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A BRIEF HISTORY AND EXPOSITION OF THE FUNDAMENTAL THEORY OF FRACTIONAL CALCULUS

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Abstract: This opening lecture is intended to serve as a propaedeutic for the papers to be presented at this conference whose nonhomogeneous audience includes scientists, mathematicians, engineers and educators. This expository and developmental lecture, a case study of mathematical growth, surveys the origin and development of a mathematical idea from its birth in intellectual curiosity to applications. The fundamental structure of fractional calculus is outlined. The possibilities for the use of fractional calculus in applicable mathematics is indicated. The lecture closes with a statement of the purpose of the conference.

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Fractional calculus has its origin in the question of the extension of meaning. A well known example is the extension of meaning of real numbers to complex numbers, and another is the extension of meaning of factorials of integers to factorials of complex numbers. In generalized integration and differentiation the question of the extension of meaning is: Can the meaning of derivatives of integral order $d^n y/dx^n$ be extended to have meaning where n is any number---irrational, fractional or complex?

Leibnitz invented the above notation. Perhaps, it was naive play with symbols that prompted L'Hospital to ask Leibnitz about the possibility that n be a fraction. "What if n be $\frac{1}{2}$?", asked L'Hospital. Leibnitz [1] in 1695 replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn." In 1697, Leibnitz, referring to Wallis's infinite product for $\pi/2$, used the notation $d^{\frac{1}{2}}y$ and stated that differential calculus might have been used to achieve the same result.

In 1819 the first mention of a derivative of arbitrary order appears in a text. The French mathematician, S. F. Lacroix [2], published a 700 page text on differential and integral calculus in which he devoted less than two pages to this topic.

Starting with $y = x^n$, n a positive integer, he found the *m*th derivative to be

$$\frac{d^m y}{dx^m} = \frac{n!}{(n-m)!} x^{n-m}.$$

Using Legendre's symbol Γ which denotes the generalized factorial, and by replacing m by $1/2$ and n by any positive real number a, in the manner typical of the classical formalists of this period, Lacroix obtained the formula

$$\frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = \frac{\Gamma(a+1)}{\Gamma(a+\frac{1}{2})} x^{a-\frac{1}{2}}$$

which expresses the derivative of arbitrary order $1/2$ of the function x^a. He gives the example for $y = x$ and gets

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(x) = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

because $\Gamma(3/2) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$ and $\Gamma(2) = 1$. This result is the same yielded by the present day Riemann-Liouville definition of a fractional derivative. It has taken 279 years since L'Hospital first raised the question for a text to appear solely devoted to this topic, [3].

Euler and Fourier made mention of derivatives of arbitrary order but they gave no applications or examples. So the honor of making the first application belongs to Niels Henrik Abel [4] in 1823. Abel applied the fractional calculus in the solution of an integral equation which arises in the formulation of the tautochrone problem. This problem, sometimes called the isochrone problem, is that of finding the shape of a frictionless wire lying in a vertical plane such that the time of slide of a bead placed on the wire slides to the lowest point of the wire in the *same time* regardless of where the bead is placed. The brachistochrone problem deals with the *shortest time* of slide.

Abel's solution was so elegant that it is my guess it attracted the attention of Liouville [5] who made the first major attempt to give a logical definition of a fractional derivative. He published three long memoirs in 1832 and several more through 1855.

Liouville's starting point is the known result for derivatives of integral order

$$D^m e^{ax} = a^m e^{ax}$$

which he extended in a natural way to derivatives of arbitrary order

$$D^v e^{ax} = a^v e^{ax} .$$

He expanded the function $f(x)$ in the series

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \tag{1}$$

and assumed the derivative of arbitrary order $f(x)$ to be

$$D^v f(x) = \sum_{n=0}^{\infty} c_n a_n^v e^{a_n x} . \tag{2}$$

This formula is known as Liouville's [6] first definition and has the obvious disadvantage that v must be restricted to values such that the series converges.

Liouville's second method was applied to explicit functions of the form x^{-a}, $a > 0$. He considered the integral

$$I = \int_0^{\infty} u^{a-1} e^{-xu} du. \tag{3}$$

The transformation $xu = t$ gives the result

$$x^{-a} = \frac{1}{\Gamma(a)} I. \tag{4}$$

Then, with the use of (1) he obtained, after operating on both sides of (4) with D^v, the result

$$D^v x^{-a} = \frac{(-1)^v \Gamma(a+v)}{\Gamma(a)} x^{-a-v} . \quad [7] \tag{5}$$

Liouville was successful in applying these definitions to problems in potential theory. "These concepts were too narrow to last," said Emil Post [8]. The first definition is restricted to certain values of v and the second method is not suitable to a wide class of functions.

Between 1835 and 1850 there was a controversy which centered on two definitions of a fractional derivative. George Peacock [9] favored Lacroix's generalization of a case of integral order. Other mathematicians favored Liouville's definition. Augustus De Morgan's [10] judgement proved to be accurate when he stated that the two versions may very possibly be parts of a more general system. In 1850 William Center [11] observed that the discrepancy between the two versions of a fractional derivative focused on the fractional derivative of a constant. According to the Peacock-Lacroix version the fractional derivative of a constant yields a result other than zero while according to Liouville's formula (5) the fractional derivative of a constant equals zero because $\Gamma(0) = \infty$.

The state of affairs in the mid-nineteenth century is now cleared up. Harold Thayer Davis [12] states, "The mathematicians at that time were aiming for a plausible definition of generalized differentiation but, in fairness to them, one should note they lacked the tools to examine the consequences of their definition in the complex plane."

Riemann [13] in 1847 while a student wrote a paper published posthumously in which he gives a definition of a fractional operation. It is my guess that Riemann was influenced by one of Liouville's memoirs in which Liouville wrote, "The ordinary differential equation

$$\frac{d^n y}{dx^n} = 0$$

has the complementary solution

$$y_c = c_o + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} .$$

Thus

$$\frac{d^u}{dx^u} f(x) = 0$$

should have a corresponding complementary solution." So, I am inclined to believe Riemann saw fit to add a complementary function to his definition of a fractional integration:

$$D^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)\,dt + \psi(x). \tag{6}$$

Cayley [13] remarked in 1880 that Riemann's complementary function is of indeterminate nature.

The development of mathematical ideas is not without error. Peacock made several errors in the topic of fractional calculus when he misapplied the *Principle of the Permanence of Equivalent Forms* which is stated for algebra and which did not always apply to the theory of operators. Liouville made an error when he failed to note in his discussion of a complementary function that the specialization of one of the parameters led to an absurdity. Riemann became hopelessly entangled with an indeterminate complementary function. Two different versions of a fractional derivative yielded different results when applied to a constant. Thus, I suggest that when Oliver Heaviside published his work in the last decade of the nineteenth century, he was met with haughty silence and disdain not only because of the hilarious jibes he made at mathematicians but also because of the distrust mathematicians had in the general concept of fractional operators.

The subject of notation cannot be minimized. The succinctness of notation of fractional calculus adds to its elegance. In the papers that follow in this text, various notations are used. The notation I prefer was invented by Harold T. Davis. All the information can be conveyed by the symbols

$$ {}_cD_x^{-v} f(x), \qquad v \geqq 0, $$

denoting integration of arbitrary order along the x-axis. The subscripts c and x denote the limits (terminals) of integration of a definite integral which defines fractional integration. The adjoining of these subscripts becomes a vital part of the operator symbol to avoid ambiguities in applications.

We now consider the mathematical problem of defining fractional integration and differentiation. It is clear that the mathematicians mentioned so far were not merely formalizing but were trying to solve a problem which they well understood but did not explicitly formulate. Briefly what is wanted is this: for every function $f(z)$, $z = x + iy$, of a sufficiently wide class, and every number v, irrational, fractional or complex, a function ${}_cD_z^v f(z) = g(z)$, or ${}_cD_x^v f(x) = g(x)$ when z is purely real, should be assigned subject to the following criteria:

1. If $f(z)$ is an analytic function of the complex variable z, the derivative ${}_cD_z^v f(z)$ is an analytic function of v and z.

2. The operation ${}_cD_x^v\, f(x)$ must produce the same result as ordinary differentiation when v is a positive integer. If v is a negative integer, say $v = -n$, then ${}_cD_x^{-n}\, f(x)$ must produce the same result as ordinary n-fold integration and ${}_cD_x^{-n}\, f(x)$ must vanish along with its n-1 derivatives at $x = c$.

3. The operation of order zero leaves the function unchanged:

$$ {}_cD_x^0\, f(x) = f(x) $$

4. The fractional operators must be linear:

$$ {}_cD_x^{-v}\, [af(x) + bg(x)] = a\, {}_cD_x^{-v}\, f(x) + b\, {}_cD_x^{-v}\, g(x) . $$

5. The law of exponents for integration of arbitrary order holds:

$$ {}_cD_x^{-u}\, {}_cD_x^{-v}\, f(x) = {}_cD_x^{-u-v}\, f(x) . $$

A definition which fulfills these criteria named in honor of Riemann and Liouville is

$$ {}_cD_x^{-v}\, f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)\, dt. \tag{7} $$

This definition for integration of arbitrary order is the same as Riemann's definition but has no complementary function. When $c = 0$ we have Riemann's definition and when $c = -\infty$, (7) is equivalent to Liouville's definitions (see [6], pp. 176-178). Although (7) can be shown to fulfill the above stated criteria, it might be of interest to establish a set of criteria that will characterize (7) uniquely. This question is discussed later in this text p. 379.

The definition (7) can be obtained in at least four different ways. Euler had shown that

$$ \int_0^x (x-t)^b t^d\, dt = \frac{\Gamma(b+1)\Gamma(d+1)}{\Gamma(b+d+2)}\, x^{b+d+1}, \quad b \text{ and } d > -1 \tag{8} $$

For $b = 3$ and $d = 4$, (8) gives the result

$$ \frac{\Gamma(4)}{8\cdot 7\cdot 6\cdot 5}\, x^8 . $$

If one were to integrate the function x^4 four times and take the constant of integration each time to be zero, the result will be

$$\frac{1}{8\cdot 7\cdot 6\cdot 5}\ x^8 .$$

Inquisitive experimentation of this type might lead one to guess that the above two results may be connected by the expression:

$${}_0D_x^{-4}\ x^4 = \frac{1}{\Gamma(4)} \int_0^x (x-t)^3\ t^4\ dt,$$

or in general

$$(9)\qquad {}_0D_x^{-n}\ f(x) = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1}\ f(t)\ dt.$$

The above is generalized by letting $n = v$.

The same result can be obtained by considering the n-fold iterated integral

$$F(x) = \int_c^x dx_1 \int_c^{x_1} dx_2 \cdots \int_c^{x_{n-2}} dx_{n-1} \int_c^{x_{n-1}} f(x_n)\ dx_n .$$

This iterated integral can be written as a single integral by the method devised by Dirichlet, that is, by integrating over an appropriate triangular region [14]. The result is

$$F(x) = \frac{1}{\Gamma(n)} \int_c^x (x-x_n)^{n-1}\ f(x_n)\ dx_n\ .$$

If we denote the operators of differentiation and of integration as

$$D_x \qquad \text{and} \qquad D_x^{-1} = \int_c^x \cdots\ dx,$$

We may write $F(x) = {}_cD_x^{-n}\ f(x)$. Then letting $x_n = t$ and generalizing by replacing n with v we again arrive at (7).

A third approach to (7) may be deduced using the theory of linear differential equations. Let

$$L = p_o(x)\ \frac{d^n}{dx^n} + p_1(x)\ \frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x)$$

be a linear differential operator whose coefficients p_k, $0 \leqq k \leqq n$ are continuous on some closed finite interval $I = [a,b]$ and $p_o(x) > 0$ on I. Let H be the one-sided Green's function for L.

Then if f is any function continuous on I, and x_0 is any point in I, then for all $x \in I$,

$$g(x) = \int_{x_0}^{x} H(x,\xi) f(\xi)\, d\xi$$

is the solution of the nonhomogeneous equation $Ly = f(x)$ which satisfies the boundary conditions

$$g^{(k)}(x_0) = 0 \quad , \quad 0 \leqq k \leqq n-1 \ .$$

[For further details see, for example, K. S. Miller, *Linear Differential Equations in the Real Domain,* W. W. Norton and Co., Inc., New York (1963); Chapter 3.]

The Green's function H is given explicitly by

$$H(x,\xi) = \frac{(-1)^{n-1}}{p_0(\xi)W(\xi)} \begin{vmatrix} \phi_1(x) & \phi_2(x) & \cdots & \phi_n(x) \\ \phi_1(\xi) & \phi_2(\xi) & \cdots & \phi_n(\xi) \\ \phi_1'(\xi) & \phi_2'(\xi) & \cdots & \phi_n'(\xi) \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \\ \phi_1^{(n-2)}(\xi) & \phi_2^{(n-2)}(\xi) & \cdots & \phi_n^{(n-2)}(\xi) \end{vmatrix}$$

where $\{\phi_k | 1 \leqq k \leqq n\}$ is a fundamental set of solutions of $Ly = 0$, and W is their Wronskian.

Now suppose

$$L = D^n \equiv \frac{d^n}{dx^n} \ .$$

Then $\{1, x, x^2, \ldots, x^{n-1}\}$ is a fundamental set of solutions of $D^n y = 0$ and

$$W(\xi) = \begin{vmatrix} \phi_1(\xi) & \phi_2(\xi) & \cdots & \phi_n(\xi) \\ \phi_1'(\xi) & \phi_2'(\xi) & \cdots & \phi_n'(\xi) \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \\ \phi_1^{(n-1)}(\xi) & \phi_2^{(n-1)}(\xi) & \cdots & \phi_n^{(n-1)}(\xi) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 0 & 1 & 2\xi & & (n-1)\xi^{n-2} \\ 0 & 0 & 2 & & (n-1)(n-2)\xi^{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \\ 0 & 0 & 0 & & (n-1)! \end{vmatrix}$$

$$= \quad (n-1)!!$$

where

$$(n-1)!! = \prod_{k=0}^{n-1} k! \quad .$$

Thus in this special case

$$H(x,\xi) = \frac{(-1)^{n-1}}{(n-1)!!} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 0 & 1 & 2\xi & \cdots & (n-1)\xi^{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & (n-1)!\ \xi \end{vmatrix}$$

is a polynomial of degree $n-1$ in x with leading coefficient

$$\frac{(-1)^{n-1}}{(n-1)!!}[(-1)^{n+1}(n-2)!!] = \frac{1}{(n-1)!} \quad .$$

But

$$\frac{\partial^k}{\partial x^k} H(x,\xi) \Bigg|_{x=\xi} = 0 \ , \ 0 \leqq k \leqq n-2 \ .$$

Thus ξ is a zero of multiplicity $n-1$ and

$$H(x,\xi) = \frac{1}{(n-1)!}(x-\xi)^{n-1} \quad . \tag{10}$$

Hence if $x_o = a$,

$$g(x) = \frac{1}{(n-1)!} \int_a^x (x-\xi)^{n-1} f(\xi)\,d\xi \tag{11}$$

is the unique solution of the differential equation

$$\frac{d^n y}{dx^n} = f(x)$$

which assumes the initial values $g^{(k)}(a) = 0$, $0 \leqq k \leqq n-1$. We may write (11) as

$$_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-\xi)^{n-1} f(\xi)\,d\xi \;.$$

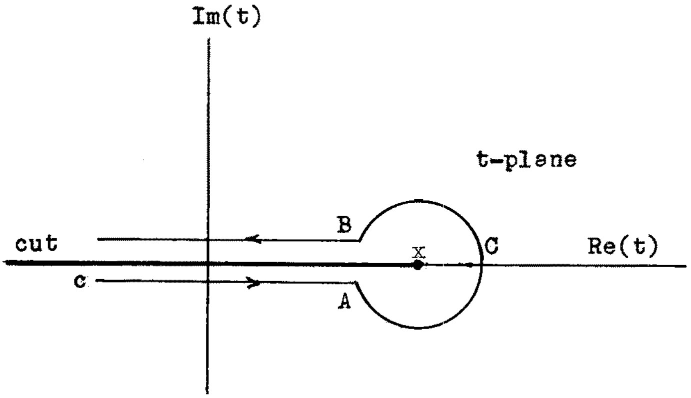
Now, of course, we replace n by ν (with $\operatorname{Re} \nu > 0$) in the above formula, [15].

The fourth method of arriving at definition (7) is by contour integration in the complex plane. It is a curious fact that these generalized operators and their connection with the Cauchy integral formula have succeeded in securing for themselves only passing references in standard works in complex variable theory. P. A. Nekrassov in 1888 used a contour starting at the origin as did A. Krug in 1890. Laurent in 1884 used a contour that started and ended at $-\infty$, now called a Laurent loop.

Cauchy's integral formula is

$$f^{(n)}(z) = D_z^n\, f(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}}\, dt .$$

In the previous three methods of obtaining (7) the generalization of n to fractional values creates no difficulties because $\nu! = \Gamma(\nu+1)$. But here $1/(t-z)^{\nu+1}$ no longer contains a pole but a branch point. To keep the function single valued, we let the branch cut be the semi-infinite line starting at $t = x > 0$ to negative infinity on the real t axis as in the figure below. Let C be the open contour (or loop) which starts a the point c, $c < x$, on the lower edge of the cut, then goes along the real axis to A, around the circle $|t-x| < \varepsilon$ in the positive sense to B, and then back to c along the upper edge of the cut.



Generalizing the Cauchy integral formula to arbitrary values of n gives

$$D^{v} f(z) = \frac{\Gamma(v+1)}{2\pi i} \int_{c}^{x+} (t-z)^{-v-1} f(t)\,dt$$

where we define

$$(t-x)^{-v-1} = e^{(-v-1)\ln(t-x)},$$

and where $\ln(t-x)$ is real when $t-x$ is a positive real number.

By standard methods of contour integration we are again led to (7) (see [6] pp. 198-202).

The general validity of definition (7) for $v = n$, n a positive integer, can be established by mathematical induction. Here we are concerned with criterion 2 which stipulates that the definition must produce the same result as ordinary integration. There is no loss of generality by taking the lower limit of integration to be zero. We have

$${}_{0}D_{x}^{-n} f(x) = \frac{1}{\Gamma(n)} \int_{0}^{x} (x-t)^{n-1} f(t)\,dt.$$

The above is obviously true for $n = 1$, for

$${}_{0}D_{x}^{-1} f(x) = \int_{0}^{x} f(t)\,dt.$$

Now assume the formula true for $n = k$:

$${}_0D_x^{-k}\ f(x) = \frac{1}{\Gamma(k)} \int_0^x (x-t)^{k-1} f(t)\,dt.$$

Replace k with k+1:

$${}_0D_x^{-(k+1)} f(x) = \frac{1}{k\Gamma(k)} \int_0^x (x-t)^{k} f(t)\,dt.$$

Operate on both sides of the above with ${}_0D_x = d/dx$ and we have

$$(12) \qquad {}_0D_x^{-k}\ f(x) = \frac{d}{dx} \frac{1}{k\Gamma(k)} \int_0^x (x-t)^{k} f(t)\,dt.$$

Applying Leibnitz's rule for the derivative of an integral gives

$${}_0D_x^{-k}\ f(x) = \frac{1}{k\Gamma(k)} \int_0^x \frac{\partial}{\partial m} (x-t)^{k} f(t)\,dt$$

$$- g(0,x)\,\frac{d(0)}{dx} + g(x,x)\,\frac{dx}{dx}\ ,$$

where the function g is the integrand $(x-t)^k f(t)$ in (12).

The last two terms on the right above vanish because $d(0)/dx = 0$, and $g(x,x) = 0$ because of the factor $(x-x)$. Then we have

$${}_0D_x^{-k}\ f(x) = \frac{k}{k\Gamma(k)} \int_0^x (x-t)^{k-1} f(t)\,dt,$$

and since $k = n$ we have by mathematical induction

$${}_0D_x^{-n}\ f(x) = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} f(t)\,dt.$$

This result is the same as (9) obtained heuristically.

The definition for differentiation of arbitrary order will be shown later to be an integration followed by ordinary differentiation. It follows that (7) fulfills criterion 2 for differentiation and integration.

Criterion 3 states that the operation of order zero leaves the function unchanged, that is, ${}_cD_x^0\ f(x) = 0$. The investigation of whether ${}_cD_x^v\ f \to f$ as $v \to 0$ shows a concern with the continuity of the ${}_cD_x^{-v}$ operator at $v = 0$. We have

(13) $$ {}_cD_x^0\ f(x) = \frac{1}{\Gamma(0)} \int_c^x (x-t)^{0-1} f(t)\,dt, $$

as a consequence of letting $v = 0$ in (7). The factor $1/\Gamma(0)$ can be taken equal to zero because $\Gamma(0) = \infty$. The integral would, in general, be divergent and we have to deal with the indeterminate form $0\cdot\infty$. There are several ways of handling this situation.

We assume $f(t)$ is expansible in a Taylor's series with remainder:

$$ f(t) = f(x) + (t-x)f'(x) + (t-x)^2 f''(x)/2! + \cdots , $$

and taking limits of both sides of (7) as $v \to 0$ we have

$$ \lim_{v\to 0}\ {}_0D_x^{-v} f(x) = \lim_{v\to 0} \Bigg[\int_0^x \frac{(x-t)^{v-1}}{\Gamma(v)} f(x)\,dt $$

$$ - \int_0^x \frac{(x-t)^{v}}{\Gamma(v)} f'(x)\,dt + \cdots + (-1)^n \int_0^x \frac{(x-t)^{v+n-1}}{n!\,\Gamma(v)} f^{(n)}(x)\,dt $$

$$ + (-1)^{n+1} \frac{f^{(n+1)}(\Theta)}{(n+1)!} \int_0^x \frac{(x-t)^{v+n}}{\Gamma(v)}\,dt \Bigg] . $$

Because $\lim_{v\to 0} \Gamma(v) = \infty$, all the terms on the right above vanish except the first because the first integral has the value

$$ \frac{x^v f(x)}{\Gamma(v+1)} $$

due to the gamma function relation $v\Gamma(v) = \Gamma(v+1)$. We note there is no loss of generality using the lower limit of integration 0 instead of c.

Thus

$$ \lim_{v\to 0}\ {}_0D_x^{-v} f(x) = \lim_{v\to 0} \frac{x^v f(x)}{\Gamma(v+1)} . $$

$$ {}_0D_x^0 f(x) = f(x) . $$

We can also arrive at the same result in the following manner.

Let the function f be continuous on the interval (c,x). The integral (7) can then be written as the sum of two integrals:

$$ {}_cD_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x \Big[f(t) - f(x)\Big] (x-t)^{v-1} dt + \frac{f(x)}{\Gamma(v)} \int_c^x (x-t)^{v-1} dt. \tag{14} $$

The first integral on the right in (14) will be shown to tend to zero as v tends to zero. It can be divided into two subintervals of integration:

$$ \frac{1}{\Gamma(v)} \int_c^{x-\delta} \Big[f(t) - f(x)\Big] (x-t)^{v-1} dt + \frac{1}{\Gamma(v)} \int_{x-\delta}^{x} \Big[f(t) - f(x)\Big] (x-t)^{v-1} dt \tag{15} $$

where δ is any small positive number. Let us designate these integrals as

$$ A + B. $$

In B denote the maximum of $|f(t) - f(x)|$ by ε. Thus,

$$ |f(t) - f(x)| \leqq \varepsilon(\delta) $$

in integral B, where ε depends on δ, and $\lim\limits_{\delta\to 0} \varepsilon(\delta) = 0$ because f is continuous. We then have

$$ \begin{aligned} |B| &\leqq \frac{1}{\Gamma(v)} \varepsilon(\delta) \int_{x-\delta}^{x} (x-t)^{v-1} dt \\ &\leqq \frac{1}{\Gamma(v)} \varepsilon(\delta) \cdot \frac{\delta^v}{v} = \frac{\varepsilon(\delta)\delta^v}{\Gamma(v+1)} . \end{aligned} $$

Then, for all $v \geqq 0$, we have

$$ \lim_{\delta\to 0} |B| = 0. $$

After evaluating the second integral on the right in (14), Eq. (14) is written as

$$ {}_cD_x^{-v} f(x) = A + B + \frac{(x-c)^v}{\Gamma(v+1)} f(x) \tag{16} $$

where $|B| \to 0$ with δ.

We can now consider the integral A in (15). Denote the maximum of $|f(t) - f(x)|$ by M. Then

$$|A| \leqq \frac{M}{\Gamma(v)} \int_c^{x-\delta} (x-t)^{v-1} dt$$

$$\leqq \frac{M}{\Gamma(v+1)} (\delta^v - (x-c)^v).$$

Let ε be any arbitrary positive number. Now choose δ so that $|B| < \varepsilon$ for all $v \geqq 0$. For this fixed δ, $|A| \to 0$ as $v \to 0$. To both sides of (16) add $-f(x)$. Then

$$\left| {}_cD_x^{-v} f(x) - f(x) \right| \leqq |A| + |B| + |f(x)| \left[\frac{(x-c)^v}{\Gamma(v+1)} -1 \right].$$

Because $|B| < \varepsilon$, we have

$$\limsup_{v \to 0} \left| {}_cD_x^{-v} f(x) - f(x) \right| \leqq 0 + \varepsilon + 0.$$

Since ε can be chosen as small as we wish, it follows that

$$\lim_{v \to 0} \left| {}_cD_x^{-v} f(x) - f(x) \right| = 0,$$

or

$$\lim_{v \to 0} {}_cD_x^{-v} f(x) = f(x).$$

Another approach to the above result using the theory of Laplace transforms might be of interest. If we define $f(x)$ only in $[0,L]$ then f can be taken as zero in $x > L$. Let $f(x)$ be such that

$$\int_v^\infty e^{-ax} f(x) dx$$

exists for some real a. Then it follows that

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt$$

is an analytic function of s in $Re(s) > a$, that, with $v > -1$,

$$\int_0^\infty e^{-sx} \int_0^x \frac{1}{\Gamma(v)} (x-t)^{v-1} f(t) dt$$

exists in $\operatorname{Re}(s) > a$, and in fact, that

$$\bar{g}(s,v) = \int_0^\infty e^{-sx} g(x,v)\, dx \equiv s^{-v}\, \bar{f}(s),$$

where $g(x,v)$ denotes the right side of (7).

It is also true that

$$f_1(x) = \frac{1}{2\pi i} \int_G e^{sx}\, \bar{f}(s)\, ds$$

where G is any vertical path lying in $\operatorname{Re}(s) > 0$ and where $f_1(x)$ differs from $f(x)$ on, at most, a countable number of points.

Furthermore,

$$g(x,v) = \frac{1}{2\pi i} \int_G e^{sx}\, \bar{g}(s,v)\, ds.$$

But, for such a path G, everything is uniformly bounded and

$$\lim_{v \to 0} g(x,v) = \frac{1}{2\pi i} \int_G e^{st}\, \bar{f}(s)\, ds = f_1(x)$$

which is the result wanted, [16].

We now consider criterion 5: ${}_cD_x^{-u}\ {}_cD_x^{-v}\ f(x) = {}_cD_x^{-u-v}\ f(x)$. By definition (7) we have

$$(17) \qquad {}_cD_x^{-u} \left[{}_cD_x^{-v}\ f(x) \right] = \frac{1}{\Gamma(u)} \int_c^x (x-s)^{u-1} ds \bullet \frac{1}{\Gamma(v)} \int_c^s (s-t)^{v-1}\, f(t)\, dt.$$

The repeated integral above corresponds to a double integral to which Dirichlet's formula, mentioned earlier, may be applied. We have

$$(18) \qquad {}_cD_x^{-u}\ {}_cD_x^{-v}\ f(x) = \frac{1}{\Gamma(u)\Gamma(v)} \int_c^x f(t)\, dt \bullet \int_t^x (x-s)^{u-1} (s-t)^{v-1}\, ds.$$

When either u or v is on the interval (0,1), the passage from (17) to (18) can be justified by a minor modification of the Dirichlet proof over a smaller triangle.

Make the transformation $y = (s-t)/(x-t)$. The second integral on the right in (18) is then

$$(x-t)^{u+v-1} \int_0^1 (1-y)^{u-1} y^{v-1} dy$$

which is a beta integral that has the value

$$\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} (x-t)^{u+v-1}.$$

When this is substituted into (18), we obtain

$${}_cD_x^{-u}\ {}_cD_x^{-v}\ f(x) = \frac{1}{\Gamma(u+v)} \int_c^x (x-t)^{u+v-1} f(t)dt.$$

The integral on the right above is definition (7) with $u+v$ playing the role of arbitrary order. We then have the required result.

A subtle mathematical problem arises when one seeks to extend the law of indices stated for integration of arbitrary order to derivatives of arbitrary order. If we follow the preceding method, we will get the divergent integral

$${}_cD_x^{u}\ {}_cD_x^{v} f(x) = \frac{1}{\Gamma(-u-v)} \int_c^x (x-t)^{-(u+v)-1} f(t)dt.$$

To establish the relation

$$(19) \qquad {}_cD_x^{u}\ {}_cD_x^{v} f(x) = {}_cD_x^{u+v}\ f(x)$$

it will be required to impose the restriction that f be a function which vanishes at the lower limit of integration, namely $f(c) = 0$. This proof is omitted here but details can be found in [6].

The restriction that f vanishes at $x = c$ and at its n-1 derivatives, as stated in criterion 2, is necessary to justify the interchange of the order of operations used in the proof of establishing (19). For example, the relation

$$DD^{-1}\ f(x) = D^0 f(x) = f(x)$$

always holds. But the relation

$$D^{-1}D\ f(x) = D^{0}f(x) = f(x) \tag{20}$$

is not always valid. For, by definition (7)

$$\begin{aligned} {}_cD_x^{-1}\ {}_cD_x f(x) &= {}_cD_x^{-1}\ f'(x) \\ &= \frac{1}{\Gamma(1)} \int_c^x (x-t)^0 f'(t)dt \\ &= f(x) - f(c), \end{aligned}$$

and (20) holds only when $f(c) = 0$.

The definition (7) is for integration of arbitrary order. For differentiation of arbitrary order it cannot be used directly. However, by means of a simple trick, we can find a convergent expression. Let $v = m-p$ where for convenience m is the least integer greater than v, and $0 < p \leqq 1$. Then for differentiation of arbitrary order we have

$$\begin{aligned} {}_cD_x^{v}\ f(x) &= {}_cD_x^{m}\ {}_cD_x^{-p}\ f(x) \\ &= \frac{d^m}{dx^m} \frac{1}{\Gamma(p)} \int_c^x (x-t)^{p-1} f(t)dt, \end{aligned} \tag{21}$$

where we take advantage of the knowledge that ${}_cD_x^{m}$ is an ordinary *m*th derivative operator d^m/dx^m. We have assumed for purposes of this definition that $D^{m-p} = D^m D^{-p}$.

The simple trick referred to above, namely $D^v = D^{m-p}$ results from the fact that D^{m-p} is the analytic continuation of the fractional operator D^{-v}. It is obvious that criterion 1 which required analyticity, and also the other four criteria, were established by hindsight. The question of extending the definition (7) for integration of arbitrary order to differentiation of arbitrary order is answered by letting v be real and greater than zero. We have

$$\emptyset(v,x) = {}_0D_x^{-v}f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt \tag{22}$$

which is in general convergent for $v > 0$. For any v we can write

$$\psi(v,x) = {}_0D_x^{-v}\ f(x) = {}_0D_x^{m}\ {}_0D_x^{-p}\ f(x)$$
$$= \frac{d^m}{dx^m}\frac{1}{\Gamma(p)}\int_0^x (x-t)^{p-1} f(t)\,dt,$$

where $-v = m-p$, $m = 0, 1, 2, \ldots$.

When $v > 0$ choose $m = 0$. Thus $v = p$ and $\emptyset = \psi$. Now, (22) can be written

$$\emptyset(v,x) = \frac{d}{dx}\int_0^x \left[\frac{1}{\Gamma(v)}\int_0^x (x-t)^{v-1} f(t)\,dt\right] dx.$$

By Dirichlet's formula, we have

$$\emptyset(v,x) = \frac{d}{dx}\frac{1}{\Gamma(v+1)}\int_0^x (x-t)^{v} f(t)\,dt$$

which is convergent for $v > -1$. We then have

$$\emptyset(v,x) = \psi(v,x) \quad \text{for} \quad m = 1.$$

This process can be repeated for $v > -n$, n a positive integer. Now $\emptyset$ is analytic in R_1 where $v > 0$ and ψ is analytic in R_2 for $v > -n$. Since $\emptyset = \psi$ on a set of points in $R_1 \cap R_2$ with a limit point in the right half plane, then ψ is the analytic continuation of $\emptyset$. This justifies the trick of writing D^{m-p} for D^v.

Some explicit examples of fractional derivatives will be useful. For the fractional derivative of a constant k, we have by letting $v = m-p$, m the least integer $> v$, and the use of (21), the formula

$$(23) \qquad {}_0D_x^{v}\ k = \frac{k}{\Gamma(1-v)}\ x^{-v}.$$

Another example is the integration and differentiation of arbitrary order of the natural logarithm.

By definition (7) we have

$$(24) \qquad {}_0D_x^{-v}\ \ln x =$$

$$\frac{1}{\Gamma(v)}\int_0^x (x-t)^{v-1}\ln t\ dt, \qquad v > 0.$$

Let $$t = x + t - x, \qquad x > 0$$
$$= x\left(1 + \frac{t-x}{x}\right).$$

Then $$\ln t = \ln x + \ln\left(1 + \frac{t-x}{x}\right)$$

with the restriction
$$-1 < \frac{t-x}{x} \leqq 1.$$

Using the Taylor's series expansion for $\ln(1+\theta)$, we get
$$\ln t = \ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(t-x)^n}{nx^n}$$

where the interval of convergence is $0 < t \leqq 2x$. Substituting the right side of the above into the right side of (24) gives
$$\frac{\ln x}{\Gamma(v)} \int_0^x (x-t)^{v-1} dt$$
$$- \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} \sum_{n=1}^{\infty} \frac{(x-t)^n}{nx^n}\, dt.$$

Term by term integration, permissible because of uniform convergence, gives the result
$${}_0D_x^{-v} \ln x = \frac{x^v \ln x}{\Gamma(v+1)} - \frac{x^v}{\Gamma(v)} \sum_{k=1}^{\infty} \frac{1}{k(v+k)} .$$

In terms of the psi function, the above result can be written
$${}_0D_x^{-v} \ln x = \frac{x^v}{\Gamma(v+1)} [\ln x - C - \psi(v+1)]$$

where C is Euler's constant. For differentiation of arbitrary order of $\ln x$ we have
$${}_0D_x^{v} \ln x = {}_0D_x^{m-p} \ln x$$
$$= \frac{d^m}{dx^m}\left[\frac{x^p \ln x}{\Gamma(p+1)} - \frac{x^p}{\Gamma(p)} \sum_{k=1}^{\infty} \frac{1}{k(p+k)}\right]$$

where the usual criteria for termwise differentiation is to be applied.

Although we now know how to interpolate between integral orders of the derivative of functions such as $\ln x$, little is known where such procedures might be applicable. In this connection this writer submitted a problem to the *American Mathematical Monthly* to appear in winter 1974-75, concerning ${}_0D_x^v \ln \Gamma(x)$. This will permit interpolation between integral orders of the psi function and might have use in the summation of series of the form $\overset{\infty}{\Sigma} 1/(1+x)^u$.

Eric Russell Love [17] has defined integration of *pure imaginary* order in such a way as to extend the properties of integration and differentiation of arbitrary order η where $\mathrm{Re}(\eta) > 0$ to the case where $\mathrm{Re}(\eta) = 0$. Francis H. Northover makes the claim that the Riemann-Liouville definition (7) can be connected to the Fourier cosine and Fourier sine transforms by means of derivatives of pure imaginary order as follows.

$$ {}_cD_x^{-v} F(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} F(t)\,dt, \qquad \mathrm{Re}(v) > 0. $$

Make the transformation

$$ t = x - (x-c)e^{-\phi}. $$

The limits (terminals) of integration then become 0 and ∞, and we have

$$ {}_cD_x^{-v} F(x) = \frac{(x-c)^v}{\Gamma(v)} \int_0^\infty e^{-v\phi} F(\phi)\, d\phi, $$

$$ {}_cD_x^{-v} F(x) = \frac{(x-c)^v}{\Gamma(v)} \mathcal{L}\{F(\phi)\}. $$

Now let $v = -i\eta$, and assume ${}_cD_x^{i\eta} F(x)$ exists. Then

$$ (26) \qquad {}_cD_x^{i\eta} F(x) = \frac{(x-c)^{-i\eta}}{\Gamma(-i\eta)} \int_0^\infty e^{i\eta\phi} F(\phi)\, d\phi $$

$$ = \frac{(x-c)^{-i\eta}}{\Gamma(-i\eta)} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \{C(\eta) + i\, S(\eta)\}, $$

where
$$ C(\eta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty F(\phi) \cos \eta\phi\, d\phi $$

and
$$ S(\eta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty F(\phi) \sin \eta\phi\, d\phi. $$

Love has shown that suitably restricted functions have derivatives of all orders v where $\mathrm{Re}(v) = 0$ but have *no* derivative of any order v for $\mathrm{Re}(v) > 0$. He has also cited an example of a function which is locally integrable but does not possess a derivative of any imaginary order. For this reason, caution was exercised in the preceding paragraph where it was stated that ${}_cD_x^{in}\, F(x)$ is assumed to exist.

Consider now

$$I = {}_aD_x^{-v}\, f(x) = \frac{1}{\Gamma(v)} \int_a^x (x-t)^{v-1} f(t)\,dt.$$

Assume $f(t)$ is expansible in a Taylor's series

$$f(t) = \sum_{n=0}^{\infty} \frac{(-1)^n\, f^{(n)}(x)\, (x-t)^n}{n!}.$$

The substitution of the series for $f(t)$ in the integrand above gives

(26a)
$$I = \frac{1}{\Gamma(v)} \sum_{n=0}^{\infty} \frac{(-1)^n\, f^{(n)}(x)\, (x-a)^{v+n}}{(v+n)\; n!}.$$

Now if $f(x) = (x-a)^p$, $p > -1$, then

$${}_aD_x^{-v}(x-a)^p = \frac{\Gamma(p+1)}{\Gamma(p+v+1)}\,(x-a)^{v+p},$$

where we have noted without proof the identity

$$\frac{\Gamma(v)}{\Gamma(p+v+1)} = \frac{1}{v\Gamma(p+1)} - \frac{1}{(v+1)\Gamma(p)} + \frac{1}{(v+2)\,2!\,\Gamma(p-1)} - \cdots .$$

If $f(x) = (x-b)^p$, $p > -1$, then from (26a)

$${}_aD_x^{-v}(x-b)^p = \frac{(x-a)^v (x-b)^p}{\Gamma(v)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x-a}{x-b}\right)^n \cdot \frac{\Gamma(p+1)}{(v+n)\; n!\; \Gamma(p-n+1)},$$

for $0 < b < a$.

We recall the laws of exponents or indices ${}_aD_x^{-v}\; {}_aD_x^{-u}\, f(x) = {}_aD_x^{-u-v}\, f(x)$ is written for the case when both terminals of integration are the same. With the results just given one can investigate a measure of deviation of the index rule, say for example, f(x) = x: ${}_aD_x^{-v}\ {}_bD_x^{-u}\ x$. (See *Open Questions*, # 3, p. 376this text).

Some special functions can be represented as an integral of arbitrary order of an elementary function. We wish to show the connection with the Bessel function:

$${}_0D_u^{-(p+\frac{1}{2})}\ \frac{\cos\sqrt{u}}{\sqrt{u}} = 2^p \sqrt{\pi}\ u^{p-2} J_o(\sqrt{u}).$$

For $\mathrm{Re}(p) > -\frac{1}{2}$, we have

$$J_p(x) = \frac{2(x/2)^p}{\sqrt{\pi}\ \Gamma(p+\frac{1}{2})} \int_0^1 (1-t^2)^{p-\frac{1}{2}} \cos xt\ dt. \ [18]$$

Make the transformation $xt = w$, the above becomes

$$J_p(x) = \frac{2}{(2x)^p \sqrt{\pi}\ \Gamma(p+\frac{1}{2})} \int_0^x (x^2-w^2)^{p-\frac{1}{2}} \cos w\ dw.$$

Let $x^2 = u$, $w^2 = v$, and the above becomes

$$2^p \sqrt{\pi}\ u^{p/2} J_p(\sqrt{u}) = \frac{1}{\Gamma(p+\frac{1}{2})} \int_0^u (u-v)^{p-\frac{1}{2}}\ \frac{\cos\sqrt{v}}{\sqrt{v}}\ dv.$$

These transformations have given us an integral which conforms to our definition (7), of arbitrary order $p+\frac{1}{2}$, and $f(u) = \frac{\cos\sqrt{u}}{\sqrt{u}}$. So, the above may be written in the form

$$2^p \sqrt{\pi}\ u^{p/2} J_p(\sqrt{u}) = {}_0D_u^{-(p+\frac{1}{2})}\ \frac{\cos\sqrt{u}}{\sqrt{u}},$$

which is the result we sought to verify.

Here we show how a hypergeometric function can be represented by the fractional operation of a product of elementary functions.

$$(27) \qquad 1 + \frac{ab}{1!g}\, x + \frac{a(a+1)b(b+1)}{2!g(g+1)}\, x^2 + \cdots$$

is called a hypergeometric series because it is a generalization of the geometric series $1 + x + x^2 \cdots$. The following notations are in common use:

$$(r)_n = (r+1)(r+2)\cdots(r+n-1),$$

$${}_2F_1(a,b;g;x).$$

The subscript 2 preceding F denotes two parameters in the numerator. The subscript 1 denotes one parameter in the denominator. Using this notation, (27) can conveniently be written in summation form:

$$(28)\qquad {}_2F_1(a,b;g;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!\,(g)_n} x^n .$$

Some properties of the gamma and beta functions which will be needed later are briefly outlined.

$$(29)\qquad \frac{(b)_n}{(g)_n} = \frac{b(b+1)\cdots(b+n-1)}{g(g+1)\cdots(g+n-1)} = \frac{\Gamma(b+n)}{\Gamma(b)} \bullet \frac{\Gamma(g)}{\Gamma(g+n)} .$$

Using the gamma-beta relation $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$, (29) becomes

$$(30)\qquad \frac{(b)_n}{(g)_n} = \frac{B(b+n,\ g-b)}{B(b,\ g-b)}$$

Thus, (28) becomes

$$(31)\qquad {}_2F_1(a,b;g;x) = \frac{1}{B(b,\ g-b)} \sum_{n=0}^{\infty} \frac{(a)_n B(b+n,g-b)}{n!} x^n$$

where the factor $1/[B(b,g-b)]$ is placed before the summation sign because it is independent of n.

Writing $B(b+n,\ g-b)$ as a beta integral, and using the symbol ${}_2F_1$ instead of ${}_2F_1(a,b;g;x)$, we then have

$$(32)\qquad {}_2F_1 = \frac{1}{B(b,g-b)} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n \int_0^1 (1-t)^{g-b-1} t^{b+n-1} dt .$$

The interchange of the summation sign and the integral sign is permissible because of the uniform convergence of the series:

$$(33)\qquad {}_2F_1 = \frac{1}{B(b,g-b)} \int_0^1 (1-t)^{g-b-1} t^{b-1} \sum_{n=0}^{\infty} \frac{(a)_n (xt)^n}{n!} dt .$$

Using the fact that

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} (xt)^n = (1-xt)^{-a},$$

we find that (33) becomes

$${}_2F_1 = \frac{1}{B(b,g-b)} \int_0^1 (1-t)^{g-b-1} t^{b-1} (1-xt)^{-a} dt,$$

valid if $|x| < 1$, and $g,b > 0$.

All that is required now is to transform the integral on the right above to an integral of the form of the definition (7). To do this let $xt = s$, and we have

$${}_2F_1 = \frac{x^{-g+1}}{B(b,g-b)} \int_0^x (x-s)^{g-b-1} s^{b-1} (1-s)^{-a} ds.$$

Using the relation $B(b,g-b) = \Gamma(b)\Gamma(g-b)/\Gamma(g)$, and writing the integral above in operator notation, we obtain the result

$$\frac{x^{g-1}\Gamma(b)}{\Gamma(g)} {}_2F_1(a,b;g;x) = {}_0D_x^{-(g-b)} x^{(b-1} (1-x)^{-a}.$$

Before turning our attention to some applications of fractional calculus, it will be useful to mention another definition of fractional integration and another access to a fractional derivative. There appears to be two representations of Hermann Weyl's definition. One is

$${}_xW_\infty^{-v} f(x) = \frac{1}{\Gamma(v)} \int_x^\infty (t-x)^{v-1} f(t)\, dt, \ \mathrm{Re}(v) > 0.$$

The significant differences between this definition and the Riemann-Liouville definition are the terminals of integration and the kernel function here being $(t-x)^{v-1}$. When the Weyl integral exists, $W^\alpha W^\beta = W^{\alpha+\beta}$ for all α and β. Kenneth S. Miller derives the Weyl integral in the following way.

Let L be the linear differential operator

$$L = p_0(x) \frac{d^n}{dx^n} + p_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + p_n(x),$$

whose coefficients p_k, $0 \leqq k \leqq n$, are of class C^∞ on some closed finite interval $I = [a,b]$ and $p_0(x) > 0$ on I. Let L^* be the adjoint of L and $H^*(x,\xi)$ its one-sided Green's function. Then if f is any function continuous on I, and x_0 is any point in I, then for all $x \in I$,

$$(36) \qquad g(x) = \int_{x_0}^{x} H^*(x,\xi) f(\xi)\, d\xi$$

is the solution of the nonhomogeneous equation $L^* y = f(x)$ which satisfies the boundary conditions

$$g^{(k)}(x_0) = 0, \qquad 0 \leqq k \leqq n-1.$$

Now let $x_0 = b$ and recall that $H^*(x,\xi) = -H(\xi,x)$ where $H(x,\xi)$ is the one-sided Green's function for L. (See p. 37 of Miller's text cited on p.90 .) Then if we let $x_0 = b$

$$g(x) = \int_{x}^{b} H(\xi,x) f(\xi)\, d\xi$$

is the solution of $L^* y = f(x)$ with initial conditions $g^{(k)}(b) = 0$, $0 \leqq k \leqq n-1$.

Now if

$$L = \frac{d^n}{dx^n} \quad ,$$

then L is formally self-adjoint since $L^* = (-1)^n L$. We recall that for this particular L, (as in (10)),

$$H(x,\xi) = \frac{1}{(n-1)!} (x-\xi)^{n-1}$$

Thus

$$g(x) = \frac{1}{\Gamma(n)} \int_{x}^{b} (\xi - x)^{n-1} f(\xi)\, d\xi$$

is the unique solution of the adjoint equation

$$(-1)^n \frac{d^n y}{dx^n} = f(x)$$

(with the initial conditions $g^{(k)}(b) = 0$, $0 \leqq k \leqq n-1$.) So we may call

$$ {}_xW_b^{-\nu}f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (\xi - x)^{\nu-1} f(\xi)\,d\xi \quad , \quad \mathrm{Re}\ \nu > 0 \quad , $$

the *adjoint fractional integral* (unless someone else has already named it).

Now for x fixed, a *sufficient* condition that

$$ \lim_{b\to\infty} {}_xW_b^{-\nu}f(x) $$

exists is

$$ f(x) = 0 \quad , \quad x < 0 $$

and

$$ \int_0^\infty |x^\nu|^2 f^2(x)\,dx < \infty \ . $$

(Apply the Cauchy-Schwarz inequality.)

Formally

$$ \frac{d}{dx}\, {}_xW_\infty^{-\nu}f(x) = -{}_xW_\infty^{-(\nu-1)}f(x) \ . $$

and, for example,

$$ {}_xW_\infty^{-(\nu-1)}e^{-x} = -\frac{d}{dx}\frac{1}{\Gamma(\nu)}\int_x^\infty (\xi - x)^{\nu-1}e^{-\xi}d\xi, \qquad x > 0. $$

Make the transformation $\xi - x = y$ and we have

$$ {}_xW_\infty^{-(\nu-1)}e^{-x} = -\frac{d}{dx}\frac{e^{-x}}{\Gamma(\nu)}\int_0^\infty y^{\nu-1}e^{-y}dy $$

$$ = -\frac{d}{dx}\frac{e^{-x}}{\Gamma(\nu)} \cdot \Gamma(\nu) $$

$$ = e^{-x}. $$

One notes that

$$ \frac{d^m}{dx^m}\, {}_xW_\infty^{-\nu}\ f(x) = (-1)^m {}_xW_\infty^{m-\nu}\ f(x) $$

so that

$$ {}_xW_\infty^{-\frac{1}{2}}\ e^{-x} = e^{-x}, $$

and

$$ {}_xW_\infty^{m-\frac{1}{2}}\, e^{-x} = e^{-x} $$

for any nonnegative integer m.

The laws of exponents hold for $\operatorname{Re}\mu > 0$ and $\operatorname{Re}\nu > 0$. The argument is similar to (17) and (18):

$$ {}_xW_\infty^{-\mu}[{}_xW_\infty^{-\nu} f(x)] = \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_x^\infty (t-x)^{\mu-1} dt \int_t^\infty (\xi-t)^{\nu-1} f(\xi)\, d\xi $$

$$ = \frac{1}{\Gamma(\mu)\Gamma(\nu)} \int_x^\infty f(\xi)\, d\xi \int_x^\infty (t-x)^{\mu-1}(\xi-t)^{\nu-1} dt $$

$$ = \frac{B(\mu,\nu)}{\Gamma(\mu)\Gamma(\nu)} \int_x^\infty f(\xi)(\xi-x)^{\mu+\nu-1} d\xi $$

$$ = {}_xW_\infty^{-(\mu+\nu)} f(x) \;, $$

which is a law of exponents. See also the paper by Kenneth S. Miller, this text, pp. 80 - 90.

Another form of Weyl's definition is that of an infinite series involving periodic functions which will be mentioned by Miklós Mikolás in his paper this afternoon. These definitions have been used by A. Zygmund in the treatment of certain Fourier series.

One of the most recent methods of defining derivatives of fractional order is by the limit of a fractional different quotient. Paul L. Butzer and Ursula Westphal are, unfortunately, unable to attend but their paper appears later in this text. Using the notation of Butzer and Westphal, the derivative of arbitrary order α of the nonperiodic function $x^\gamma/\Gamma(\gamma+1)$ is obtained as follows. We define

$$ \Delta_t^\alpha f(t) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(x-tj) $$

$$ f_\gamma(x) = \begin{cases} \dfrac{x^\gamma}{\Gamma(\gamma+1)} & (x > 0) \\ 0 & (x < 0) \end{cases} $$

$$\mathcal{L}[t^{-\alpha}\Delta_t^\alpha f_\gamma(x)]\ (s) = t^{-\alpha}\frac{1}{s^{\gamma+1}}(1-e^{-st})^\alpha$$

$$= \frac{1}{s^{\gamma-\alpha+1}}\left(\frac{1-e^{-st}}{st}\right)^\alpha$$

$$= \mathcal{L}[(f_{\gamma-\alpha}(u) * \tfrac{1}{t}\,p_\alpha(\tfrac{u}{t}))(x)](s),$$

where the function $p_\alpha(x)$ is defined by

$$p_\alpha(x) = \frac{1}{\Gamma(\alpha)}\sum_{0\leq j<x}(-1)^j\binom{\alpha}{j}(x-j)^{\alpha-1}, \qquad x > 0.$$

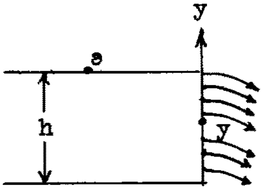
$p_\alpha(x)$ belongs to $L^1(0,\infty)$ and has the Laplace transform $s^{-\alpha}(1-e^{-s})^\alpha$. Thus we have $t^{-\alpha}\Delta_t^\alpha f_\gamma(x) \to f_{\gamma-\alpha}$ for $t \to 0+$.

The following problem, admittedly a bit contrived, serves the purposes of showing how the kernel function of the form $(x-t)^v$ and an integral equation is formulated in the physical sciences. Its frequency deserves consideration for it is not too far fetched to use a backdrop of economics, demand and money flow instead of fluid flow. The integral equation is solved by fractional operations.

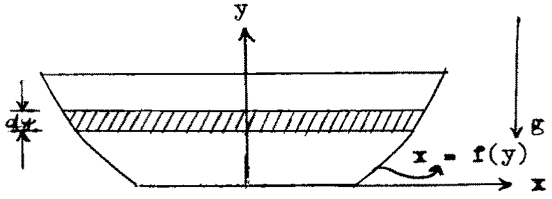
The problem is to determine the shape $f(y)$ of a weir notch, an opening in a dam, in which the volume flow rate of fluid, Q, through the notch is expressed as a function of the height h of the notch, [19]. We first establish the equation

$$Q(h) = c\int_0^h (h-y)^{\frac{1}{2}}f(y)\,dy.$$

Consider front and side views of the notch below:



Side view Fig. 1



Front view Fig. 2

Assuming that Bernoulli's equation can be applied between the points a and y in Fig. 1, [20], we obtain

(37) $$\frac{p_a}{\rho} + gh + \frac{V_a^2}{2} = \frac{p_y}{\rho} + gy + \frac{V_y^2}{2} ,$$

where p_a, p_y, V_a, V_y are the pressures and velocities at points a and y; g is the gravitational acceleration and ρ is the fluid density.

The pressures at a and y are both taken to be nearly atmospheric so that $p_a = p_y$, and the velocity at a is assumed to be negligible ($V_a = 0$) since the fluid behind the notch is slow moving. Thus, (37) becomes

(38) $$gh = gy + V_y^2/2,$$

so that

(39) $$V_y = (2g)^{\frac{1}{2}}(h-y)^{\frac{1}{2}}$$

gives the velocity of the fluid at distance y above the notch floor (x-axis).

The elemental area (shaded region in Fig. 2) is given by

(40) $$dA = 2f(y)\,dy.$$

So, by definition, the elemental volume flow rate through dA is

(41) $$dQ = V_y dA = 2(2g)^{\frac{1}{2}}(h-y)^{\frac{1}{2}}f(y)\,dy.$$

Denoting $2(2g)^{\frac{1}{2}}$ by c and integrating (41) from $y = 0$ to $y = h$ gives the total volume flow rate through the notch:

(42) $$Q(h) = c\int_0^h (h-y)^{\frac{1}{2}}f(y)\,dy.$$

We find f(y) by finding f(h). By the definition of fractional integration (7), Eq. (42) can be written in the form

(43) $$\frac{Q(h)}{c} = \Gamma(\tfrac{3}{2})\ {}_0D_h^{-3/2}\ f(h).$$

Operating on both sides of (43) with ${}_0D_h^{3/2}$ gives the result

(44) $$f(h) = \frac{1}{\Gamma(3/2)}\ {}_0D_h^{3/2}\ \frac{Q(h)}{c} .$$

But ${}_0D_h^{3/2} = {}_0D_h^2\ {}_0D_h^{-\frac{1}{2}}$ where ${}_0D_h^2$ is d^2/dh^2. Denoting $Q(h)/c$ by $g(h)$, we can write (44) as follows:

$$f(h) = \frac{1}{\Gamma(3/2)} \frac{d^2}{dh^2} \frac{1}{\Gamma(\frac{1}{2})} \int_0^h (h-y)^{-\frac{1}{2}} g(y)\,dy. \tag{45}$$

Since $Q(h)/c$ is known, then $g(h)$ and $g(y)$ are known. Then, after evaluating the beta integral on the right above, and taking its second derivative we obtain $f(h)$. Thus, we have $f(y)$.

As a specific example let $g(h) = h^a$. Then (45) yields

$$f(h) = \frac{2}{\pi^{\frac{1}{2}}} \frac{\Gamma(a+1)}{\Gamma(a-\frac{1}{2})} h^{a-3/2}$$

valid for $a > -\frac{1}{2}$. If $a = 7/2$, the weir is shaped like a parabola, and if $a = 3/2$, the weir is a rectangle.

In his communication to this writer, Robert M. Hashway, Warwick, R.I. suggests a similar problem. A fluid reservoir of circular symmetry is to be designed. The minimum and maximum heights of the reservoir above the ground are h and H respectively. The fluid exits through an orifice at H. The time for the fluid to reach a particular height above the ground, say height z, is given by $t(z)$ where $t(H) = 0$. Determine the shape of the reservoir.

We will now consider a generalization of Leibnitz's rule for the *n*th derivative of a product:

$${}_0D_x^v f(x) g(x) = \sum_{n=0}^{\infty} \binom{v}{n}\ {}_0D_x^{(n)} f(x)\ {}_0D_x^{(v-n)} g(x). \tag{46}$$

$D^{(n)}$ is ordinary differentiation and $D^{(v-n)}$ is fractional differentiation. Consider the identity

$$x^{a+b} = x^a x^b$$

Operate on both sides with D^v. Treat the left hand side as a fractional derivative in accord with the definition (21) and treat the right hand side in accord with (46). By equating coefficients of like powers, we get an infinite series of gamma functions:

$$\frac{\Gamma(a+b+1)}{\Gamma(a+b+v+1)} = \Gamma(b+1) \left[\frac{1}{\Gamma(b+v+1)} - \frac{va}{\Gamma(b+v+2)} + \frac{v(v+1)a(a-1)}{2!\,\Gamma(b+v+3)} - \cdots \right].$$

The case for fractional calculus might well lie in the simplicity it offers in the solution of certain integral equations of the Volterra type. Consider the problem of finding $f(x)$ explicitly given the equation

$$xf(x) = \int_0^x (x-t)^{-\frac{1}{2}} f(t)dt.$$

By definition (7) the right hand side above is $\Gamma(\frac{1}{2})\ {}_0D_x^{-\frac{1}{2}} f(x)$.

Omitting subscripts for convenience we have

(48) $$xf(x) = \sqrt{\pi}\ D^{-\frac{1}{2}} f(x).$$

Operating on both sides of the above with $D^{\frac{1}{2}}$ yields

(49) $$D^{\frac{1}{2}}xf(x) = \sqrt{\pi}\ f(x).$$

Apply formula (46) to get

(50) $$xD^{\frac{1}{2}}f(x) + \frac{1}{2}D^{-\frac{1}{2}}f(x) = \sqrt{\pi}\ f(x).$$

Substituting (48) into (50) gives

(51) $$xD^{\frac{1}{2}}f(x) + \frac{xf(x)}{2\sqrt{\pi}} = \sqrt{\pi}\ f(x).$$

We can get an expression for $D^{\frac{1}{2}}f(x)$ by operating on both sides of (48) with D:

(52) $$D[xf(x)] = \sqrt{\pi}\ D^{\frac{1}{2}} f(x),$$

or

(53) $$xf'(x) + f(x) = \sqrt{\pi}\ D^{\frac{1}{2}} f(x).$$

Our objective has been reached when (53) is substituted into (51). We arrive at the ordinary differential equation

$$x^2f'(x) + (\frac{3x}{2} - \pi)f(x) = 0$$

which has the solution

$$f(x) = ke^{-\pi/x}x^{-3/2}.$$

Murray R. Spiegel, author of *Laplace Transforms* and other texts in the *Schaum's* outline series, suggests the following solution to the previous problem.

$$xF(x) = \int_0^x (x-u)^{-\frac{1}{2}} F(u)\,du = x^{-\frac{1}{2}} * F(x).$$

$$-\frac{d}{ds} f(s) = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} f(s) = \frac{\sqrt{\pi}}{s^{\frac{1}{2}}} f(s),$$

$$-f'(s)/f(s) = \sqrt{\pi}\, s^{-\frac{1}{2}}$$

$$-\ln f(s) = 2\sqrt{\pi}\, s^{\frac{1}{2}} + c_1$$

$$f(s) = ce^{-2\sqrt{\pi s}}.$$

$$F(x) = c\mathcal{L}^{-1}\left\{e^{-2\sqrt{\pi s}}\right\} = \frac{2\sqrt{\pi}}{2\sqrt{\pi x^3}} e^{-4\pi/4x}$$

$$= cx^{-3/2}e^{-\pi/x}.$$

No claim can be made that the fractional calculus approach is better than some other approach. However, to paraphrase Theodore Parker Higgins who confided in me that he paraphrased A. Erdélyi, there is a succinctness of notation and simplicity of formulation in the fractional calculus that might suggest a solution to a complicated functional equation that is not readily obtained by other means.

In 1940 and 1941 Erdélyi and Kober investigated properties of a generalization of the Riemann-Liouville and of the Weyl definitions. Professor Sneddon will survey some of these results. The topic of fractional calculus lay relatively dormant from 1941 to the early nineteen sixties, when a modest resurgence began. More papers were published by Erdélyi, Higgins, Mikolás, Al-Bassam, Osler and others in the 1960's and early 1970's. Of particular interest to the applied mathematician in the last decade was the development of some formal techniques for the solution, by means of fractional operations, of dual and triple integral equations that stem from *mixed* boundary value problems of mathematical physics.

The pair of equations

$$\int_0^\infty K(x,t)G(t)f(t)\,dt = g(x) \qquad 0 < x < 1,$$

$$\int_0^\infty K(x,t)f(t)dt = h(x) \qquad x > 1$$

where the kernel K(x,t), G(t), g(x), h(x) are known functions and f(t) is to be determined are known as dual integral equations. The idea is to reduce a specific physical problem to a pair of dual integral equations, for example, in finding an expression for the potential in the field of an electrified disc where different boundary conditions hold over two different parts of the same boundary.

When the problem is such that different conditions hold over three different parts of the same boundary, it is often convenient to determine the solution by constructing a set of triple integral equations. Professor Mikolás will discuss further trends in the theory and applications of fractional calculus in his lectures.

Fractional calculus is old but studied little. Many mathematicians and scientists are unfamiliar with this topic. The wide variety of papers to be presented at this conference will help to fill this void. This conference has several singular purposes, singular being taken in the sense of Sherlock Holmes. One obvious purpose is to popularize the topic in the hope it will induce scientists and mathematicians to include it in their research and curricula. Another purpose is to exchange and impart information which may serve to suggest new areas of research.

Fractional calculus can be categorized as applicable mathematics. The properties and theory of these fractional operators are proper objects of study in their own right. Scientists and applied mathematicians, in the last decade, found the fractional calculus useful in various fields: rheology, quantitative biology, electrochemistry, scattering theory, diffusion, transport theory, probability, potential theory and elasticity. However, many mathematicians and scientists are unfamiliar with this topic possibly because they have not been exposed to its applications. Thus, while the theory of fractional calculus has developed, its use has lagged behind. So, another objective of this conference is to encourage attempts to discover additional formal methods of representing physical phenomena with mathematical models that can be treated with the elegance of fractional calculus.

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[15] I am indebted to Dr. Kenneth S. Miller, Riverside Research Institute, New York City, for this contribution.

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