Theory of Oscillators

Andronov, Vitt and Khaikin

Dover (1966)

Thus all our results admit a simple interpretation. In the cases of a synchronous motor, and of a.c. machines working in parallel this interpretation will be more complicated.

§ 4. ZHUKOVSKII'S PROBLEM OF GLIDING FLIGHT (1891)

In concluding the chapter, we shall consider Zhukovskii's problem [64] on gliding flight taking place in a vertical plane (Fig. 340). We shall introduce the notation: ϑ—the angle of slope of the trajectory, v—the velocity of the centre of gravity of the glider, m—the mass of the glider,

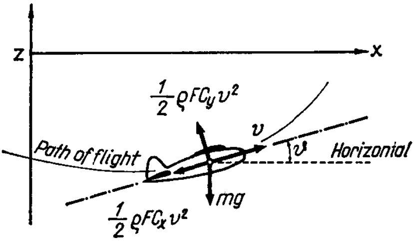

FIG. 340

F—the area of its wings, g—the acceleration of gravity, ϱ—the density of the air and C_x and C_y—the aerodynamical drag and lift coefficients of the bird or glider. Then the equations of motion of the centre of gravity to the glider for tangential and centripetal components of the acceleration are

$$\left.\begin{aligned} m\frac{dv}{dt} &= -mg\sin\vartheta - \frac{1}{2}\varrho F C_x v^2, \\ mv\frac{d\vartheta}{dt} &= -mg\cos\vartheta + \frac{1}{2}\varrho F C_y v^2. \end{aligned}\right\} \tag{7.13}$$

Let the moment of inertia of the glider (with respect to the centre of gravity) be so small, and the stabilizing moment of the forces developed by the tail unit be so large, that we can neglect the variations of the angle of attack of the glider (the angle between its longitudinal axis and the trajectory of its centre of gravity) and assume it to be constant. Then the coefficients C_x and C_y in the equation (7.13) will also be constant. On introducing the new variables

$$v = v_0 y,$$

where $v_0 = (2mg/\varrho FC_y)^{\frac{1}{2}}$ is the velocity of horizontal flight for which the weight of the glider is equalled by the lift force, and

$$t = \frac{v_0}{g}\, t_{\text{new}}.$$

we shall reduce the equations (7.13) to the following non-dimensional form:

$$\left.\begin{aligned} \dot{y} &= -\sin\vartheta - ay^2 = F(\vartheta, y), \\ \dot{\vartheta} &= \frac{-\cos\vartheta + y^2}{y} = \Phi(\vartheta, y), \end{aligned}\right\} \tag{7.14}$$

where a dot denotes differentiation with respect to the new time and

$$a = \frac{C_x}{C_y}.$$

Since the states $(\vartheta+2\pi, y)$ and (ϑ, y) are physically coincident (the right-hand sides of the equations (7.14) are periodic functions of the angle ϑ with period 2π), we take a phase cylinder and plot the quantity y, proportional to the velocity v, along its axis, and the angle around the axis. For forward flight, we can consider phase paths only on the upper half of the cylinder ($y \geqslant 0$). The equation of the integral curves on the cylinder is clearly

$$\frac{dy}{d\vartheta} = \frac{y(\sin\vartheta + ay^2)}{\cos\vartheta - y^2}. \tag{7.15}$$

Note that this equation has the integral curve $y=0$, which is a *singular* phase path of the system (7.14) and corresponds to an *instantaneous tip-over* of the glider into the position $\vartheta = -\pi/2$ as soon as the velocity v (or y) reduces to zero. For according to (7.14) when $y=0$, $\dot{\vartheta} = +\infty$, if $-3\pi/2 < \vartheta < -\pi/2$ and $\dot{\vartheta} = -\infty$ if $-\pi/2 < \vartheta < +\pi/2$.

The appearance of this singular phase path, corresponding to physically impossible somersaulting of the glider at the instant of rest ($v=0$) is caused by assuming a constant angle of attack. This assumption, as is wellknown, is not satisfied for small velocities of motion of the glider for then the stabilizing moment developed by the tail unit is also small. For large velocities this moment ensures a nearly constant angle of attack.

I. Just as in the previous problem, we shall begin our analysis with the conservative case $a=0$ (no forces of resistance). This has been analysed in detail by N. E. Zhukovskii [64, 171]. The differential equation of the integral curves (7.15) has the integral

$$\frac{y^3}{3}-y\cos\vartheta = C(=\text{const}) \tag{7.16}$$

and the three singular points: (1) $\vartheta=0$, $y=+1$; (2) $\vartheta=+\pi/2$, $y=0$ and (3) $\vartheta=-\pi/2$, $y=0$. Only the first of these is a state of equilibrium of equations (7.14) (for $a=0$),

$$\dot{y} = -\sin\vartheta, \quad \dot{\vartheta} = \frac{-\cos\vartheta+y^2}{y}, \tag{7.14a}$$

and corresponds to horizontal flight of the glider with constant velocity $v=v_0$. The other two singular points lie on the singular integral curve $y=0$

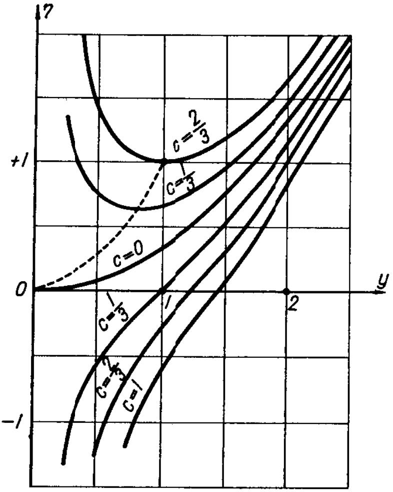

FIG. 341

which we have discussed above and are not states of equilibrium of the system (7.14a) since at these points $\dot{y}\neq 0$.

To construct the integral curves note that equation (7.16) can be solved with respect to θ

$$\vartheta = \pm\arccos\eta, \tag{7.16a}$$

where

$$\eta = \frac{y^2}{3}-\frac{C}{y} \quad (|\eta|\leqslant 1).$$

Fig. 341 shows a family of auxiliary curves $\eta=\eta(y, C)$ for $y<0$[†], and in Figs. 342 and 343 the phase paths on the development of the cylinder and on the phase cylinder itself. The value $C=-\frac{2}{3}$ corresponds to the singular point of the centre type $\vartheta=0$, $y=+1$, and a state of equilibrium of the system of equations (7.14a). The remaining phase paths are closed: the phase paths for which $-\frac{2}{3}<C<0$ encircle the centre but do not encircle the cylinder, while the paths with $C>0$ encircle it[‡]. The first of them corresponds to flight of the glider along "wave-shaped" lines and the second ones to a flight for which the glider performs "dead" loops. The flight paths of the glider are shown in Fig. 344[††].

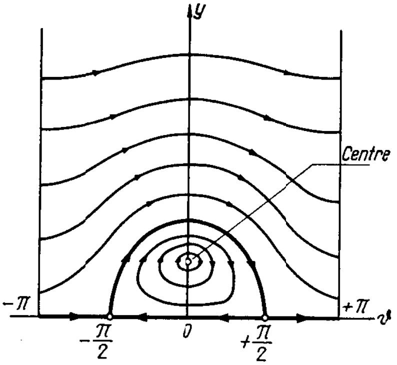

FIG. 342

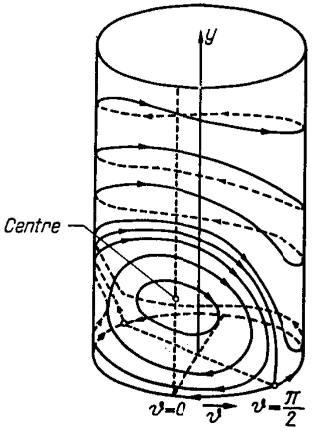

FIG. 343

[†] The curves $\eta=\eta(y, C)$ are monotonic for $C>0$, and have minima that lie on the parabola $y^2=\eta$ for $C<0$, for, as is easily seen

$$\frac{d\eta}{dy}=\frac{2}{3}y+\frac{C}{y^2}=\frac{1}{y}(y^2-\eta);$$

For $C<-\frac{2}{3}$ these curves lie entirely above the straight line $\eta=+1$.

[‡] These two types of closed phase paths are separated by the integral curve $C=0$, consisting of the circle $y=0$ and of the separatrix of the saddle points (the equation of the latter has the form: $\vartheta=\cos^{-1}y^{\frac{2}{3}}$).

[††] The equations of these symmetric flight paths of the glider, in the absence of air resistance, were studied by N. E. Zhukovskii. These paths were later called "phugoids",

II. Let us pass now to the qualitative analysis of the flight taking into account the air resistance ($a > 0$) [166]. As before there is a unique state of equilibrium of the system of equations (7.14); its coordinates will be

$$\left.\begin{aligned} \vartheta_0 &= -\arc\tan a \qquad \left(-\frac{\pi}{2} < \vartheta_0 < 0\right), \\ y_0 &= \frac{1}{\sqrt[4]{1+a^2}} \qquad (0 < y_0 < 1). \end{aligned}\right\}$$

This state of equilibrium of the system (7.14) corresponds to a flight along a *descending* straight line with constant velocity $v < v_0$. On linearizing the equations (7.14) in a neighbourhood of the state of equilibrium (ϑ_0, y_0) it is easily verified that the latter is always *stable* and for sufficiently small a $\left(\text{for } a < 8^{\frac{1}{2}}\right)$ it is a focus.

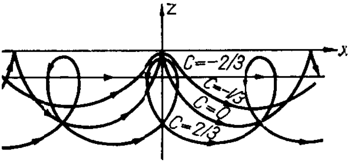

FIG. 344

We shall use Dulac's criterion to prove the absence of closed integral curves (except the circle $y=0$). On taking y as the multiplier $B(\vartheta, y)$ we obtain from the equations (7.14)

$$\frac{\partial}{\partial y}[yF] + \frac{\partial}{\partial \vartheta}[y\Phi] = -3ay^2 \leqslant 0, \tag{7.18}$$

and may be obtained in the following manner. First of all, since $v \sin\vartheta = dz/dt$, we obtain from the equations (7.13), for the case $C_x = 0$,

$$\frac{1}{2}d(v^2) = -g\,dz, \quad \frac{1}{2}v^2 = -gz \quad \text{or} \quad y^2 = -\frac{2gz}{v_0^2} \qquad (\alpha)$$

(the constant of integration is equal to zero, as the height z is measured from the level that corresponds to the velocity $v=0$). Further,

$$\frac{dz}{dx} = \tan\vartheta \quad \text{or} \quad dx = \frac{dz}{\tan\vartheta} = \frac{dz}{R(z, C)} \qquad (\beta)$$

where $R(z, C)$ is a function of z that is obtained if we express $\tan\vartheta$ in terms of z by means of the relations (7.16a) and (α). On integrating the equation (β) by any approximate method (the integral of the right-hand side is not expressible in terms of elementary functions) we have x in terms of z, the graphs of which are shown in Fig. 344.

the equality to zero being true only on the circle $y=0$. Hence there are no closed integral curves (or closed contours consisting of integral curves) that do not encircle the phase cylinder, and there is not more than one closed integral curve encircling the cylinder. Since such a closed integral curve encircling the phase cylinder is the circle $y=0$ (it corresponds, as in the conservative case, to an instantaneous turn-over of the glider at $v=0$), we can assert that the system of equations (7.14) (for $a>0$) does not have any closed phase paths encircling the cylinder on the upper half of the phase cylinder ($y>0$). In other words, this system does not have any periodic oscillations.

The equation of the integral curves (7.15), as in the case $a=0$, has in addition to the state of equilibrium (ϑ_0, y_0) two more singular points of the saddle type $(-\pi/2,0)$ and $(\pi/2,0)$ that are not states of equilibrium of the system (7.14). However, in contrast to the case $a=0$, the separatrix of the saddle point $(-\pi/2,0)$, in the upper half of the phase cylinder, can no longer reach the saddle point $(\pi/2,0)$[†]. Also note that all the circles $y=\text{const}\geqslant a^{-\frac{1}{2}}$ are such that on them $\dot{y}\leqslant 0$. Hence, all the phase paths go from distant regions of the upper half of the cylinder into the region comprised between the circle $y=0$ and $y=a^{-\frac{1}{2}}$ and containing the state of equilibrium (ϑ_0, y_0). We can assert, since there are no closed integral curves (except the circle $y=0$), that *all phase paths approach asymptotically the stable state of equilibrium*, the point (ϑ_0, y_0).

Enough has been discovered to construct the phase portrait on the phase cylinder for the system (7.14) for any $a>0$. Such a picture is shown in Fig. 345.

In the presence of air resistance the glider can have a unique stable flight with constant velocity $v=v_0y_0$ along a descending straight line at an angle ϑ_0 to the horizontal. This flight path can arise for any initial conditions. If the initial velocity of the glider is sufficiently large, then the glider first performs a number of "dead" loops (this number being determined by the initial conditions) and then approaches along a "wave-shaped" curve the final rectilinear flight path. Such a flight path is shown in Fig. 345[‡].

[†] Should this separatrix arrive at the saddle point $(\pi/2, 0)$, then on the phase cylinder there would be two closed contours consisting of integral curves (of the separatrix of the saddle point and of one or other semicircle $y=0$) and not encircling the cylinder, which is impossible since the conditions of Dulac's criterion are verified.

[‡] In contrast to the conservative case $a=0$ the equations of the flight paths in the x, z plane are no longer obtainable by quadratures, since when $a>0$, the integral (7.16) is not true, nor is the equation (α) in the footnote on p. 440.

It is necessary in certain problems to introduce other types of phase surface, differing from the plane and the cylinder, for example a torus or surfaces with many sheets. The systems with a phase surface in the form of a torus exceed the scope of this book. Certain systems with a phase surface of many sheets will be considered in the following chapter†.

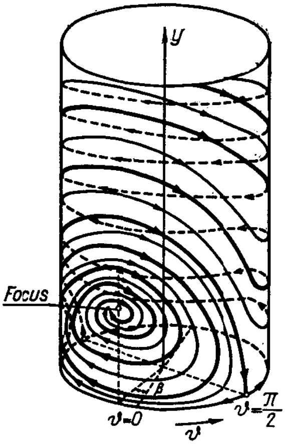

FIG. 345

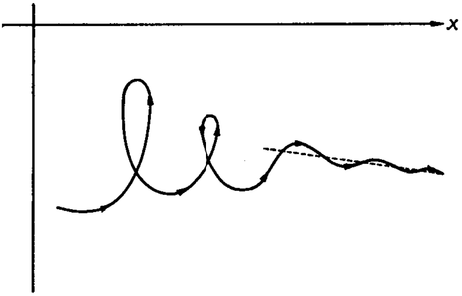

FIG. 346

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† One more dynamic system with a cylindrical phase surface (a simplified model of a steam engine) will be considered in the following chapter (in Section 10).