2 + \frac{d^2\Omega}{ds^2}Ds^2 + \frac{2\Delta^2}{DuDs}\right]\sigma = 0.$$

In order to remove the term involving $D\sigma$, a last transformation will be made; we put

$$\sigma = \frac{\mathbf{w}}{\sqrt{DuDs}}.$$

Then the differential equation, determining $\mathbf{w}$, is

$$D^2\mathbf{w} = \theta\mathbf{w},$$

in which

$$\theta = \frac{2}{DuDs}\left[\frac{d^2\Omega}{du^2}Du^2 + \frac{d^2\Omega}{ds^2}Ds^2\right] + \left(\frac{2\Delta}{DuDs}\right)^2 + \frac{D^2(DuDs)}{2DuDs} - \left[\frac{D(DuDs)}{2DuDs}\right]^2$$

$$= \frac{2}{DuDs}\left[\frac{d^2\Omega}{du^2}Du^2 + \frac{d^2\Omega}{ds^2}Ds^2\right] + \left(\frac{2\Delta}{DuDs}\right)^2 - \frac{D^2\Omega}{DuDs} - \left[\frac{D\Omega}{DuDs}\right]^2.$$

But we have

$$D\Omega = \frac{d\Omega}{du}Du + \frac{d\Omega}{ds}Ds,$$

$$D^2\Omega = \frac{d^2\Omega}{du^2}Du^2 + 2\frac{d^2\Omega}{duds}DuDs + \frac{d^2\Omega}{ds^2}Ds^2 + 2\mathbf{m}\Delta - 2\mathbf{m}^2DuDs - 4\frac{d\Omega}{du}\frac{d\Omega}{ds},$$

in which, from the latter equation, have been eliminated the second derivatives of u and s, by means of their values obtained from equations (11). From these is obtained

$$D^2\Omega + \frac{[D\Omega]^2}{DuDs} = \frac{d^2\Omega}{du^2}Du^2 + 2\frac{d^2\Omega}{duds}DuDs + \frac{d^2\Omega}{ds^2}Ds^2 + \frac{\Delta^2}{DuDs} - \mathbf{m}^2DuDs,$$

on substitution of which in the value of θ, there results

$$(16)\qquad \theta = \frac{1}{DuDs}\left[\frac{d^2\Omega}{du^2}Du^2 - 2\frac{d^2\Omega}{duds}DuDs + \frac{d^2\Omega}{ds^2}Ds^2\right] + 3\left(\frac{\Delta}{DuDs}\right)^2 + \mathbf{m}^2.$$

The partial derivatives of Ω, involved in this expression, have the values

$$\frac{d\Omega}{du} = -\frac{1}{2}\frac{\varkappa}{r^3}s + \frac{3}{4}\mathbf{m}^2(u+s),$$

$$\frac{d\Omega}{ds} = -\frac{1}{2}\frac{\varkappa}{r^3}u + \frac{3}{4}\mathbf{m}^2(u+s),$$

$$\frac{d^2\Omega}{du^2} = \frac{3}{4}\frac{\varkappa}{r^5}s^2 + \frac{3}{4}\mathbf{m}^2,$$

$$\frac{d^2\Omega}{duds} = \frac{1}{4}\frac{\varkappa}{r^3} + \frac{3}{4}\mathbf{m}^2,$$

$$\frac{d^2\Omega}{ds^2} = \frac{3}{4}\frac{\varkappa}{r^5}u^2 + \frac{3}{4}\mathbf{m}^2,$$

where, for us, has been written r^2, the square of the moon's radius vector. After the substitution of these, it will be found that we can write

$$(17) \qquad \theta = \frac{\varkappa}{r^3} + \frac{3}{8}\,\frac{\frac{\varkappa}{r^5}[uDs - sDu]^2 + \mathbf{m}^2(Du - Ds)^2}{C - \Omega} + \frac{3}{4}\left[\frac{\Delta}{C - \Omega}\right]^2 + \mathbf{m}^2,$$

in which

$$\Delta = \left[-\frac{1}{2}\frac{\varkappa}{r^3} + \frac{3}{4}\mathbf{m}^2\right][uDs - sDu] - \frac{3}{4}\mathbf{m}^2(uDu - sDs) + 2\mathbf{m}(C - \Omega).$$

This expression for θ, from which all derivatives of u and s, higher than the first, have been eliminated whenever they presented themselves, is suitable for development in infinite series, when the method of special values is employed. The quadrant being divided into a certain number of equal parts with reference to τ, we compute the values of the four variables u, s, Du, and Ds of which θ is a function, for these special values of τ, and by substitution ascertain the corresponding values of θ. From the last, by the wellknown process, are derived the several coefficients of the periodic terms of θ. A discussion of the lunar inequalities, which are independent of every thing but the parameter $\mathbf{m}$, shows that the values of u and s have the form

$$u = \Sigma_i \mathbf{a}_i \zeta^{2i+1}, \qquad s = \Sigma_i \mathbf{a}_i \zeta^{-2i-1},$$

where i receives all integral values from $-\infty$ to $+\infty$, zero included, and the coefficients $\mathbf{a}_i$ are constant, being equivalent each to the same constant multiplied by a function of $\mathbf{m}$ which is of the $2i^{\text{th}}$ order with respect to this parameter.

By taking the derivatives

$$Du = \Sigma_i (2i + 1)\mathbf{a}_i \zeta^{2i+1}, \qquad Ds = -\Sigma_i (2i + 1)\mathbf{a}_i \zeta^{-2i-1}.$$

It will be seen from these equations that, in the terms where i is large, we will be subjected to the inconvenience of having the errors, with which the coefficients $\mathbf{a}_i$ are necessarily affected, multiplied by large numbers.

This will be avoided by employing, in the computation of θ, the formula

$$uDs - sDu = 2\mathbf{m}r^2 - \frac{3}{2}\mathbf{m}^2 D^{-1}(u^2 - s^2),$$

where D^{-1} denotes the inverse operation of D. This does not give the constant term of $uDs - sDu$, but this can be obtained from the expression

$$-2\Sigma_i(2i+1)\mathbf{a}_i^2,$$

which is not subject to the difficulty mentioned above. Wherever Du and Ds occur elsewhere in the formula for θ, they are multiplied by the small factor $\mathbf{m}^2$, and, in consequence, the given formulae suffice.

This mode of proceeding will give only a numerical result: if we wish to have $\mathbf{m}$ left indeterminate in the development of θ, it will be advantageous to give the latter another form. In this case there is no objection to the appearance of second and third derivatives of u and s in the expression of θ.

From the value of $D^2\Omega$, previously given, it is easy to conclude that

$$\frac{2}{DuDs}\left[\frac{d^2\Omega}{du^2}Du^2 + \frac{d^2\Omega}{ds^2}Ds^2\right]$$

$$= -4\frac{d^2\Omega}{duds} - 2\left(\frac{\Delta}{DuDs}\right)^2 + 2\mathbf{m}^2 - \frac{D^2(DuDs)}{DuDs} + \frac{1}{2}\left[\frac{D(DuDs)}{DuDs}\right]^2.$$

If this is substituted in the expression first given for θ, and we note that

$$4\frac{d^2\Omega}{duds} = \frac{\varkappa}{r^3} + 3\mathbf{m}^2,$$

$$\Delta = \frac{1}{2}[DuD^2s - DsD^2u] - \mathbf{m}DuDs,$$

the latter being obtained by substituting in the previously given value of Δ, the values of the partial derivatives of Ω given by equations (11), we get

$$(18) \qquad \theta = -\left[\frac{\varkappa}{r^3} + \mathbf{m}^2\right] + 2\left[\frac{1}{2}\left(\frac{D^2u}{Du} - \frac{D^2s}{Ds}\right) + \mathbf{m}\right]^2$$

$$-\left[\frac{1}{2}\left(\frac{D^2u}{Du} + \frac{D^2s}{Ds}\right)\right]^2 - D\left[\frac{1}{2}\left(\frac{D^2u}{Du} + \frac{D^2s}{Ds}\right)\right].$$

For the development of the first term of this expression, we can employ either of the following equations which result from equations (11),

$$\frac{\kappa}{r^3} + \mathbf{m}^2 = \frac{D^2 u + 2\mathbf{m}Du + \frac{3}{2}\mathbf{m}^2 s}{u} + \frac{5}{2}\mathbf{m}^2$$

$$= \frac{D^2 s - 2\mathbf{m}Ds + \frac{3}{2}\mathbf{m}^2 u}{s} + \frac{5}{2}\mathbf{m}^2,$$

which, if one studies symmetry of expression, may be written

$$\frac{\kappa}{r^3} + \mathbf{m}^2 = \left[\frac{Du}{u} + \mathbf{m}\right]^2 + D\left[\frac{Du}{u} + \mathbf{m}\right] + \frac{3}{2}\mathbf{m}^2\left[1 + \frac{s}{u}\right]$$

$$= \left[\frac{Ds}{s} - \mathbf{m}\right]^2 + D\left[\frac{Ds}{s} - \mathbf{m}\right] + \frac{3}{2}\mathbf{m}^2\left[1 + \frac{u}{s}\right];$$

and if half the sum of the second members is substituted for the first term in (18) we shall have a singularly symmetrical expression for θ.

If the values of u and s in terms of ζ are substituted in the first of these equations, we get

$$\frac{\kappa}{r^3} + \mathbf{m}^2 = 1 + 2\mathbf{m} + \frac{5}{2}\mathbf{m}^2 + \frac{\Sigma_i\left[4i(i + 1 + \mathbf{m})\mathbf{a}_i + \frac{3}{2}\mathbf{m}^2\mathbf{a}_{-i-1}\right]\zeta^{2i}}{\Sigma_i \mathbf{a}_i \zeta^{2i}}.$$

Let the last term of the second member of this equation be denoted by the series

$$\Sigma_i R_i \zeta^{2i};$$

since r is a series of cosines, we must have, in consequence of the equations of condition which the $\mathbf{a}_i$ satisfy, $R_{-i} = R_i$, and the equations, which determine these coefficients, can be obtained from the formula

$$\Sigma_j \mathbf{a}_{i-j} R_j = 4i(i + 1 + \mathbf{m})\mathbf{a}_i + \frac{3}{2}\mathbf{m}^2\mathbf{a}_{-i-1},$$

when we attribute to i, in succession, all integral values from $i = 0$ to $i = \infty$, or which is preferable, from $i = 0$ to $i = -\infty$. The following are all the equations and terms which need be retained when it is proposed to neglect quantities of the same order of smallness as $\mathbf{m}^{10}$;

$$\mathsf{a}_0 R_0 + (\mathsf{a}_1 + \mathsf{a}_{-1}) R_1 + (\mathsf{a}_2 + \mathsf{a}_{-2}) R_2 = \frac{3}{2}\mathsf{m}^2 \mathsf{a}_{-1},$$

$$\mathsf{a}_{-1} R_0 + (\mathsf{a}_0 + \mathsf{a}_{-2}) R_1 + \mathsf{a}_1 R_2 = -4\mathsf{m}\mathsf{a}_{-1} + \frac{3}{2}\mathsf{m}^2 \mathsf{a}_0,$$

$$\mathsf{a}_{-2} R_0 + (\mathsf{a}_{-1} + \mathsf{a}_{-3}) R_1 + \mathsf{a}_0 R_2 + \mathsf{a}_1 R_3 = 8(1-\mathsf{m})\mathsf{a}_{-2} + \frac{3}{2}\mathsf{m}^2 \mathsf{a}_1,$$

$$\mathsf{a}_{-2} R_1 + \mathsf{a}_{-1} R_2 + \mathsf{a}_0 R_3 = 12(2-\mathsf{m})\mathsf{a}_{-3} + \frac{3}{2}\mathsf{m}^2 \mathsf{a}_2,$$

$$\mathsf{a}_{-3} R_1 + \mathsf{a}_{-2} R_2 + \mathsf{a}_{-1} R_3 + \mathsf{a}_0 R_4 = 16(3-\mathsf{m})\mathsf{a}_{-4} + \frac{3}{2}\mathsf{m}^2 \mathsf{a}_3.$$

For the purpose of illustrating the present method, we content ourselves with giving the following approximate formula: —

$$\frac{\varkappa}{r^3} + \mathsf{m}^2$$

$$= 1 + 2\mathsf{m} + \frac{5}{2}\mathsf{m}^2 - \frac{3}{2}\mathsf{m}^2\mathsf{a}_1 + 4\mathsf{m}\mathsf{a}_{-1}(\mathsf{a}_1 + \mathsf{a}_{-1}) + \left[\frac{3}{2}\mathsf{m}^2 - 4\mathsf{m}\mathsf{a}_{-1}\right](\zeta^2 + \zeta^{-2})$$

$$+ \left[8(1-\mathsf{m})\mathsf{a}_{-2} + \frac{3}{2}\mathsf{m}^2(\mathsf{a}_1 - \mathsf{a}_{-1}) + 4\mathsf{m}\mathsf{a}_{-1}^2\right](\zeta^4 + \zeta^{-4}),$$

where, for convenience in writing, it has been assumed that $\mathsf{a}_0 = 1$, and consequently that a_i denotes here the ratio to a_0, which, as has been mentioned above, is a function of m. The absolute term and the coefficient of $\zeta^4 + \zeta^{-4}$ are affected with errors of the eighth order, while the coefficient of $\zeta^2 + \zeta^{-2}$ is affected with one of the sixth order.

We attend now to the remaining terms of θ. If we put

$$\frac{D^2 u}{Du} = \frac{\Sigma_i (2i+1)^2 \mathsf{a}_i \zeta^{2i}}{\Sigma_i (2i+1) \mathsf{a}_i \zeta^{2i}} = \Sigma_i U_i \zeta^{2i},$$

it is plain that we shall have

$$\frac{D^2 s}{Ds} = -\frac{\Sigma_i (2i+1)^2 \mathsf{a}_i \zeta^{-2i}}{\Sigma_i (2i+1) \mathsf{a}_i \zeta^{-2i}} = -\Sigma_i U_i \zeta^{-2i},$$

and in consequence,

$$\frac{1}{2}\left(\frac{D^2u}{Du} - \frac{D^2s}{Ds}\right) = \Sigma_i \frac{1}{2}(U_i + U_{-i})\zeta^{2i},$$

$$\frac{1}{2}\left(\frac{D^2u}{Du} + \frac{D^2s}{Ds}\right) = \Sigma_i \frac{1}{2}(U_i - U_{-i})\zeta^{2i}.$$

From this it will be seen that the development of $\frac{D^2u}{Du}$ will suffice for obtaining all the remaining terms of θ. Let us put

$$h_i = (2i + 1)\mathbf{a}_i.$$

The equations which determine the coefficients U_i are given by the formula

$$\Sigma_j h_{i-j} U_j = (2i + 1)h_i,$$

but, in order to exhibit some of their properties, I write a few, *in extenso,* thus:

$$(19)\quad \begin{cases} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ldots + h_0 U_{-2} + h_{-1} U_{-1} + h_{-2}(U_0 - 1) + h_{-3} U_1 + h_{-4} U_2 + \ldots = -4h_{-2}, \\ \ldots + h_1 U_{-2} + h_0\ U_{-1} + h_{-1}(U_0 - 1) + h_{-2} U_1 + h_{-3} U_2 + \ldots = -2h_{-1}, \\ \ldots + h_2 U_{-2} + h_1\ U_{-1} + h_0\ (U_0 - 1) + h_{-1} U_1 + h_{-2} U_2 + \ldots = \quad 0\quad, \\ \ldots + h_3 U_{-2} + h_2\ U_{-1} + h_1\ (U_0 - 1) + h_0\ U_1 + h_{-1} U_2 + \ldots = \quad 2h_1\ , \\ \ldots + h_4 U_{-2} + h_3\ U_{-1} + h_2\ (U_0 - 1) + h_1\ U_1 + h_0\ U_2 + \ldots = \quad 4h_2\ , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

When the subscripts of both the h and U in these equations are negatived, and the signs of the right-hand members reversed, the system of equations is the same as before. Hence, if we have found the value of U_i, which is a function of the h, the value of U_{-i} will be got from it by simply negativing the subscripts of all the h involved in it and reversing the sign of the whole expression. When this operation is applied to the particular unknown $U_0 - 1$, we get the condition

$$U_0 - 1 = -(U_0 - 1);$$

whence we have, rigorously,

$$U_0 = 1.$$

This result can also be established by the aid of a definite integral. The absolute term, in the development of $\frac{D^{\nu+1}u}{D^\nu u}$ in powers of ζ, is given by the definite integral

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{D^{\nu+1}u}{D^\nu u}\,d\tau = \frac{1}{2\pi\sqrt{-1}}\int_0^{2\pi} \frac{\frac{d^{\nu+1}u}{d\tau^{\nu+1}}}{\frac{d^\nu u}{d\tau^\nu}}\,d\tau.$$

The indefinite integral of the expression under the sign of integration is

$$\log\frac{d^\nu u}{d\tau^\nu} = \log\left[\frac{d^\nu x}{d\tau^\nu} + \frac{d^\nu y}{d\tau^\nu}\sqrt{-1}\right],$$

and if, for the moment, we take ρ and φ such that

$$\frac{d^\nu x}{d\tau^\nu} = \rho\cos\varphi, \qquad \frac{d^\nu y}{d\tau^\nu} = \rho\sin\varphi,$$

this integral takes the shape

$$\log\rho + \varphi\sqrt{-1}.$$

The first term of this has the same value for $\tau = 0$ and $\tau = 2\pi$, and consequently contributes nothing to the value of the definite integral. Thus we have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{D^{\nu+1}u}{D^\nu u}\,d\tau = \frac{1}{2\pi}\left[\varphi\right]_{\tau=0}^{\tau=2\pi}.$$

When $\tau = 0$, let φ be assumed between 0 and 2π: it will be found that φ has the value 0 or $\frac{\pi}{2}$ or π or $\frac{3}{2}\pi$ according as ν is of the form 4μ or $4\mu + 1$ or $4\mu + 2$ or $4\mu + 3$. Moreover, when τ augments, φ also augments, and when τ has passed over one circumference, φ has also augmented by a circumference. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{D^{\nu+1}u}{D^{\nu}u}\,d\tau = 1.$$

It follows, therefore, that ν denoting zero or a positive integer, the absolute term of the development of $\frac{D^{\nu+1}u}{D^{\nu}u}$ in integral powers of ζ is 1. And, in like manner, the absolute term of $\frac{D^{\nu+1}s}{D^{\nu}s}$ is -1.

Equations (19) are readily solved by successive approximations, and when terms of the tenth order are neglected, we can write

$$\begin{aligned}
\frac{D^2u}{Du} = 1 &+ 2[h_1 - h_{-1}h_2 + h_1 h_1 h_{-1}]\zeta^2 \\
&- 2[h_{-1} - h_1 h_{-2} + h_{-1}h_{-1}h_1]\zeta^{-2} \\
&+ 2[2h_2 - h_1 h_1 - 2h_{-1}h_3 + 4h_1 h_{-1}h_2 - 2h_1 h_1 h_1 h_{-1}]\zeta^4 \\
&- 2[2h_{-2} - h_{-1}h_{-1} - 2h_1 h_{-3} + 4h_{-1}h_1 h_{-2} - 2h_{-1}h_{-1}h_{-1}h_1]\zeta^{-4} \\
&+ 2[3h_3 - 3h_1 h_2 + h_1 h_1 h_1]\zeta^6 \\
&- 2[3h_{-3} - 3h_{-1}h_{-2} + h_{-1}h_{-1}h_{-1}]\zeta^{-6} \\
&+ 2[4h_4 - 4h_1 h_3 + 4h_1 h_1 h_2 - 2h_2 h_2 - h_1 h_1 h_1 h_1]\zeta^8 \\
&- 2[4h_{-4} - 4h_{-1}h_{-3} + 4h_{-1}h_{-1}h_{-2} - 2h_{-2}h_{-2} - h_{-1}h_{-1}h_{-1}h_{-1}]\zeta^{-8},
\end{aligned}$$

where we have supposed again that $h_0 = \mathbf{a}_0 = 1$.

With the same degree of approximation we have used for $\frac{x}{r^3} + \mathbf{m}^2$, θ can be written

$$\begin{aligned}
\theta = 1 &+ 2\mathbf{m} - \frac{1}{2}\mathbf{m}^2 + \frac{3}{2}\mathbf{m}^2\mathbf{a}_1 + 54\mathbf{a}_1^2 + (12 - 4\mathbf{m})\mathbf{a}_1\mathbf{a}_{-1} + (6 - 4\mathbf{m})\mathbf{a}_{-1}^2 \\
&+ \left[(6 + 12\mathbf{m})\mathbf{a}_1 + (6 + 8\mathbf{m})\mathbf{a}_{-1} - \frac{3}{2}\mathbf{m}^2\right](\zeta^2 + \zeta^{-2}) \\
&+ \Big[20\mathbf{m}\mathbf{a}_2 + (16 + 20\mathbf{m})\mathbf{a}_{-2} - (9 + 40\mathbf{m})\mathbf{a}_1^2 + 6\mathbf{a}_1\mathbf{a}_{-1} \\
&\qquad + (7 + 4\mathbf{m})\mathbf{a}_{-1}^2 - \frac{3}{2}\mathbf{m}^2(\mathbf{a}_1 - \mathbf{a}_{-1})\Big](\zeta^4 + \zeta^{-4}).
\end{aligned}$$

In the determination of the terms of the lunar coördinates which depend only on the parameter $\mathbf{m}$, it has been found[1] that, with errors of the sixth order,

$$\mathbf{a}_1 = \frac{3}{16}\frac{6+12\mathbf{m}+9\mathbf{m}^2}{6-4\mathbf{m}+\mathbf{m}^2}\mathbf{m}^2,$$

$$\mathbf{a}_{-1} = -\frac{3}{16}\frac{38+28\mathbf{m}+9\mathbf{m}^2}{6-4\mathbf{m}+\mathbf{m}^2}\mathbf{m}^2,$$

and, with errors of the eighth order,

$$\mathbf{a}_2 = \frac{27}{256}\frac{2+4\mathbf{m}+3\mathbf{m}^2}{[6-4\mathbf{m}+\mathbf{m}^2][30-4\mathbf{m}+\mathbf{m}^2]}\left[238+40\mathbf{m}+9\mathbf{m}^2-32\frac{29-35\mathbf{m}}{6-4\mathbf{m}+\mathbf{m}^2}\right]\mathbf{m}^4,$$

$$\mathbf{a}_{-2} = \frac{27}{64}\frac{2+4\mathbf{m}+3\mathbf{m}^2}{[6-4\mathbf{m}+\mathbf{m}^2][30-4\mathbf{m}+\mathbf{m}^2]}\left[-28-7\mathbf{m}+24\frac{7-\mathbf{m}}{6-4\mathbf{m}+\mathbf{m}^2}\right]\mathbf{m}^4.$$

No use will be made of these formulæ in the sequel of this memoir: they are given only that we may at need easily deduce an approximate literal expansion for the important function θ.

III.

In the preceding discussion it has been established that the determination of the lunar inequalities, which have the simple power of the eccentricity as factor, depends on the integration of the linear differential equation

$$D^2\mathbf{w} = \theta\mathbf{w};$$

to the treatment of which we accordingly proceed. We assume that the development of θ, in a series of the form

$$\theta = \Sigma_i \theta_i \zeta^{2i},$$

has been obtained. Here we have the condition $\theta_{-i} = \theta_i$. If θ_1, θ_2, etc.,

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[1] These expressions are established in another memoir. See American Journal of Mathematics, Vol. I, p. 138.

are, to a considerable degree, smaller than θ_0, an approximate statement of the equation is

$$D^2\mathbf{w} = \theta_0 \mathbf{w};$$

the complete integral of which is

$$\mathbf{w} = K\zeta^{\mathbf{c}} + K'\zeta^{-\mathbf{c}},$$

K and K' being the arbitrary constants and $\mathbf{c}$ being written for $\sqrt{\theta_0}$. When the additional terms of θ are considered, the effect is to modify this value of $\mathbf{c}$, and also to add to $\mathbf{w}$ new terms of the general form $A\zeta^{\pm\mathbf{c}+2i}$. It is plain, therefore, that we may suppose

$$\mathbf{w} = Kf(\zeta, \mathbf{c}) + K'f(\zeta, -\mathbf{c}),$$

and may take, as a particular integral

$$\mathbf{w} = \Sigma_i \mathbf{b}_i \zeta^{\mathbf{c}+2i},$$

$\mathbf{b}_i$ being a constant coefficient. If this equivalent of $\mathbf{w}$ is substituted in the differential equation, we get the equation

$$(20) \qquad [\mathbf{c} + 2j]^2 \mathbf{b}_j - \Sigma_i \theta_{j-i} \mathbf{b}_i = 0,$$

which holds for all integral values for j, positive and negative. These conditions determine the ratios of all the coefficients $\mathbf{b}_i$ to one of them, as $\mathbf{b}_0$, which may then be regarded as the arbitrary constant. They also determine $\mathbf{c}$, which is the ratio of the synodic to the anomalistic month. For the purpose of exhibiting more clearly the properties of the equations represented generally by (20), I write a few of them *in extenso:* for convenience let

$$[i] = (\mathbf{c} + 2i)^2 - \theta_0;$$

then

$$(21) \quad \begin{cases} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ldots + [-2]\mathbf{b}_{-2} - \theta_1\mathbf{b}_{-1} - \theta_2\mathbf{b}_0 - \theta_3\mathbf{b}_1 - \theta_4\mathbf{b}_2 - \ldots = 0, \\ \ldots - \theta_1\mathbf{b}_{-2} + [-1]\mathbf{b}_{-1} - \theta_1\mathbf{b}_0 - \theta_2\mathbf{b}_1 - \theta_3\mathbf{b}_2 - \ldots = 0, \\ \ldots - \theta_2\mathbf{b}_{-2} - \theta_1\mathbf{b}_{-1} + [0]\mathbf{b}_0 - \theta_1\mathbf{b}_1 - \theta_2\mathbf{b}_2 - \ldots = 0, \\ \ldots - \theta_3\mathbf{b}_{-2} - \theta_2\mathbf{b}_{-1} - \theta_1\mathbf{b}_0 + [1]\mathbf{b}_1 - \theta_1\mathbf{b}_2 - \ldots = 0, \\ \ldots - \theta_4\mathbf{b}_{-2} - \theta_3\mathbf{b}_{-1} - \theta_2\mathbf{b}_0 - \theta_1\mathbf{b}_1 + [2]\mathbf{b}_2 - \ldots = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

If, from this group of equations, infinite in number, and the number of terms in each equation also infinite, we eliminate all the $\mathbf{b}$ except one, we get a symmetrical determinant involving $\mathbf{c}$, which, equated to zero, determines this quantity. This equation we will denote thus: —

$$\mathfrak{D}(\mathbf{c}) = 0. \tag{22}$$

If, in (20), we put $-\mathbf{c}$ for $\mathbf{c}$, $-j$ for j, and suppose that $\mathbf{b}_j$ is now denoted by $\mathbf{b}_{-j}$, the equation is the same as at first; hence the determinant just mentioned remains unchanged, when for $\mathbf{c}$ in it we substitute $-\mathbf{c}$, and

$$\mathfrak{D}(-\mathbf{c}) = \mathfrak{D}(\mathbf{c}),$$

or, in other words, $\mathfrak{D}(\mathbf{c})$ is a function of $\mathbf{c}^2$. Again, in the same equation, let $\mathbf{c} + 2\nu$ be substituted for $\mathbf{c}$, ν being any positive or negative integer, and write $j - \nu$ for j, and suppose that $\mathbf{b}_j$ is now denoted by $\mathbf{b}_{j+\nu}$. The equation is again the same as at first, and hence the determinant suffers no change when $\mathbf{c} + 2\nu$ is written in it for $\mathbf{c}$. That is,

$$\mathfrak{D}(\mathbf{c} + 2\nu) = \mathfrak{D}(\mathbf{c}).$$

It follows from all this that if (22) is satisfied by a root $\mathbf{c} = \mathbf{c}_0$, it will also have, as roots, all the quantities contained in the expression

$$\pm\, \mathbf{c}_0 + 2i,$$

where i denotes any positive or negative integer or zero. And these are all the roots the equation admits; for each of the expressions denoted by $[i]$ is of two dimensions in $\mathbf{c}$, and may be regarded as introducing into the equation the two roots $2i + \mathbf{c}_0$ and $2i - \mathbf{c}_0$. Consequently the roots are either all real or all imaginary, and it is impossible that the equation should have any equal roots unless all the roots are integral. But in the last case the inequalities we treat would evidently coalesce with those having the argument of the variation, and could not be separated from them; hence this case may be set aside as practically not occurring.

It is evident from the foregoing remarks that, in an analytical point of view, it is indifferent which of the roots of (22) is taken as the value of $\mathbf{c}$; in every case we get the same value for $\mathbf{w}$. For denoting the mean anomaly of the moon by ξ, we have the infinite series of arguments

$$\ldots,\ \xi - 4\tau,\quad \xi - 2\tau,\quad \xi,\quad \xi + 2\tau,\quad \xi + 4\tau,\ \ldots$$

each of which can be made to play the same *rôle* as ξ, and analysis knows no distinction between them. Hence the equation, which determines the motion of ξ, must, of necessity, also give the motions of all the arguments of the series above, as well as of their negatives.[1] One has, however, been in the habit of taking for $\mathbf{c}$ the root which approximates to $\sqrt{\Theta_0}$.

It may be well to notice here the modifications which the addition to the investigation of terms of higher orders produces in equation (22). This may be written

$$\Pi(x \pm \mathbf{c}_0 + 2i) = 0,$$

where x is the unknown quantity and Π is a symbol denoting the product of the infinite number of factors obtained by attributing to i all integral values positive and negative, zero included, and taking in succession the ambiguous sign in both significations. Had the terms, involving higher powers of e, been included in the investigation, the equation would have been

$$\Pi(x + j\mathbf{c}_0 + 2i) = 0,$$

where j receives all integral values positive and negative. If, furthermore, we had included all terms involving the argument τ and its odd multiples, the equation would have been

$$\Pi(x + j\mathbf{c}_0 + i) = 0.$$

If to these we had added all terms depending on the solar eccentricity, the equation would have been

$$\Pi(x + j\mathbf{c}_0 + i + k\mathbf{m}) = 0,$$

where k is also to receive all integral values positive and negative.

[1] A similar condition of things occurs in many less complex problems; for instance, in the determination of the principal axes of rotation of a rigid body. Although there is but one set of such axes, yet the final equation, solving the question, is of the third degree, all because analysis knows no distinction between the axes of x, y and z.

A similar thing is true in the general planetary problem. Professor NEWCOMB says,[1] »The quantities **b**», where **b** is of similar signification with $\mathbf{c}_0$ above, »ought, perhaps, to appear as the roots of an equation of the $3n^{\text{th}}$ degree». But it is plain, from the foregoing remarks, that not only does this equation contain the $3n$ roots $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{3n}$, but also every root given by the general integral linear function of the **b**

$$i_1\mathbf{b}_1 + i_2\mathbf{b}_2 + \ldots + i_{3n}\mathbf{b}_{3n},$$

for which, in the analysis, the corresponding argument

$$i_1\lambda_1 + i_2\lambda_2 + \ldots + i_{3n}\lambda_{3n}$$

can play the same *rôle* as any one of the individual arguments λ. Hence this equation, in all cases but the problem of two bodies, must be regarded as transcendental or of infinite degree.

The equations which determine the coefficients $\mathbf{b}_i$ and the quantity $\mathbf{c}$, having the form of normal equations in the method of least squares, can be solved by the process usually adopted for the latter. Let two of these equations be written

$$[j]\mathbf{b}_j - \Sigma_i\,\theta_{j-i}\mathbf{b}_i = 0,$$

$$[\nu]\mathbf{b}_\nu - \Sigma_i\,\theta_{\nu-i}\mathbf{b}_i = 0,$$

where, in the first, the summation does not include the value $i = j$, or in the second the value $i = \nu$. The result of the elimination of $\mathbf{b}_\nu$ from these is

$$\left[[j] - \frac{\theta_{j-\nu}\,\theta_{j-\nu}}{[\nu]}\right]\mathbf{b}_j - \sum\nolimits_i\left[\theta_{j-i} + \frac{\theta_{j-\nu}\,\theta_{i-\nu}}{[\nu]}\right]\mathbf{b}_i = 0,$$

where, in the summation, i does not receive the values j and ν. This equation may be written

$$[j]^{(\nu)}\mathbf{b}_j - \Sigma_i\,\theta^{(\nu)}_{j-i}\mathbf{b}_i = 0.$$

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[1] *On the general integrals of planetary motion,* Smithsonian Contributions to Knowledge, No. 281, p. 31.

In like manner we may eliminate from the system of equations a second unknown $\mathbf{b}_{\nu'}$. And the general form of equation obtained may be written

$$[j]^{(\nu,\nu')}\mathbf{b}_j - \Sigma_i \theta_{j-i}^{(\nu,\nu')}\mathbf{b}_i = 0,$$

where, in the summation, i receives neither of the values j, ν and ν'. This process may be continued until all the $\mathbf{b}$, having sensible values but $\mathbf{b}_0$, are eliminated; and the single equation remaining, after division by $\mathbf{b}_0$, may be written

$$[0]^{(\ldots -2, -1, 1, 2, \ldots)} = 0.$$

This determines $\mathbf{c}$; when we pursue the method of numerical substitutions, it will be the most advantageous course to perform the preceding elimination twice, using two values for $\mathbf{c}$, slightly different, but each quite approximate. The last equation will then, in neither case, be exactly satisfied, but, by a comparison of the errors, one will discover the value of $\mathbf{c}$ which makes the left member sensibly zero. By a similar interpolation between the values of the $\mathbf{b}$, given severally by the first and second eliminations, we get the sensibly exact values of these quantities.

When it is proposed to neglect terms of the same order as $\mathbf{m}^6$, the equation for $\mathbf{c}$ may be written

$$[-1][0][1] - \theta_1^2\left[[-1] + [1]\right] = 0;$$

or, when we substitute for the symbols their significations,

$$[(\mathbf{c}^2 + 4 - \theta_0)^2 - 16\mathbf{c}^2][\mathbf{c}^2 - \theta_0] - 2\theta_1^2[\mathbf{c}^2 + 4 - \theta_0] = 0.$$

But, as $\mathbf{c}^2 - \theta_0$ is a quantity of the third order, we may neglect the cube of it in the first term, and the product of it by θ_1^2 in the second. Thus reduced, the equation becomes

$$[\mathbf{c}^2 - \theta_0]^2 + 2[\theta_0 - 1][\mathbf{c}^2 - \theta_0] + \theta_1^2 = 0;$$

whose solution gives

$$\mathbf{c} = \sqrt{1 + \sqrt{(\theta_0 - 1)^2 - \theta_1^2}}.$$

This is a remarkably simple expression for obtaining an approximate value of the motion of the lunar perigee. The actual numerical values of the two elements entering into this formula are

$$\theta_0 = 1.1588439, \qquad \theta_1 = -0.0570440.$$

θ_1 is therefore more than one third of $\theta_0 - 1$, which explains why such an erroneous value is obtained for the motion of the lunar perigee, when we neglect it and take $\mathbf{c} = \sqrt{\theta_0}$. The numbers being substituted in the formula, we get $\mathbf{c} = 1.0715632$; and as the ratio of the motion of the perigee to the sidereal mean motion of the moon is given by the equation

$$\frac{1}{n}\frac{d\omega}{dt} = 1 - \frac{\mathbf{c}}{1 + \mathbf{m}},$$

we get

$$\frac{1}{n}\frac{d\omega}{dt} = 0.008591.$$

This is about $\frac{1}{60}$ in excess of the value 0.008452 given by observation. The difference is caused, in the main, by our neglect of the inclination of the lunar orbit. The solar force is less effective in producing motion in the perigee than it would be if the moon moved in the plane of the ecliptic.

It will occur immediately to every one that the properties we have stated of the roots of $\mathfrak{D}(\mathbf{c}) = 0$ are precisely those of the transcendental equation

$$\cos(\pi x) - a = 0;$$

of which, if x_0 is one of the roots, the whole series of roots is represented by

$$\pm x_0 + 2i.$$

Hence we must necessarily have, identically,

$$\mathfrak{D}(\mathbf{c}) = A[\cos(\pi\mathbf{c}) - \cos(\pi\mathbf{c}_0)],$$

A being some constant independent of $\mathbf{c}$. As is the general custom, we assume that the positive sign is given to the element of the determinant formed by the product of the diagonal line of constituents containing $\mathbf{c}$. When, therefore, the determinant $\mathfrak{D}(\mathbf{c})$ is developed in powers of $\mathbf{c}$, using only a finite number of constituents in it, the coefficient of the highest power of $\mathbf{c}$ in it is always positive unity: hence we may assume that this is the value of the coefficient when the number of constituents is increased without limit. But from the well-known equation

$$\cos(\pi\mathbf{c}) = \left(1 - \frac{4}{1}\mathbf{c}^2\right)\left(1 - \frac{4}{9}\mathbf{c}^2\right)\left(1 - \frac{4}{25}\mathbf{c}^2\right)\ldots,$$

we gather that the coefficient of the highest power of $\mathbf{c}$, in the development of $\cos(\pi\mathbf{c})$ in powers of $\mathbf{c}$, may be regarded as represented by the infinite product

$$-\frac{4}{1}\cdot-\frac{4}{9}\cdot-\frac{4}{25}\cdot\ldots$$

If then the row of constituents of $\mathfrak{D}(\mathbf{c})$, containing $[0]$, is multiplied by -4, the rows containing $[-1]$ and $[1]$ by $\frac{4}{4^2-1}$, the rows containing $[-2]$ and $[2]$ by $\frac{4}{8^2-1}$, and, in general, the row containing $[i]$ by $\frac{4}{(4i)^2-1}$, we shall have the constituents of a second determinant, which may be designated as $\nabla(\mathbf{c})$, And the equation

$$\nabla(\mathbf{c}) = 0,$$

having the same roots as $\mathfrak{D}(\mathbf{c}) = 0$, will serve our purposes as well as the latter. We evidently now have

$$\nabla(\mathbf{c}) = \cos(\pi\mathbf{c}) - \cos(\pi\mathbf{c}_0).$$

As this is an identical equation, it holds when any special value is attributed to $\mathbf{c}$, and we are thus furnished with an elegant method of obtaining the value of the absolute term of the equation $\cos(\pi\mathbf{c}_0)$. For example, substituting for $\mathbf{c}$, in succession, the values 0, $\frac{1}{2}$, 1, $\sqrt{\theta_0}$, we have our choice between the values

$$\begin{aligned}\cos(\pi\mathbf{c}_0) &= 1 - \nabla(0)\\ &= -\nabla\left(\frac{1}{2}\right)\\ &= -1 - \nabla(1)\\ &= \cos(\pi\sqrt{\theta_0}) - \nabla(\sqrt{\theta_0}).\end{aligned}$$

As the determinant $\nabla(\mathrm{o})$ appears the simplest, we retain the first expression. Then, dropping the now useless subscript $(_0)$, the equation which determines $\mathbf{c}$ may be written

$$\cos(\pi\mathbf{c}) = 1 - \nabla(\mathrm{o}).$$

This is certainly a remarkable equation: it virtually amounts to a general solution of the equation $\mathfrak{D}(\mathbf{c}) = 0$. It also affords us immediately the criterion for the reality of the roots of the latter. Using the phrase of CAUCHY, if the modulus of the quantity $1 - \nabla(\mathrm{o})$ does not exceed unity, the roots are all real; in the contrary case, they are all imaginary. The criterion for deciding whether the variable $\mathbf{w}$ is always contained between definite limits, or is capable of increasing or diminishing beyond every limit, is the same. In the first case, it is developable in a series of circular cosines; in the second, in a series of potential cosines.

As, in the particular case, where Θ_1, Θ_2, &c., all vanish, the proper value of $\mathbf{c}$ is $\sqrt{\Theta_0}$, it follows that the element of the determinant $\nabla(\mathrm{o})$, formed by the product of the diagonal line of constituents involving Θ_0, is

$$1 - \cos(\pi\sqrt{\Theta_0}) = 2\sin^2\left(\frac{\pi}{2}\sqrt{\Theta_0}\right).$$

If therefore each row of constituents of the determinant $\nabla(\mathrm{o})$ is divided by the constituent of it which lies in the just-mentioned diagonal line, we shall have a set of constituents forming a third determinant $\square(\mathrm{o})$, such that

$$\nabla(\mathrm{o}) = 2\sin^2\left(\frac{\pi}{2}\sqrt{\Theta_0}\right).\square(\mathrm{o}).$$

In consequence the equation, determining $\mathbf{c}$, can be put in the form

$$\frac{\sin^2\left(\frac{\pi}{2}\mathbf{c}\right)}{\sin^2\left(\frac{\pi}{2}\sqrt{\Theta_0}\right)} = \square(\mathrm{o}).$$

For the sake of exhibiting more clearly the significance of this equation, I write a few of the central constituents of the determinant $\square(0)$, from which the rest can be easily inferred.

$$\square(0) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & +1 & -\frac{\theta_1}{4^2-\theta_0} & -\frac{\theta_2}{4^2-\theta_0} & -\frac{\theta_3}{4^2-\theta_0} & -\frac{\theta_4}{4^2-\theta_0} & \cdots \\ \cdots & -\frac{\theta_1}{2^2-\theta_0} & +1 & -\frac{\theta_1}{2^2-\theta_0} & -\frac{\theta_2}{2^2-\theta_0} & -\frac{\theta_3}{2^2-\theta_0} & \cdots \\ \cdots & -\frac{\theta_2}{0^2-\theta_0} & -\frac{\theta_1}{0^2-\theta_0} & +1 & -\frac{\theta_1}{0^2-\theta_0} & -\frac{\theta_2}{0^2-\theta_0} & \cdots \\ \cdots & -\frac{\theta_3}{2^2-\theta_0} & -\frac{\theta_2}{2^2-\theta_0} & -\frac{\theta_1}{2^2-\theta_0} & +1 & -\frac{\theta_1}{2^2-\theta_0} & \cdots \\ \cdots & -\frac{\theta_4}{4^2-\theta_0} & -\frac{\theta_3}{4^2-\theta_0} & -\frac{\theta_2}{4^2-\theta_0} & -\frac{\theta_1}{4^2-\theta_0} & +1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

The question of the convergence, so to speak, of a determinant, consisting of an infinite number of constituents, has nowhere, so far as I am aware, been discussed. All such determinants must be regarded as having a central constituent; when, in computing in succession the determinants formed from the 3^2, 5^2, 7^2 &c., constituents symmetrically situated with respect to the central constituent, we approach, without limit, a determinate magnitude, the determinant may be called convergent, and the determinate magnitude is its value.

In the present case, there can scarcely be a doubt that, as long as the series $\Sigma_i \theta_i \zeta^{2i}$ is a legitimate expansion of θ, the determinant $\nabla(0)$ must be regarded as convergent.

We will give another equation for determining **c**. We have

$$\cos(\pi \mathbf{c}) = \cos(\pi\sqrt{\theta_0}) - \nabla(\sqrt{\theta_0}).$$

The diagonal line of constituents in $\nabla(\sqrt{\theta_0})$ is represented in general by the formula

$$\frac{16i(i+\sqrt{\theta_0})}{(4i)^2-1};$$

and when the factor corresponding to $i = 0$ is omitted, the product

$$\prod_{i=-\infty}^{i=+\infty} \frac{16i(i+\sqrt{\theta_0})}{(4i)^2-1} = \frac{\pi \sin(\pi\sqrt{\theta_0})}{8\sqrt{\theta_0}}.$$

Consequently if we put

$$\square(\sqrt{\theta_0}) = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots + 1 & -\frac{\theta_1}{8(2-\sqrt{\theta_0})} & -\frac{\theta_2}{8(2-\sqrt{\theta_0})} & -\frac{\theta_3}{8(2-\sqrt{\theta_0})} & -\frac{\theta_4}{8(2-\sqrt{\theta_0})} & \cdots \\ \cdots -\frac{\theta_1}{4(1-\sqrt{\theta_0})} & +1 & -\frac{\theta_1}{4(1-\sqrt{\theta_0})} & -\frac{\theta_2}{4(1-\sqrt{\theta_0})} & -\frac{\theta_3}{4(1-\sqrt{\theta_0})} & \cdots \\ \cdots + \theta_2 & +\theta_1 & +0 & +\theta_1 & +\theta_2 & \cdots \\ \cdots -\frac{\theta_3}{4(1+\sqrt{\theta_0})} & -\frac{\theta_2}{4(1+\sqrt{\theta_0})} & -\frac{\theta_1}{4(1+\sqrt{\theta_0})} & +1 & -\frac{\theta_1}{4(1+\sqrt{\theta_0})} & \cdots \\ \cdots -\frac{\theta_4}{8(2+\sqrt{\theta_0})} & -\frac{\theta_3}{8(2+\sqrt{\theta_0})} & -\frac{\theta_2}{8(2+\sqrt{\theta_0})} & -\frac{\theta_1}{8(2+\sqrt{\theta_0})} & +1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

a determinant, which, having 0 for its central constituent, presents some facilities in its computation, we shall have, for determining **c**, the equation

$$\frac{\sin^2\left(\frac{\pi}{2}\mathbf{c}\right)}{\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)} = 1 + \frac{\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{2\sqrt{\theta_0}} \square(\sqrt{\theta_0}).$$

In the lunar theory θ_i is a quantity of the $2i^{\text{th}}$ order, and $1-\sqrt{\theta_0}$ a quantity of the first order; hence it is clear that, if we are willing to admit an error of the seventh order in **c**, the determinant

$$\square(\sqrt{\theta_0}) = -\frac{1}{2}\frac{\theta_1^2}{\theta_0 - 1}.$$

If, neglecting then quantities of the seventh order, we put

$$\frac{\pi\theta_1^2}{8\sqrt{\theta_0}(\theta_0-1)} = \operatorname{tg}\theta,$$

θ will be a small angle, and **c** will result from the equation

$$\sin\left(\frac{\pi}{2}\,\mathbf{c}\right) = \frac{\sin\left(\frac{\pi}{2}\sqrt{\Theta_0} - \theta\right)}{\cos\theta}\,.$$

This formula, although it involves the same coefficients Θ_0 and Θ_1 as the approximate formula previously given, is two orders more exact. A greater degree of approximation can be arrived at only by including the additional coefficient Θ_2. Employing the numerical values already attributed to Θ_0 and Θ_1, we find

$$\theta = 25'\,41''.395, \qquad \mathbf{c} = 1.0715815, \qquad \frac{1}{n}\frac{d\omega}{dt} = 0.008574.$$

It is better, however, to employ the equations

$$\frac{\pi\Theta_1^2}{4\sqrt{\Theta_0(\Theta_0 - 1)}} = \text{tg}\,\theta, \qquad \cos(\pi\mathbf{c}) = \frac{\cos(\pi\sqrt{\Theta_0} - \theta)}{\cos\theta}\,,$$

which give

$$\theta = 51'\,22''.6185, \qquad \mathbf{c} = 1.07158837865, \qquad \frac{1}{n}\frac{d\omega}{dt} = 0.0085721020.$$

The determinants $\square(0)$ and $\square(\sqrt{\Theta_0})$ can be replaced by infinite series proceeding according to ascending powers and products of the coefficients Θ_1, Θ_2, &c.

Let us take the first, as being in more respects the simpler. It is plain that the element of the determinant formed by the product of the diagonal line of constituents is the only term of the zero order in it. Then one exchange always produces terms of the 4th or higher orders, two exchanges terms of the 8th or higher orders, three exchanges terms of the 12th or higher orders, and so on. Now let i, i', i'', ... be positive or negative integers, of which no two are identical, written in the order of their algebraical magnitude, and let $\{i\}$ stand for $(2i)^2 - \Theta_0$. Then all the terms of $\square(0)$, which are obtained by 0, 1, 2, and 3 exchanges, are contained in the following expression, which is, consequently affected with an error of the 16th order,

$$
\begin{aligned}
\square(0) = 1 \; - & \sum_{i,\, i'} \frac{\theta^2_{i'-i}}{\{i\}\{i'\}} \\
+ & \sum_{i,\, i',\, i'',\, i'''} \frac{\theta^2_{i'-i}\,\theta^2_{i'''-i''}}{\{i\}\{i'\}\{i''\}\{i'''\}} \\
- & \, 2 \sum_{i,\, i',\, i''} \frac{\theta_{i'-i}\,\theta_{i''-i'}\,\theta_{i''-i}}{\{i\}\{i'\}\{i''\}} \\
- & \sum_{i,\, i',\, i'',\, i''',\, i^{\mathrm{IV}},\, i^{\mathrm{V}}} \frac{\theta^2_{i'-i}\,\theta^2_{i'''-i''}\,\theta^2_{i^{\mathrm{V}}-i^{\mathrm{IV}}}}{\{i\}\{i'\}\{i''\}\{i'''\}\{i^{\mathrm{IV}}\}\{i^{\mathrm{V}}\}} \\
+ & \, 2 \sum_{i,\, i',\, i'',\, i''',\, i^{\mathrm{IV}}} \frac{\left\{\begin{array}{l} \theta_{i'-i}\,\theta^2_{i'''-i''}\,\theta_{i^{\mathrm{IV}}-i}\,\theta_{i^{\mathrm{IV}}-i'} \\ + \; \theta^2_{i'-i}\,\theta_{i'''-i''}\,\theta_{i^{\mathrm{IV}}-i''}\,\theta_{i^{\mathrm{IV}}-i'''} \end{array}\right\}}{\{i\}\{i'\}\{i''\}\{i'''\}\{i^{\mathrm{IV}}\}} \\
- & \, 2 \sum_{i,\, i',\, i'',\, i'''} \frac{\left\{\begin{array}{l} \theta_{i'-i}\,\theta_{i''-i'}\,\theta_{i'''-i''}\,\theta_{i'''-i} \\ + \; \theta_{i''-i}\,\theta_{i''-i'}\,\theta_{i'''-i'}\,\theta_{i'''-i} \\ + \; \theta_{i'-i}\,\theta_{i''-i}\,\theta_{i'''-i'}\,\theta_{i'''-i''} \end{array}\right\}}{\{i\}\{i'\}\{i''\}\{i'''\}} .
\end{aligned}
$$

Particularizing the summations in this expression, and retaining only terms which are of lower orders than the 16^{th}, we get

$$
(23) \quad \left\{
\begin{aligned}
\square(0) = 1 \; - & \; \theta_1^2 \sum_i \frac{1}{\{i\}\{i+1\}} - \theta_2^2 \sum_i \frac{1}{\{i\}\{i+2\}} - \theta_3^2 \sum_i \frac{1}{\{i\}\{i+3\}} \\
+ & \; \theta_1^4 \sum_{i,\, i'} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}} \\
+ & \; \theta_1^2 \theta_2^2 \sum_{i,\, i'} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+2\}} \\
- & \; 2\,\theta_1^2 \theta_2 \sum_i \frac{1}{\{i\}\{i+1\}\{i+2\}} \\
- & \; 2\,\theta_1 \theta_2 \theta_3 \sum_i \frac{1}{\{i\}\{i+1\}\{i+3\}} \\
- & \; 2\,\theta_1 \theta_2 \theta_3 \sum_i \frac{1}{\{i\}\{i+2\}\{i+3\}} \\
- & \; \theta_1^6 \sum_{i,\, i',\, i''} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}\{i''\}\{i''+1\}} \\
+ & \; 2\,\theta_1^4 \theta_2 \sum_{i,\, i'} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}\{i'+2\}} \\
- & \; 2\,[\theta_1^2 \theta_2^2 + \theta_1^3 \theta_3] \sum_i \frac{1}{\{i\}\{i+1\}\{i+2\}\{i+3\}} .
\end{aligned}
\right.
$$

The functions of θ_0, which are represented by the summations, can all be replaced by finite expressions. For brevity, let us put $\theta_0 = 4\theta^2$, then resolving the expression into partial fractions, i being taken as the variable, we have, for instance,

$$\sum_i \frac{1}{\{i\}\{i+k\}} = \frac{1}{16}\sum_i \frac{1}{(\theta+i)(\theta-i)(\theta+i+k)(\theta-i-k)}$$

$$= \frac{1}{16}\sum_i \left[\frac{A}{\theta+i} + \frac{B}{\theta-i} + \frac{C}{\theta+i+k} + \frac{D}{\theta-i-k}\right].$$

where A, B, C and D are determined by the equations

$$2k\theta(2\theta-k)A = 1, \qquad -2k\theta(2\theta+k)B = 1,$$

$$-2k\theta(2\theta+k)C = 1, \qquad 2k\theta(2\theta-k)D = 1.$$

But, as is well known,

$$\sum_i \frac{1}{\theta+i} = \sum_i \frac{1}{\theta-i} = \sum_i \frac{1}{\theta+i+k} = \sum_i \frac{1}{\theta-i-k} = \pi \operatorname{cotg} \pi\theta.$$

Consequently

$$\sum_i \frac{1}{\{i\}\{i+k\}} = \frac{1}{16}(A+B+C+D)\pi \operatorname{cotg} \pi\theta$$

$$= \frac{\pi \operatorname{cotg} \pi\theta}{8\theta(4\theta^2-k^2)}$$

$$= \frac{\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}(\theta_0-k^2)}.$$

In like manner will be found

$$\sum_i \frac{1}{\{i\}\{i+k\}\{i+k'\}}$$

$$= -\frac{1}{16}\,\frac{3\theta_0-(k^2-kk'+k'^2)}{\sqrt{\theta_0}(\theta_0-k^2)(\theta_0-k'^2)[\theta_0-(k-k')^2]}\,\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right),$$

$$\sum_i \frac{1}{\{i\}\{i+1\}\{i+k\}\{i+k+1\}}$$

$$= \frac{1}{32}\,\frac{5\theta_0-(k^2+1)}{\sqrt{\theta_0}(\theta_0-1)(\theta_0-k^2)[\theta_0-(k+1)^2][\theta_0-(k-1)^2]}\,\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right).$$

By attributing, in these equations, special integral values to k, will be obtained the values of all the single summations appearing in the preceding expression for $\square(0)$. With regard to the double summations, we may proceed as follows: substitute $i + k$ for i', then resolve the expression under consideration into partial fractions with respect to i as variable, and sum between the limits $-\infty$ and $+\infty$; the fractions occurring in the result thus obtained are next resolved into partial fractions with reference to k, and the summations, with reference to this integer, are taken between the limits 2 and $+\infty$; or which is the same thing, between the limits 0 and $+\infty$, and the terms corresponding to $k = 0$ and $k = 1$ subtracted from the result. The single triple summation may be treated in an analogous manner. Thus we get

$$\sum_{i,i'} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}}$$

$$= \frac{\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{32\sqrt{\theta_0}(1-\theta_0)^2}\left[\frac{\pi \operatorname{cotg}(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}} - \frac{1}{\theta_0} + \frac{2}{1-\theta_0} + \frac{9}{2(4-\theta_0)}\right],$$

$$\sum_{i,i'} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+2\}}$$

$$= \frac{\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{16\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)}\left[\frac{\pi \operatorname{cotg}(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}} - \frac{1}{\theta_0} + \frac{2}{1-\theta_0} + \frac{2}{4-\theta_0} + \frac{5}{9-\theta_0}\right],$$

$$\sum_{i,i'} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}\{i'+2\}}$$

$$= \frac{3\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{64\sqrt{\theta_0}(1-\theta_0)^2(4-\theta_0)}\left[\frac{\pi \operatorname{cotg}(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}} - \frac{1}{\theta_0} + \frac{2}{1-\theta_0} + \frac{2}{4-\theta_0} + \frac{20}{3(9-\theta_0)}\right],$$

$$\sum_{i,i',i''} \frac{1}{\{i\}\{i+1\}\{i'\}\{i'+1\}\{i''\}\{i''+1\}}$$

$$= -\frac{\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{128\sqrt{\theta_0}(1-\theta_0)^3}\left\{\left[-\frac{1}{\theta_0} + \frac{2}{1-\theta_0} + \frac{9}{2(4-\theta_0)}\right]\frac{\pi \operatorname{cotg}(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}\right.$$

$$\left. -\frac{25}{8\theta_0} + \frac{1}{\theta_0^2} + \frac{2}{1-\theta_0} + \frac{4}{(1-\theta_0)^2} - \frac{9}{8(4-\theta_0)} + \frac{9}{(4-\theta_0)^2} - \frac{4}{9-\theta_0} - \frac{\pi^2}{3\theta_0}\right\}.$$

From which it follows that

$$
(24)\quad
\begin{aligned}
\square(0) = 1 &+ \frac{\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}}\left[\frac{\theta_1^2}{1-\theta_0}+\frac{\theta_2^2}{4-\theta_0}+\frac{\theta_3^2}{9-\theta_0}\right] \\
&+ \frac{\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{32\sqrt{\theta_0}(1-\theta_0)^2}\left[\frac{\pi \operatorname{cotg}(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{9}{2(4-\theta_0)}\right]\theta_1^4 \\
&+ \frac{3\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{8\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)}\,\theta_1^2\theta_2 \\
&+ \frac{\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{128\sqrt{\theta_0}(1-\theta_0)^3}\left\{\left[-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{9}{2(4-\theta_0)}\right]\frac{\pi \operatorname{cotg}(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}\right. \\
&\qquad -\frac{25}{8\theta_0}+\frac{1}{\theta_0^2}+\frac{2}{1-\theta_0}+\frac{4}{(1-\theta_0)^2}-\frac{9}{8(4-\theta_0)} \\
&\qquad \left.+\frac{9}{(4-\theta_0)^2}-\frac{4}{9-\theta_0}-\frac{\pi^2}{3\theta_0}\right\}\theta_1^6 \\
&+ \frac{3\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{32\sqrt{\theta_0}(1-\theta_0)^2(4-\theta_0)}\left[\frac{\pi \operatorname{cotg}(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}\right. \\
&\qquad \left.+\frac{2}{4-\theta_0}+\frac{20}{3(9-\theta_0)}\right]\theta_1^4\theta_2 \\
&+ \frac{\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{16\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)}\left[\frac{\pi \operatorname{cotg}(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}\right. \\
&\qquad \left.+\frac{2}{4-\theta_0}+\frac{10}{9-\theta_0}\right]\theta_1^2\theta_2^2 \\
&+ \frac{(7-3\theta_0)\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)(9-\theta_0)}\,\theta_1^2\theta_2\theta_3 \\
&+ \frac{5\pi \operatorname{cotg}\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{16\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)(9-\theta_0)}\,\theta_1^3\theta_3 .
\end{aligned}
$$

This is the same result as would be obtained if, setting out with the equation $\mathfrak{D}(\mathbf{c}) = 0$, and assuming that $\mathbf{c} = \sqrt{\theta_0}$ is an approximate value, we should expand the function $\sin^2\left(\frac{\pi}{2}\mathbf{c}\right)$ in ascending powers and products of the coefficients θ_1, θ_2, &c.

IV.

In order to obtain a numerical result from the preceding investigation, we assume

$$n = 17325594''.06085, \qquad n' = 1295977''.41516$$

whence

$$\mathbf{m} = 0.08084\,89338\,08311.6.$$

From an investigation[1] of the corresponding values of the $\mathbf{a}_i$, we have

$$\begin{aligned}
2h_1 &= +\,0.00909\,42448\,77375.5 \\
4h_2 &= +\,0.00011\,75731\,31569.1 \\
6h_3 &= +\,0.00000\,12613\,28523.8 \\
8h_4 &= +\,0.00000\,00126\,19314.9 \\
10h_5 &= +\,0.00000\,00001\,21722.9 \\
12h_6 &= +\,0.00000\,00000\,01147.9 \\
14h_7 &= +\,0.00000\,00000\,00010.6
\end{aligned}$$

$$\begin{aligned}
-\,2h_{-1} &= -\,0.01739\,14939\,23079.4 \\
-\,4h_{-2} &= +\,0.00000\,19654\,85829.2 \\
-\,6h_{-3} &= +\,0.00000\,00738\,11780.8 \\
-\,8h_{-4} &= +\,0.00000\,00006\,87885.7 \\
-\,10h_{-5} &= +\,0.00000\,00000\,05777.1 \\
-\,12h_{-6} &= +\,0.00000\,00000\,00047.5 \\
-\,14h_{-7} &= +\,0.00000\,00000\,00000.4
\end{aligned}$$

[1] See American Journal of Mathematics, Vol. I, p. 247.

The values of the U_i derived from these are

$U_1 = + 0.00909\ 40932\ 76038.2$	$U_{-1} = - 0.01739\ 21860\ 78260.6$
$U_2 = + 0.00007\ 62192\ 02104.5$	$U_{-2} = + 0.00015\ 32094\ 08075.6$
$U_3 = + 0.00000\ 06474\ 24628.8$	$U_{-3} = - 0.00000\ 12670\ 56302.6$
$U_4 = + 0.00000\ 00055\ 23086.8$	$U_{-4} = + 0.00000\ 00115\ 67648.9$
$U_5 = + 0.00000\ 00000\ 47209.0$	$U_{-5} = - 0.00000\ 00000\ 95049.5$
$U_6 = + 0.00000\ 00000\ 00403.9$	$U_{-6} = + 0.00000\ 00000\ 00867.3$
$U_7 = + 0.00000\ 00000\ 00003.4$	$U_{-7} = - 0.00000\ 00000\ 00007.2$

In combination with the values of the R_i, which has been given elsewhere,[1] these afford the following periodic series for θ: —

$$
\begin{aligned}
\theta = \quad & 1.15884\ 39395\ 96583 \\
& - 0.11408\ 80374\ 93807 \cos 2\tau \\
& + 0.00076\ 64759\ 95109 \cos 4\tau \\
& - 0.00001\ 83465\ 77790 \cos 6\tau \\
& + 0.00000\ 01088\ 95009 \cos 8\tau \\
& - 0.00000\ 00020\ 98671 \cos 10\tau \\
& + 0.00000\ 00000\ 12103 \cos 12\tau \\
& - 0.00000\ 00000\ 00211 \cos 14\tau
\end{aligned}
$$

The values of the coefficients θ_0, θ_1, θ_2, &c., are the halves of these coefficients, except θ_0 which is equal to the first coefficient.

On substituting the numerical values of these quantities in (24), and separating the sum of the terms into groups according to their order for the sake of exhibiting the degree of convergence, we get

Term of the zero order,	$1.00000\ 00000\ 00000\ 0$
Term of the 4^{th} order,	$+ 0.00180\ 46110\ 93422\ 7$
Sum of the terms of the 8^{th} order,	$+ 0.00000\ 01808\ 63109\ 9$
Sum of the terms of the 12^{th} order,	$+ 0.00000\ 00000\ 64478\ 6$
	$\square(0) = 1.00180\ 47920\ 21011\ 2$

As far as we can judge from induction, the value of $\square(0)$ would be affected, only in the 14^{th} decimal, by the neglected remainder of the

[1] See American Journal of Mathematics, Vol. I, p. 249.

series, which is of the 16^{th} order. An error in $\square(0)$ is multiplied by 2.8 nearly in $\mathbf{c}$.

The value, which is derived thence for $\mathbf{c}$, is

$$\mathbf{c} = 1.07158\ 32774\ 16016.$$

In order that nothing may be wanting in the exact determination of this quantity, we will employ the value just obtained as an approximate value in the elimination between equations (21). The coefficients $[i]$, as many of them as we have need for, have the following values: —

$$[-4] = 46.8, \qquad [1] = 8.27577\ 98905\ 1,$$
$$[-3] = 23.13045, \qquad [2] = 24.56211\ 3,$$
$$[-2] = 7.41678\ 05615\ 1, \qquad [3] = 48.85.$$
$$[-1] = -0.29688\ 63288\ 2300,$$

If the quantities $\mathbf{b}_i$ are eliminated from equations (21) in the order $\mathbf{b}_{-1}$, $\mathbf{b}_1$, $\mathbf{b}_{-2}$, $\mathbf{b}_2$, $\mathbf{b}_{-3}$, $\mathbf{b}_3$ and $\mathbf{b}_{-4}$, it will be found that the coefficient of $\mathbf{b}_0$, in the principal equation, undergoes the following successive depressions,

$$
\begin{aligned}
&[0] &&= -0.01055\ 32191\ 58933,\\
&[0]^{(-1)} &&= +0.00040\ 72723\ 11650,\\
&[0]^{(-1,1)} &&= +0.00001\ 50888\ 08423,\\
&[0]^{(-2,-1,1)} &&= +0.00000\ 00253\ 21700,\\
&[0]^{(-2,-1,1,2)} &&= +0.00000\ 00009\ 20420,\\
&[0]^{(-3,-2,-1,1,2)} &&= +0.00000\ 00000\ 03941,\\
&[0]^{(-3,-2,-1,1,2,3)} &&= +0.00000\ 00000\ 00155,\\
&[0]^{(-4,-3,-2,-1,1,2,3)} &&= +0.00000\ 00000\ 00008.
\end{aligned}
$$

The last number is not sensibly changed by the elimination of any of the $\mathbf{b}_i$, beyond $\mathbf{b}_{-4}$ on the one side, or $\mathbf{b}_3$ on the other. This residual is so small that it will not be necessary to repeat the computation with another value of $\mathbf{c}$: it will suffice to subtract half of it from the assumed value of $\mathbf{c}$. Thus we have as the final result

$$\mathbf{c} = 1.07158\ 32774\ 16012;$$

and, consequently,

$$\frac{1}{n}\frac{d\omega}{dt} = 0.00857\ 25730\ 04864.$$

Let us compare this value with that obtained from Delaunay's literal expression [1]

$$\frac{1}{n}\frac{d\omega}{dt}=\frac{3}{4}m^2+\frac{225}{32}m^3+\frac{4071}{128}m^4+\frac{265493}{2048}m^5+\frac{12822631}{24576}m^6$$
$$+\frac{1273925965}{589824}m^7+\frac{71028685589}{7077888}m^8+\frac{32145882707741}{679477248}m^9,$$

where m denotes the ratio of the mean motions of the sun and moon. On the substitution of the numerical values we have employed for these quantities, this series gives, term by term,

$$\frac{1}{n}\frac{d\omega}{dt}=0.00419\,6429+0.00294\,2798+0.00099\,5700+0.00030\,3577$$
$$+0.00009\,1395+0.00002\,8300+0.00000\,9836$$
$$+0.00000\,3468=0.00857\,1503.$$

From the comparison, it appears that the sum of the remainder of Delaunay's series is 0.00000 1070, somewhat less than would be inferred by induction from the terms of the series itself. And, although Delaunay has been at the great pains of computing 8 terms of this series, they do not suffice to give correctly the first 4 significant figures of the quantity sought. On the other hand, the terms of the highest order, computed in the expression for □(0), were of the 12th order only; and yet, as we have seen, they have sufficed for giving **c** exact nearly to the 15th decimal. As well as can be judged from induction, it would be necessary to prolong the series, in powers of m, as far as m^{27}, in order to obtain an equally precise result. Allowing that the two last figures of the foregoing value of $\frac{1}{n}\frac{d\omega}{dt}$ may be vitiated by the accumulation of error arising from the very numerous operations, we may, I think, assert that 13 decimals correctly correspond to the assumed value of **m**. It may be stated that all the computations have been made twice, and no inconsiderable portion of them three times.

[1] Comptes rendus de l'académie des sciences de Paris, Tome LXXIV, p. 19.