STABILITY RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO CONTROL PROCESSING

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ABSTRACT

In this paper, stability results of main concern for control theory are given for finite-dimensional linear fractional differential systems. For fractional differential systems in state-space form, both internal and external stabilities are investigated. For fractional differential systems in polynomial representation, external stability is thoroughly examined. Our main qualitative result is that stabilities are guaranteed iff the roots of some polynomial lie outside the closed angular sector $|\arg(\sigma)| \leq \alpha\pi/2$, thus generalizing in a stupendous way the well-known results for the integer case $\alpha = 1$.

1. INTRODUCTION

Fractional differential systems have proved to be useful in control processing for the last two decades (see [21, 22]).

The notion of fractional derivative dates back two centuries; some references that have now become classical were written two decades ago (see [20, 25]). Several authors published reference books on the subject very recently: see [26] for a thorough mathematical study, and [17] for a treatment of linear fractional differential equations which slightly differs from ours in so far as the link between algebraic tools and analytic properties is not being made so clear: in our approach, only *sequential* fractional differential equations are considered, for it is the only way to justify the use of linear algebra properly (see [9, 13, 10]).

The question of stability is of main interest in control theory; it has been addressed in [2, 23, 19]: the conclusion is that no poles of the generalized non-integer system must lie in the closed right-half plane of the Laplace plane. For the particular class of fractional differential systems with integer powers of one fractional order of derivation, this question has been reconsidered and solved in [9, 13]: in this paper, we aim at focusing on this very technical point, and give the most general results on the subject (refined estimates of the aymptotic behaviour of the eigenfunctions in the case of multiple poles have recently been given in [11]).

From a theoretical point of view, questions of controllability and observability of finite-dimensional linear fractional differential systems are also interesting; they have recently been addressed in [16] (see also references therein).

This paper deals with stability properties of linear fractional differential systems, given either in state-space form or in polynomial representation.

The paper is organized as follows: in section 2, we first recall recent results in the theory of fractional differential systems and introduce some notations and definitions used throughout the paper; in section 3, we then define the problem of internal stability for state-space representations, and state theorem 2, which gives an algebraic solution to the internal stability problem, together with refined analytical estimates for the convergence rate (in the stable case); in section 4, we finally introduce the problem of external stability, and state theorem 4 for systems in state-space representation and theorem 5 for systems in polynomial form, which both give algebraic solutions to the external stability problem, and refined analytical estimates for the decay rate of the impulse response, which proves to belong to $L^1(\mathbb{R}^+, \mathbb{R}^{pm})$ when stable.

2. PRELIMINARIES ON FRACTIONAL DIFFERENTIAL SYSTEMS

We now recall the main definitions and results concerning fractional derivative operators in subsection 2.1, and concerning their eigenfunctions in subsection 2.2. We also introduce the state-space form and polynomial representation of linear fractional differential systems in subsection 2.3 and subsection 2.4 respectively.

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2.1. Fractional derivative operators

For more details, the reader can refer to [9, Appendix A], [13, 11], [7, chapter 1, section 5.5] and [26, chapter 2, section 8].

2.1.1. *Fractional integrals*

Notation. We define Y_α, the convolution kernel of order α for fractional integrals:

$$\text{for} \quad \alpha > 0, \quad Y_\alpha(t) \triangleq \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \in L^1_{\text{loc}}(\mathbb{R}^+)$$

where Γ is the well-known Euler Gamma function. The Laplace transform of Y_α is: $\mathcal{L}[Y_\alpha](s) = s^{-\alpha}$ for $\Re e(s) > 0$; we have the important convolution property $Y_\alpha \star Y_\beta$ for $\alpha > 0$ and $\beta > 0$.

With this notation, the fractional integral of order α of a continuous, even $L^1_{\text{loc}}(\mathbb{R}^+)$, causal function f is:

$$I^\alpha f \triangleq Y_\alpha \star f$$

2.1.2. *Fractional derivatives in the sense of distributions*

Notation. We define $Y_{-\alpha}$, the causal distribution — or generalized function in the sense of Schwartz (see [27, chapters II & III] and [7, chapter 1, section 3.2]) — as the unique convolutive inverse of $Y_{+\alpha}$ in the convolution *algebra* $\mathcal{D}'_+(\mathbb{R})$; with the use of δ — the Dirac distribution — which is the neutral element of convolution, this reads:

$$Y_{+\alpha} \star Y_{-\alpha} = \delta$$

The Laplace transform of $Y_{-\alpha}$ is: $\mathcal{L}[Y_{-\alpha}](s) = s^\alpha$ for $\Re e(s) > 0$; we then have the important convolution property $Y_\alpha \star Y_\beta$ for *any* real numbers α, β: the latter property ensures that derivation in the sense of causal distributions is *sequential* in the sense of [17, section VI.4].

With this notation, the fractional derivative of order α of a continuous causal function f is:

$$D^\alpha f \triangleq Y_{-\alpha} \star f$$

2.1.3. *Smooth fractional derivatives*

In order to make this definition tractable from an analytic point of view, it proves useful to define a *smooth* fractional derivation operator for continuous f, with $f' \in L^1_{\text{loc}}(\mathbb{R}^+)$; for $0 < \alpha \leq 1$:

$$d^\alpha f \triangleq D^\alpha f - f(0^+) Y_{1-\alpha} = \int_0^t Y_{1-\alpha}(t-\tau) f'(\tau)\, d\tau$$

Remark 2.1. The difference between D^α and d^α is exactly the same as the one between derivation in the sense of distributions (D^1) and classical derivation (d^1); namely, $D^1 f = d^1 f + f(0^+)\,\delta$.

Remark 2.2. Care must be taken that, contrarily to D^α, d^α does not intrinsically possess the sequentiality property: it really depends on the function under study! Hence, we will work with one basic smooth operator d^α and its successive compositions $(d^\alpha)^{\circ j}$ to ensure the sequentiality property (think, for example, that usual derivative of the second order is nothing but derivative applied twice to the function).

2.2. Eigenfunctions of the fractional derivative operators

Let us now define the eigenfunctions of the previous operators D^α and d^α (see [9, Appendix B], [13, 24, 18] for more details, and especially [11], where an extensive study of these special functions is made, and existing links with generalized Mittag-Leffler functions and their derivatives are examined).

2.2.1. *for fractional derivatives in the sense of distributions*

Notation. We define $\mathcal{E}_\alpha(\lambda, t)$ as the fundamental solution of the operator $D^\alpha - \lambda$, the Laplace transform of which is $(s^\alpha - \lambda)^{-1}$ for $\Re e(s) > a_\lambda$.

It follows that:

$$D^\alpha \mathcal{E}_\alpha(\lambda, t) = \lambda\, \mathcal{E}_\alpha(\lambda, t) + \delta$$

But in the case of multiple root, we need to define the j-th convolution of $\mathcal{E}_\alpha(\lambda, t)$, namely:

Definition 1. For integer j, we define $\mathcal{E}_\alpha^{\star j}(\lambda, t)$ as the fundamental solution of the operator $(D^\alpha - \lambda)^{\circ j}$:

$$\begin{aligned}
\mathcal{E}_\alpha^{\star j}(\lambda, t) &\triangleq \sum_{k=0}^{\infty} C_{j-1+k}^{j-1} \lambda^k \, Y_{(j+k)\alpha} \\
&= t_+^{j\alpha - 1} \sum_{k=0}^{\infty} C_{j-1+k}^{j-1} \frac{(\lambda\, t_+^\alpha)^k}{\Gamma((j+k)\alpha)}
\end{aligned}$$

the Laplace transform of which is $(s^\alpha - \lambda)^{-j}$ for $\Re e(s) > a_\lambda$.

It follows that:

$$(D^\alpha - \lambda)^{\circ j}\ \mathcal{E}_\alpha^{\star j}(\lambda, t) = \delta$$

Remark 2.3. With this extended definition, the previous notation becomes: $\mathcal{E}_\alpha(\lambda, t) = \mathcal{E}_\alpha^{\star 1}(\lambda, t)$.

2.2.2. for smooth fractional derivatives

Notation. For $0 < \alpha \leq 1$, we define $E_\alpha(\lambda\, t_+^\alpha)$ as the eigenfunction of the smooth derivation operator d^α for the eigenvalue λ, with initial value 1, the Laplace transform of which is $s^{\alpha-1}(s^\alpha - \lambda)^{-1}$ for $\Re e(s) > a_\lambda$.

It follows that:

$$d^\alpha E_\alpha(\lambda\, t_+^\alpha) = \lambda\, E_\alpha(\lambda\, t_+^\alpha) \quad \text{with } E_\alpha(0) = 1$$

Remark 2.4. In fact, there is a convolution link between these two special functions, namely: $E_\alpha(\lambda\, t_+^\alpha) = \left(Y_{1-\alpha}(.) \star \mathcal{E}_\alpha(\lambda, .)\right)(t)$.

In the case of multiple root, it then proves useful to define the analogous function of order j, namely:

Definition 2. For integer j and for $0 < \alpha \leq 1$, we define $E_\alpha^j(\lambda, t)$ as:

$$\begin{aligned} E_\alpha^j(\lambda, t) &\triangleq \left(Y_{1-\alpha}(.) \star \mathcal{E}_\alpha^{\star j}(\lambda, .)\right)(t) \\ &= \sum_{k=0}^{\infty} C_{j-1+k}^{j-1}\, \lambda^k\, Y_{1+(j-1+k)\alpha} \\ &= t_+^{(j-1)\alpha} \sum_{k=0}^{\infty} C_{j-1+k}^{j-1} \frac{(\lambda\, t_+^\alpha)^k}{\Gamma(1+(j-1+k)\alpha)} \end{aligned}$$

the Laplace transform of which is $s^{\alpha-1}(s^\alpha - \lambda)^{-j}$ for $\Re e(s) > a_\lambda$.

It follows that:

$$(d^\alpha - \lambda)^{\circ j}\; E_\alpha^j(\lambda, t) = 0, \quad \text{with } j \text{ initial conditions:}$$

$$\left\{ \begin{array}{rcl} (d^\alpha)^{\circ(j-1)}\, E_\alpha^j(\lambda, t)\Big|_{t=0} &=& 1 \\ (d^\alpha)^{\circ(j-2)}\, E_\alpha^j(\lambda, t)\Big|_{t=0} &=& 0 \\ &\vdots& \\ E_\alpha^j(\lambda, t)\Big|_{t=0} &=& 0 \end{array} \right.$$

Remark 2.5. With this extended definition, the previous notation becomes: $E_\alpha(\lambda\, t_+^\alpha) = E_\alpha^1(\lambda, t)$.

Remark 2.6. For $\alpha = 1$, the two eigenfunctions are the same causal *exponential* function, convoluated j times: $E_1^j(\lambda, t) = \mathcal{E}_1^{\star j}(\lambda, t) = \exp(\lambda t)\, Y_j(t)$.

2.3. Fractional differential equations: state-space form

In the sequel, as introduced in [9, Appendix B] and [13, 10], we will consider a system given by the following linear state-space form with finite inner dimension n:

$$\left\{ \begin{array}{rcl} d^\alpha x &=& Ax + Bu \\ y &=& Cx \end{array} \right. , \quad x(0) = x_0 \qquad (1)$$

where $0 < \alpha \leq 1$, $u \in \mathbb{R}^m$ is the control, $x \in \mathbb{R}^n$ is the state, and $y \in \mathbb{R}^p$ is the observation.

Remark 2.7. The *realization* of an input/output linear fractional differential equation naturally leads to a state vector x of the following form: $x' = \left[y \;\; d^\alpha y \;\; (d^\alpha)^{\circ 2} y \;\; \cdots \;\; (d^\alpha)^{\circ(n-1)} y\right]'$. In the case $\alpha = 1/q$, the $(1+kq)$-th component of the initial condition vector x_0 is exactly $y^{(k)}(0)$ for *integer* k, while the other components are set equal to 0 (see [13, 10]).

2.4. Fractional differential equations: polynomial representation

2.4.1. for ordinary differential systems

An *ordinary* input/output relation (with only integer derivatives) can be written in a *polynomial* representation (see e.g. [8, chapter 6], [1, chapter 8]):

$$\left\{ \begin{array}{rcl} P(\sigma)\, \xi &=& Q(\sigma)\, u \\ y &=& R(\sigma)\, \xi \end{array} \right. \qquad (2)$$

where $u \in \mathbb{R}^m$ is the control, $\xi \in \mathbb{R}^{\bar{n}}$ is the partial state, and $y \in \mathbb{R}^p$ is the observation; P, Q, R are polynomial matrices in the variable σ of dimensions $\bar{n} \times \bar{n}$, $\bar{n} \times m$ and $p \times \bar{n}$ respectively; σ can be seen either as the symbol for the usual derivative d^1 or as the Laplace variable s when *all* initial conditions are zero.

Let us briefly recall stability results for polynomial representation (2) (see e.g. [1, section 8.4]).

Property 1. If $\det(P(\sigma)) \neq 0 \quad \forall \sigma, \Re e(\sigma) \geq 0$, then system (2) is bounded–input bounded–output.

Care must be taken that some pole–zero cancellations can occur when P and Q are not left coprime (lack of controllability), or when P and R are not right coprime (lack of observability); then, defining the transfer matrix $H(\sigma)$ of the system, and computing the irreductible form of it:

$$H(\sigma) \triangleq R(\sigma)\, P^{-1}(\sigma)\, Q(\sigma) = \frac{N(\sigma)}{d(\sigma)}$$

where $N(\sigma)$ is a polynomial matrix of dimension $p \times m$ and $d(\sigma)$ is a monic polynomial of minimum degree, we have:

Property 2. System (2) is bounded–input bounded–output iff $d(\sigma) \neq 0 \quad \forall \sigma, \Re e(\sigma) \geq 0$.

2.4.2. for fractional differential systems

In a similar way, a *fractional* input/output relation can be written in a *polynomial* representation of the form (2), where σ can now be seen either as the symbol for the fractional derivative d^α or as s^α, the complex function in the Laplace variable s, when *all* initial conditions are zero.

Remark 2.8. From now on, σ and therefore representation (2) will be considered in the fractional derivative sense.

Remark 2.9. A straightforward representation (2) can be obtained from state-space form (1) by simply taking:

$$P(\sigma) = \sigma I - A, \quad Q(\sigma) = B, \quad R(\sigma) = C$$

We are now interested in stability properties of fractional differential systems given either by (1) or by (2).

3. INTERNAL STABILITY

Following [8, section 6.2] or [1, section 3.1], we propose the definition of internal stability:

Definition 3. The *autonomous* system (1)

$$d^\alpha x = Ax, \quad \text{with } x(0) = x_0$$

is said to be:

- *stable* iff $\forall x_0,\ \exists A,\ \forall t \geq 0,\ ||x(t)|| \leq A$
- *asymptotically stable* iff $\lim_{t\to+\infty} ||x(t)|| = 0$

Theorem 1. *We have the following asymptotic equivalents for $E_\alpha^j(\lambda, t)$ as t reaches infinity:*

- *for $|\arg(\lambda)| \leq \alpha\pi/2$,*

$$E_\alpha^j(\lambda,t) \sim \frac{1}{\alpha(j-1)!}\left\{\left(\frac{d}{d\sigma}\right)^{j-1} e^{\sigma^{1/\alpha}t}\right\}\Bigg|_{\sigma=\lambda}$$

  it has the structure of a polynomial of degree $j-1$ in t, multiplied by $\exp(\lambda^{1/\alpha}t)$.

- *for $|\arg(\lambda)| > \alpha\pi/2$,*

$$E_\alpha^j(\lambda,t) \sim \frac{1}{\Gamma(1-\alpha)}\,(-\lambda)^{-j}\,t^{-\alpha}$$

  which decays slowly towards 0.

The proof of the theorem requires some analytic insight; we give a sketch of it in two steps:

- **step 1**: we compute the inverse Laplace transform of $s^{\alpha-1}(s^\alpha - \lambda)^{-1}$ with a cut along $\mathbb{R}^-$, in order to tackle the multiformity of the complex function under study (see [4]). We are then left with a non zero residue term of polynomial–exponential type when $|\arg(\lambda)| < \alpha\pi$, and an integral term, namely:

$$I_\alpha^j(\lambda,t) = \int_0^{+\infty} w_{\alpha,\lambda}^j(\beta)\,e^{-\beta t}\,d\beta$$

  the latter can be viewed as a *continuous* superposition of purely damped exponentials (it is therefore sometime called the *aperiodic multimode*, see [21]).

- **step 2**: we then perform the asymptotic expansion of the integral term, which naturally proves to be strongly related to the *fractional* power series expansion at $\beta = 0$ of the weight function $w_{\alpha,\lambda}^j$ (see e.g. [6, 5]).

The complete proof of theorem 1 is to be found in [11] (see also references such as [3] therein).

We can now state the main result of this section:

Theorem 2. *The* autonomous *system (1) is:*

- *asymptotically stable iff $|\arg(\text{spec}(A))| > \alpha\pi/2$. In this case, the components of the state decay towards 0 like $t^{-\alpha}$.*

- *stable iff either it is asymptotically stable, or those critical eigenvalues which satisfy $|\arg(\text{spec}(A))| = \alpha\pi/2$ have* geometric *multiplicity one.*

The proof is straightforward. (Let us briefly recall, with e.g. [1, section 2.4], that geometric multiplicity in the minimal polynomial must not be confused with algebraic multiplicity in the characteristic polynomial).

4. EXTERNAL STABILITY

Following [8, section 6.2], [28, section 6.3] or [1, sections 7.4 & 8.4], we recall the definition of external stability:

Definition 4. An input/ouput linear system is externally stable or bounded–input bounded–output iff:

$$\forall u \in L^\infty\left(\mathbb{R}^+, \mathbb{R}^m\right), \quad y = h \star u \in L^\infty\left(\mathbb{R}^+, \mathbb{R}^p\right)$$

which is equivalent to: $h \in L^1\left(\mathbb{R}^+, \mathbb{R}^{p\times m}\right)$.

4.1. Fractional systems in state-space form

Theorem 3. *We have the following asymptotic equivalents for $\mathcal{E}_\alpha^{\star j}(\lambda, t)$ as t reaches infinity:*

- *for* $|\arg(\lambda)| \leq \alpha\pi/2$,

$$\mathcal{E}_\alpha^{\star j}(\lambda,t) \sim \frac{1}{\alpha(j-1)!}\left\{\left(\frac{d}{d\sigma}\right)^{j-1}\sigma^{\frac{1}{\alpha}-1}\,e^{\sigma^{\frac{1}{\alpha}}\,t}\right\}\Bigg|_{\sigma=\lambda}$$

it has the structure of a polynomial of degree $j-1$ *in* t*, multiplied by* $\exp(\lambda^{1/\alpha}t)$.

- *for* $|\arg(\lambda)| > \alpha\pi/2$,

$$\mathcal{E}_\alpha^{\star j}(\lambda,t) \sim \frac{\alpha}{\Gamma(1-\alpha)}\,j(-\lambda)^{-1-j}\,t^{-1-\alpha}$$

which belongs to $L^r\left([1,+\infty[,\mathbb{R}\right)$, $\forall r \geq 1$.

The proof of the theorem requires some analytic insight; as for theorem 1, it is performed in two steps. The complete proof of theorem 3 is to be found in [11].

Remark 4.1. Contrarily to $t \mapsto E_\alpha^j(\lambda,t)$, the function $t \mapsto \mathcal{E}_\alpha^{\star j}(\lambda,t)$ is *not* continous at the origin $t=0$; in fact, the following equivalent can be easily computed:

$$\mathcal{E}_\alpha^{\star j}(\lambda,t) \sim Y_{j\alpha}(t) = \frac{t_+^{j\alpha-1}}{\Gamma(j\alpha)}$$

which proves that it is locally integrable at the origin.

We can now state the first main result of this section:

Theorem 4. *If the triplet* (A,B,C) *is* minimal*, we have the equivalence: system (1) is bounded–input bounded–output iff* $|\arg(\mathrm{spec}(A))| > \alpha\pi/2$.

In general, we have the equivalence: system (1) is bounded–input bounded–output iff the controllable (relatively to the pair (A,B)*) and observable (relatively to the pair* (C,A)*) modes* λ_{min} *of the matrix of dynamics* A *satisfy* $|\arg(\lambda_{min})| > \alpha\pi/2$.

When system (1) is externally stable, each component h_{ij} *of its impulse response behaves like* $t^{-1-\alpha}$ *at infinity; thus* h *belongs to* $L^1\left(\mathbb{R}^+,\mathbb{R}^{pm}\right)$.

The proof is straightforward. In the general part of the theorem, we refer to the algebraic notions of controllability, observability and minimality for time-invariant linear ordinary differential systems in state-space form; see [16] for an analytic justification of the extension of these notions to linear fractional differential systems.

4.2. Fractional systems in polynomial representation

With the help of theorem 4, we can now state the second main result of this section, which is the following necessary and sufficient condition analogous to property 2 for polynomial representation:

Theorem 5. *If the triplet* (P,Q,R) *of polynomial matrices is* minimal*, we have the equivalence: system (2) is bounded–input bounded–output iff* $\det(P(\sigma)) \neq 0 \quad \forall\sigma,\ |\arg(\sigma)| \leq \alpha\pi/2$.

In general, we have the equivalence: system (2) is bounded–input bounded–output iff the minimum degree polynomial $d(\sigma)$ *of the denominator of the irreductible form of the transfer function* $H(\sigma)$ *satisfies* $d(\sigma) \neq 0 \quad \forall\sigma,\ |\arg(\sigma)| \leq \alpha\pi/2$.

The proof is straightforward. In the general part of the theorem, we refer to the algebraic notions of minimality of a triplet of polynomial matrices for time-invariant linear ordinary differential systems in polynomial representation; see [16] for a justification of the extension of these notions to linear fractional differential systems.

5. CONCLUSION

In this paper, we have defined the internal and external stability propreties of linear fractional differential systems of finite dimension, given either in state-space form or in polynomial representation, and we have derived the structural results from both analytic and algebraic point of views.

The main qualitative result of this paper is that stabilities are guaranteed iff the roots of some polynomial (the eigenvalues of the matrix of dynamics or the poles of the transfer matrix) lie *outside* the closed angular sector:

$$|\arg(\sigma)| \leq \alpha\frac{\pi}{2}$$

thus generalizing in a stupendous way the well-known results for the integer case $\alpha = 1$.

The main quantitative result of this paper is that stabilities are not of exponential type: in the case of stable systems, the response to initial conditions decays like $t^{-\alpha}$, whereas the impulse response decays like $t^{-1-\alpha}$, whatever the location of the roots outside the aforementionned sector — a so called *long memory* behaviour typical for fractional differential systems, which does not occur in ordinary differential systems.

Future works will first consist in designing observer based controllers for linear fractional differential systems: see [15] for complete results, which were already suggested in [13]; secondly, they will consist in extending the stability results to abstract linear systems of infinite dimension, as already done on a special case in [9, 14, 12].

6. REFERENCES

[1] B. d'Andréa-Novel et M. Cohen de Lara. *Commande Linéaire des Systèmes Dynamiques.* Collection MASC. Éditions Masson, 1994.

[2] M. Bansard. *Du formalisme de la dérivation généralisée à l'unité Outils Mathématiques du logiciel* CRONE. Thèse de Doctorat, Univ. Bordeaux I, juillet 1993.

[3] H. Bateman. *Higher transcendental functions*, volume 3, chapter XVIII, pages 206–212. McGraw Hill, New York, 1954.

[4] H. Cartan. *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes.* Collection Enseignement des Sciences. Hermann, 1961.

[5] J. Dieudonné. *Calcul infinitésimal.* Collection Méthodes. Hermann, 2nd edition, 1980.

[6] A. Erdélyi. *Asymptotic expansions.* Dover, 1956.

[7] I. M. Guelfand et G. E. Chilov. *Les distributions*, volume 1. Éditions Dunod, 1972.

[8] Th. Kailath. *Linear Systems.* Prentice-Hall, 1980.

[9] D. Matignon. *Représentations en variables d'état de modèles de guides d'ondes avec dérivation fractionnaire.* Thèse de Doctorat, Univ. Paris XI, Novembre 1994.

[10] D. Matignon. Analysis of fractional differential equations by algebraic means. 1996. in preparation

[11] D. Matignon. Fractional derivation operators: eigenfunctions and their asymptotic properties. *Control, Optimisation and Calculus of Variations*, 1996. submitted to COCV.

[12] D. Matignon. Fractional modal decomposition of a boundary-controlled-and-observed infinite-dimensional linear system. Saint Louis, Missouri, June 1996. MTNS. 5 pages accepted to the Mathematical Theory of Networks and Systems symposium.

[13] D. Matignon et B. d'Andréa-Novel. Décomposition modale fractionnaire de l'équation des ondes avec pertes viscothermiques. In *Journées d'Études : les systèmes d'ordre non entier en automatique.* GdR Auto CNRS, April 1995.

[14] D. Matignon and B. d'Andréa-Novel. Spectral and time-domain consequences of an integro-differential perturbation of the wave PDE. In *Third international conference on mathematical and numerical aspects of wave propagation phenomena*, pages 769–771. INRIA/SIAM, Apr. 1995.

[15] D. Matignon and B. d'Andréa-Novel. Control of fractional differential systems. in preparation.

[16] D. Matignon and B. d'Andréa-Novel. Some results on controllability and observability of finite-dimensional fractional differential systems. Lille, France, July 1996. IMACS, IEEE-SMC. (these proceedings).

[17] K. S. Miller and B. Ross. *An intoduction to the fractional calculus and fractional differential equations.* John Wiley & Sons, 1993.

[18] G. Mittag-Leffler. Sur la représentation analytique d'une branche uniforme d'une fonction monogène. *Acta Math.*, 29:101–168, Sept. 1904.

[19] F. Nanot. *Dérivateur généralisé et représentation généralisée des systèmes linéaires.* Thèse de Doctorat, Univ. Bordeaux I, janvier 1996.

[20] K. B. Oldham and J. Spanier. *The Fractional Calculus.* Academic Press, New York and London, 1974.

[21] A. Oustaloup. *Systèmes asservis d'ordre fractionnaire.* Éditions Masson, 1983.

[22] A. Oustaloup. *La commande* CRONE. série Automatique. Éditions Hermès, 1991.

[23] A. Oustaloup. *La dérivation non entière : théorie, synthèse, applications.* série Automatique. Éditions Hermès, 1995.

[24] I. Podlubny. The Laplace transform method for linear differential equations of the fractional order. Technical Report UEF-02-94, Inst. Exp. Phys., Slovak Acad. Sci., Kosice, 1994.

[25] B. Ross, editor. *Fractional Calculus and Its Applications*, number 457 in Lecture Notes in Mathematics, Proc. int. conf. New Haven, June 1974, Springer Verlag.

[26] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional integrals and derivatives: theory and applications.* Gordon & Breach, 1993. original in russian, 1987.

[27] L. Schwartz. *Méthodes mathématiques pour les sciences physiques.* Collection Enseignement des Sciences. Hermann, 1965.

[28] E. D. Sontag. *Mathematical Control Theory. Deterministic Finite Dimensional Systems*, volume 6 of *Texts in Applied Mathematics.* Springer Verlag, 1990.