ELEMENTARY CALCULUS

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PREFACE

The calculus was originally developed using the intuitive concept of an infinitesimal, or an infinitely small number. But for the past one hundred years infinitesimals have been banished from the calculus course for reasons of mathematical rigor. Students have had to learn the subject without the original intuition. This calculus book is based on the work of Abraham Robinson, who in 1960 found a way to make infinitesimals rigorous. While the traditional course begins with the difficult limit concept, this course begins with the more easily understood infinitesimals. It is aimed at the average beginning calculus student and covers the usual three or four semester sequence.

The infinitesimal approach has three important advantages for the student. First, it is closer to the intuition which originally led to the calculus. Second, the central concepts of derivative and integral become easier for the student to understand and use. Third, it teaches both the infinitesimal and traditional approaches, giving the student an extra tool which may become increasingly important in the future.

Before describing this book, I would like to put Robinson's work in historical perspective. In the 1670's, Leibniz and Newton developed the calculus based on the intuitive notion of infinitesimals. Infinitesimals were used for another two hundred years, until the first rigorous treatment of the calculus was perfected by Weierstrass in the 1870's. The standard calculus course of today is still based on the "ε, δ definition" of limit given by Weierstrass. In 1960 Robinson solved a three hundred year old problem by giving a precise treatment of the calculus using infinitesimals. Robinson's achievement will probably rank as one of the major mathematical advances of the twentieth century.

Recently, infinitesimals have had exciting applications outside mathematics, notably in the fields of economics and physics. Since it is quite natural to use infinitesimals in modelling physical and social processes, such applications seem certain to grow in variety and importance. This is a unique opportunity to find new uses for mathematics, but at present few people are prepared by training to take advantage of this opportunity.

Because the approach to calculus is new, some instructors may need additional background material. An instructor's volume, "Foundations of Infinitesimal

33 There are two circles of radius 2 which have centers on the line $x = 1$ and pass through the origin. Find their equations.

34 Find the equation of the circle which passes through the three points $(0, 0)$, $(0, 1)$, $(2, 0)$.

35 Find the equation of the circle which has a diameter whose endpoints are $(-1, 0)$ and $(5, 8)$.

36 Find the equation of the parabola with directrix $y = 0$ and focus $F(2, 2)$.

37 Find the equation of the parabola with directrix $x = -1$ and focus $F(0, 0)$.

38 Find the focus of the parabola with directrix $y = 1$ and vertex $(1, 2)$.

39 Find the equation of the parabola with focus $(-1, -1)$ and vertex $(-1, 0)$.

40 Find the directrix and focus of the parabola $y = x^2$.

41 Find the directrix and focus of the parabola $y = 2x^2 - 3x - 1$.

42 Find the equation of the vertical parabola which passes through the three points $(0, 0)$, $(2, 0)$, $(3, 1)$.

43 Find the equation of the vertical parabola which has the vertex $(3, 2)$ and passes through the point $(4, 0)$.

44 Prove that a parabola of the form $y = ax^2 + c$ is symmetric about the y-axis; that is, if (x, y) lies on the curve then so does $(-x, y)$.

□ 45 Prove that the vertex $(0, c)$ is the lowest point on the parabola $y = ax^2 + c$ if $a > 0$, and is the highest point if $a < 0$. *Hint:* use the fact that $x^2 \geq 0$ for all x.

46 Find the equation of the horizontal ellipse through the points $(5, 0)$ and $(0, 1)$.

47 Find the equation of the vertical ellipse through the points $(\frac{1}{4}, 0)$ and $(0, \frac{1}{3})$.

48 Find the equation of the vertical hyperbola through the points $(0, 1)$, $(1, \sqrt{2})$.

49 Find the equation of the horizontal hyperbola through the points $(2, 0)$, $(4, \sqrt{3})$.

1.4 SLOPE AND VELOCITY; THE HYPERREAL LINE

In Section 1.2 the slope of the straight line through the points (x_1, y_1) and (x_2, y_2) is shown to be the ratio of the change in y to the change in x,

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

If the line has the equation

$$y = mx + b,$$

then the constant m is the slope.

What is meant by the slope of a *curve*? The differential calculus is needed to answer this question, as well as to provide a method of computing the value of the slope. We shall do this in the next chapter. However, to provide motivation, we now describe intuitively the method of finding the slope.

Consider the parabola

$$y = x^2.$$

The slope will measure the direction of a curve just as it measures the direction of a line. The slope of this curve will be different at different points on the x-axis, because the direction of the curve changes.

If (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$ are two points on the curve, then the "average slope" of the curve between these two points is defined as the ratio of the change in y to the change in x,

$$\text{average slope} = \frac{\Delta y}{\Delta x}.$$

This is exactly the same as the slope of the straight line through the points (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$, as shown in Figure 1.4.1.

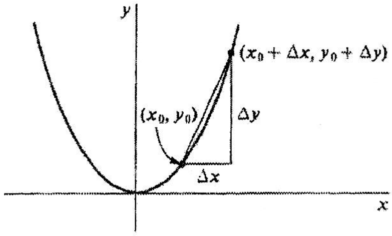


Figure 1.4.1

Let us compute the average slope. The two points (x_0, y_0) and $(x_0 + \Delta x, y_0 + \Delta y)$ are on the curve, so

$$y_0 = x_0^2,$$

$$y_0 + \Delta y = (x_0 + \Delta x)^2.$$

Subtracting,

$$\Delta y = (x_0 + \Delta x)^2 - x_0^2.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x}.$$

This can be simplified,

$$\frac{\Delta y}{\Delta x} = \frac{x_0^2 + 2x_0\,\Delta x + (\Delta x)^2 - x_0^2}{\Delta x}$$

$$= \frac{2x_0\,\Delta x + (\Delta x)^2}{\Delta x} = 2x_0 + \Delta x.$$

Thus the average slope is

$$\frac{\Delta y}{\Delta x} = 2x_0 + \Delta x.$$

Notice that this computation can only be carried out when $\Delta x \neq 0$, because at $\Delta x = 0$ the quotient $\Delta y/\Delta x$ is undefined.

Reasoning in a nonrigorous way, the actual slope of the curve at the point (x_0, y_0) can be found thus. Let Δx be very small (but not zero). Then the point $(x_0 + \Delta x, y_0 + \Delta y)$ is close to (x_0, y_0), so the average slope between these two points is close to the slope of the curve at (x_0, y_0);

$$[\text{slope at } (x_0, y_0)] \text{ is close to } 2x_0 + \Delta x.$$

We neglect the term Δx because it is very small, and we are left with

$$[\text{slope at } (x_0, y_0)] = 2x_0.$$

For example, at the point (0, 0) the slope is zero, at the point (1, 1) the slope is 2, and at the point (−3, 9) the slope is −6. (See Figure 1.4.2.)

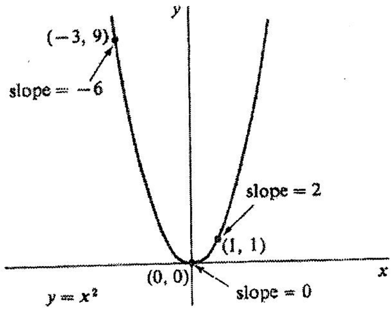


Figure 1.4.2

The whole process can also be visualized in another way. Let t represent time, and suppose a particle is moving along the y-axis according to the equation $y = t^2$. That is, at each time t the particle is at the point t^2 on the y-axis. We then ask: what is meant by the *velocity* of the particle at time t_0? Again we have the difficulty that the velocity is different at different times, and the calculus is needed to answer the question in a satisfactory way. Let us consider what happens to the particle between a time t_0 and a later time $t_0 + \Delta t$. The time elapsed is Δt, and the distance moved is $\Delta y = 2t_0\,\Delta t + (\Delta t)^2$. If the velocity were constant during the entire interval of time, then it would just be the ratio $\Delta y/\Delta t$. However, the velocity is changing during the time interval. We shall call the ratio $\Delta y/\Delta t$ of the distance moved to the time elapsed the "average velocity" for the interval;

$$v_{\text{ave}} = \frac{\Delta y}{\Delta t} = 2t_0 + \Delta t.$$

The average velocity is not the same as the velocity at time t_0 which we are after. As a matter of fact, for $t_0 > 0$, the particle is speeding up; the velocity at time t_0 will be somewhat less than the average velocity for the interval of time between t_0 and $t_0 + \Delta t$, and the velocity at time $t_0 + \Delta t$ will be somewhat greater than the average.

But for a very small increment of time Δt, the velocity will change very little, and the average velocity $\Delta y/\Delta t$ will be close to the velocity at time t_0. To get the velocity v_0 at time t_0, we neglect the small term Δt in the formula

$$v_{\text{ave}} = 2t_0 + \Delta t,$$

and we are left with the value

$$v_0 = 2t_0.$$

When we plot y against t, the velocity is the same as the slope of the curve $y = t^2$, and the average velocity is the same as the average slope.

The trouble with the above intuitive argument, whether stated in terms of slope or velocity, is that it is not clear when something is to be "neglected." Nevertheless, the basic idea can be made into a useful and mathematically sound method of finding the slope of a curve or the velocity. What is needed is a sharp distinction between numbers which are small enough to be neglected and numbers which aren't.

Actually, no real number except zero is small enough to be neglected. To get around this difficulty, we take the bold step of introducing a new kind of number, which is infinitely small and yet not equal to zero.

A number ε is said to be *infinitely small*, or *infinitesimal*, if

$$-a < \varepsilon < a$$

for every positive real number a. Then the only *real* number which is infinitesimal is zero. We shall use a new number system called the *hyperreal numbers*, which contains all the real numbers and also has infinitesimals which are not zero. Just as the real numbers can be constructed from the rational numbers, the hyperreal numbers can be constructed from the real numbers. But, as we did for the real numbers, we shall simply list the properties of the hyperreal numbers with a set of axioms. This will be done later on in this chapter.

First we shall give an intuitive picture of the hyperreal numbers and show how they can be used to find the slope of a curve. The set of all hyperreal numbers is denoted by R^*. Every real number is a member of R^*, but R^* has other elements too. The infinitesimals in R^* are of three kinds: positive, negative, and the real number 0. The symbols $\Delta x, \Delta y, \ldots$ and the Greek letters ε (epsilon) and δ (delta) will be used for infinitesimals. If a and b are hyperreal numbers whose difference $a - b$ is infinitesimal, we say that a is *infinitely close to b*. For example, if Δx is infinitesimal then $x_0 + \Delta x$ is infinitely close to x_0. If ε is positive infinitesimal, then $-\varepsilon$ will be a negative infinitesimal. $1/\varepsilon$ will be an *infinite positive number*, that is, it will be greater than any real number. On the other hand, $-1/\varepsilon$ will be an *infinite negative number*, i.e., a number less than every real number. Hyperreal numbers which are not infinite numbers are called *finite numbers*. Figure 1.4.3 shows a drawing of the hyperreal line. The circles represent "infinitesimal microscopes" which are powerful enough to show an infinitely small portion of the hyperreal line.

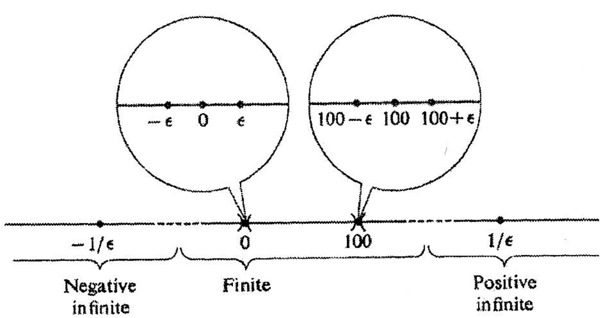


Figure 1.4.3

The set R of real numbers is scattered among the finite numbers. About each real number c is a portion of the hyperreal line composed of the numbers infinitely close to c (shown under an infinitesimal microscope for $c = 0$ and $c = 100$). The numbers infinitely close to 0 are the infinitesimals.

In Figure 1.4.3 the finite and infinite parts of the hyperreal line were separated from each other by a dotted line. Another way to represent the infinite parts of the hyperreal line is with an "infinite telescope" as in Figure 1.4.4. The field of view of an infinite telescope has the same scale as the finite portion of the hyperreal line, while the field of view of an infinitesimal microscope contains an infinitely small portion of the hyperreal line blown up.

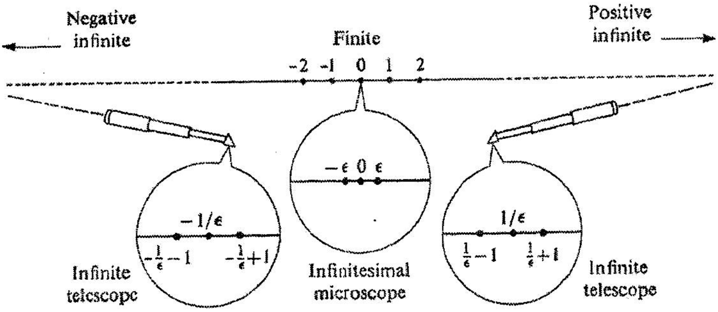


Figure 1.4.4

In discussing the real line we remarked that we have no way of knowing what a line in physical space is really like. It might be like the hyperreal line, the real line or neither. However, in applications of the calculus it is helpful to imagine a line in physical space as a hyperreal line. The hyperreal line is, like the real line, a useful mathematical model for a line in physical space.

The hyperreal numbers can be algebraically manipulated just like the real numbers. Let us try to use them to find slopes of curves. We begin with the parabola $y = x^2$.

Consider a real point (x_0, y_0) on the curve $y = x^2$. Let Δx be either a positive or a negative infinitesimal (but not zero), and let Δy be the corresponding change in y. Then the slope at (x_0, y_0) is *defined* in the following way:

$$[\text{slope at } (x_0, y_0)] = \left[\text{the real number infinitely close to } \frac{\Delta y}{\Delta x}\right].$$

We compute $\frac{\Delta y}{\Delta x}$ as before: $\frac{\Delta y}{\Delta x} = \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x} = 2x_0 + \Delta x.$

This is a hyperreal number, not a real number. Since Δx is infinitesimal, the hyperreal number $2x_0 + \Delta x$ is infinitely close to the real number $2x_0$. We conclude that

$$[\text{slope at } (x_0, y_0)] = 2x_0.$$

The process can be illustrated by the picture in Figure 1.4.5, with the infinitesimal changes Δx and Δy shown under a microscope.

The same method can be applied to other curves. The third degree curve $y = x^3$ is shown in Figure 1.4.6. Let (x_0, y_0) be any point on the curve $y = x^3$, and let Δx be a positive or a negative infinitesimal. Let Δy be the corresponding change in

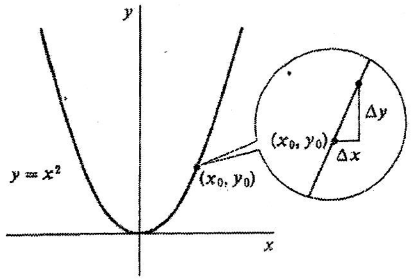


Figure 1.4.5

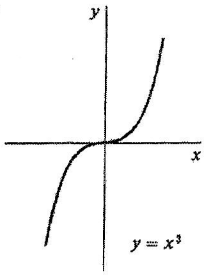


Figure 1.4.6

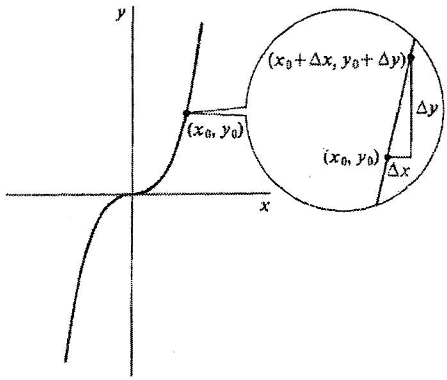


Figure 1.4.7

y along the curve. In Figure 1.4.7, Δx and Δy are shown under a microscope. We again define the slope at (x_0, y_0) by

$$[\text{slope at } (x_0, y_0)] = \left[\text{the real number infinitely close to } \frac{\Delta y}{\Delta x}\right].$$

We now compute the hyperreal number $\dfrac{\Delta y}{\Delta x}$.

$$\begin{aligned}
y_0 &= x_0^3 \\
y_0 + \Delta y &= (x_0 + \Delta x)^3 \\
\Delta y &= (x_0 + \Delta x)^3 - x_0^3 \\
\frac{\Delta y}{\Delta x} &= \frac{(x_0 + \Delta x)^3 - x_0^3}{\Delta x} \\
&= \frac{x_0^3 + 3x_0^2\,\Delta x + 3x_0(\Delta x)^2 + (\Delta x)^3 - x_0^3}{\Delta x} \\
&= \frac{3x_0^2\,\Delta x + 3x_0(\Delta x)^2 + (\Delta x)^3}{\Delta x}
\end{aligned}$$

and finally
$$\frac{\Delta y}{\Delta x} = 3x_0^2 + 3x_0\,\Delta x + (\Delta x)^2.$$

In the next section we shall develop some rules about infinitesimals which will enable us to show that since Δx is infinitesimal,

$$3x_0\,\Delta x + (\Delta x)^2$$

is infinitesimal as well. Therefore the hyperreal number

$$3x_0^2 + 3x_0\,\Delta x + (\Delta x)^2$$

is infinitely close to the real number $3x_0^2$, whence

$$[\text{slope at } (x_0, y_0)] = 3x_0^2.$$

For example, at (0, 0) the slope is zero, at (1, 1) the slope is 3, and at (2, 8) the slope is 12.

We shall return to the study of the slope of a curve in Chapter 2 after we have learned more about hyperreal numbers. From the last example it is evident that we need to know how to show that two numbers are infinitely close to each other. This is our next topic.

1.5 INFINITESIMAL, FINITE, AND INFINITE NUMBERS

Let us summarize our intuitive description of the hyperreal numbers as developed in Section 1.4. Surrounding each real number r, we introduce a collection of hyperreal numbers infinitely close to r; these hyperreal numbers are all finite. The hyperreals infinitely close to zero are called infinitesimals. We must introduce infinite hyperreal numbers as reciprocals of nonzero infinitesimals. The collection of all hyperreal numbers is to satisfy the same elementary algebraic laws as the real numbers. In the rest of this chapter we shall give a more precise description of the hyperreals and develop a facility for computation with them.

As with the real numbers, we begin with a list of basic properties called axioms. There are six axioms for the hyperreal numbers. We give the first three axioms in this section and the others later in the chapter. The complete list of axioms is repeated on the end paper pages. The first two axioms for the hyperreal numbers are exactly the same as the first two axiom groups for the real numbers, with the added statement that every real number is a hyperreal number.

I*. ALGEBRAIC AXIOMS FOR THE HYPERREAL NUMBERS

Every real number is a hyperreal number. If a and b are hyperreal numbers, so are $a + b$, ab, and $a - b$. If a is a hyperreal number and $a \neq 0$, $1/a$ is a hyperreal number.

The Commutative laws, Associative laws, Identity laws, Inverse laws, and Distributive law hold for all hyperreal numbers.

II*. ORDER AXIOMS FOR HYPERREAL NUMBERS

The Transitive law, Trichotomy law, Sum law, and Product law hold for all hyperreal numbers.

For every hyperreal number $a > 0$ and every positive integer n, there is a hyperreal number $b > 0$ such that $b^n = a$.

The positive nth root of a hyperreal number $a > 0$ is written $\sqrt[n]{a}$. Negative hyperreals, like reals, have roots of odd order which are given by

$$\sqrt[n]{-a} = -\sqrt[n]{a}, \qquad n \text{ odd}.$$

All of the additional algebraic and order rules listed on the end paper pages can be proved from Axioms I* and II* and thus are true for the hyperreal numbers as well as the real numbers. Let us work out one example.

Given any two hyperreal numbers b and c, if $b < c$ then $-c < -b$. Here is a proof using Axioms I* and II*. Assume $b < c$. Then:

$$\begin{aligned}
b - b &< c - b && \text{(Sum law)}\\
0 &< c - b && \text{(Inverse law)}\\
0 &< -b + c && \text{(Commutative law)}\\
0 - c &< (-b + c) - c && \text{(Sum law)}\\
-c &< (-b + c) - c && \text{(Identity law)}\\
-c &< -b + (c - c) && \text{(Associative law)}\\
-c &< -b + 0 && \text{(Inverse law)}\\
-c &< -b && \text{(Identity law)}
\end{aligned}$$

So far our axioms have given properties which the reals and hyperreals have in common. The next axiom will make the hyperreals differ from the reals. Before writing this axiom down, we give a careful definition of infinitesimal.

DEFINITION

A hyperreal number b is said to be:

positive infinitesimal *if b is positive but less than every positive real number.*

negative infinitesimal *if b is negative but greater than every negative real number.*

infinitesimal *if b is either positive infinitesimal, negative infinitesimal, or zero.*

By this definition, zero is an infinitesimal. However, no other real numbers are infinitesimal.

III*. INFINITESIMAL AXIOM

There exists a positive infinitesimal hyperreal number.

This axiom gives us at least one infinitesimal which is not zero, namely a positive infinitesimal ε. Starting from ε, we can construct infinitely many other hyperreal numbers which are not real. Here are some examples.

Rule 1 If ε is positive infinitesimal, then $-\varepsilon$ is negative infinitesimal.

Let us prove this from our axioms. We let s be any negative real number, so $-s$ is positive. Then

$$0 < \varepsilon \quad \text{and} \quad \varepsilon < -s,$$
$$-\varepsilon < -0 \quad \text{and} \quad -(-s) < -\varepsilon,$$

so

$$-\varepsilon < 0 \quad \text{and} \quad s < -\varepsilon.$$

Therefore $-\varepsilon$ is negative infinitesimal.

Rule 2 If ε is positive infinitesimal and r is a real number, then $r + \varepsilon$ is hyperreal but not real.

$r + \varepsilon$ cannot be real because the difference of two real numbers is real, while the difference between $r + \varepsilon$ and r,

$$(r + \varepsilon) - r = \varepsilon,$$

is not real.

Rule 3 Suppose ε is positive infinitesimal and a is any positive real number. Then the product $a\varepsilon$ is positive infinitesimal. For example, all of the hyperreal numbers

$$2\varepsilon, \quad \tfrac{1}{3}\varepsilon, \quad 1000\varepsilon, \quad \sqrt{5}\varepsilon$$

are infinitesimal.

This can be seen intuitively from the diagram in Figure 1.5.1; an infinitely thin rectangle of length a has infinitely small area.

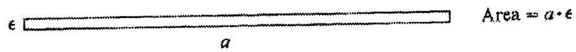


Figure 1.5.1

Again we can give a rigorous proof. $a\varepsilon$ is positive because a and ε are positive. Let r be any positive real number. Then

$$0 < \varepsilon < \frac{r}{a}.$$

Multiplying by a,

$$0 < a\varepsilon < r.$$

Therefore $a\varepsilon$ is positive infinitesimal.

Let us now study the finite and infinite hyperreal numbers (Figure 1.5.2).

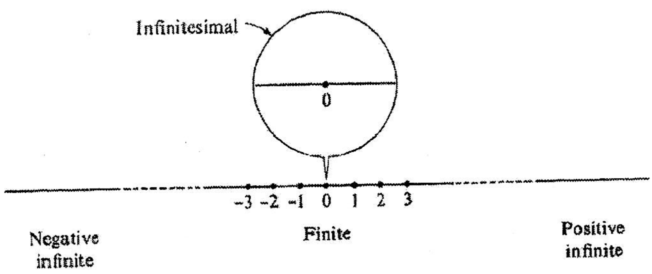


Figure 1.5.2

DEFINITION

A hyperreal number b is said to be:

Finite *if b is between two real numbers.*
Positive infinite *if b is greater than every real number.*
Negative infinite *if b is less than every real number.*

The following remarks can be seen at once.

The only infinitesimal real number is 0.
Every real number is infinite.
Every infinitesimal is finite.

Rule 4 If ε is positive infinitesimal then $1/\varepsilon$ is positive infinite and $-(1/\varepsilon)$ is negative infinite.

From experience we know that reciprocals of small numbers are large, so we intuitively expect $1/\varepsilon$ to be positive infinite. We can again give a proof from our axioms. Let r be any positive real number. Since ε is positive infinitesimal, $0 < \varepsilon < 1/r$. Using a rule from algebra, $1/\varepsilon > r$. This shows that $1/\varepsilon$ is positive infinite.

We have shown that there are positive infinite hyperreal numbers. Although all the Algebraic and Order Axioms are true of the hyperreal numbers, we see now that *the Archimedean Axiom III is not true of the hyperreal numbers.* If H is a positive infinite hyperreal number, then no positive integer is greater than H.

The last few examples are part of a set of rules which can often be used to decide whether a given hyperreal number is infinitesimal, finite but not infinitesimal, or infinite. All of these rules are intuitively plausible and can be proved using the axioms. We shall list these rules in the theorem below.

THEOREM 1 (Rules for Infinitesimal, Finite, and Infinite Numbers)

Assume that ε, δ are infinitesimals, b, c are hyperreal numbers which are finite but not infinitesimal and H, K are infinite hyperreal numbers.

(i) ***Negatives:***

$-\varepsilon$ *is infinitesimal.*
$-b$ *is finite but not infinitesimal.*
$-H$ *is infinite.*

(ii) ***Reciprocals:***

If $\varepsilon \neq 0$, $1/\varepsilon$ *is infinite.*
$1/b$ *is finite but not infinitesimal*
$1/H$ *is infinitesimal.*

(iii) ***Sums:***

$\varepsilon + \delta$ *is infinitesimal.*
$b + \varepsilon$ *is finite but not infinitesimal.*
$b + c$ *is finite (possibly infinitesimal).*
$H + \varepsilon$ *and* $H + b$ *are infinite.*

(iv) ***Products:***

$\delta \cdot \varepsilon$ and $b \cdot \varepsilon$ are infinitesimal.
$b \cdot c$ is finite but not infinitesimal.
$H \cdot b$ and $H \cdot K$ are infinite.

(v) ***Roots:***

If $\varepsilon > 0$, $\sqrt[n]{\varepsilon}$ is infinitesimal.
If $b > 0$, $\sqrt[n]{b}$ is finite but not infinitesimal.
If $H > 0$, $\sqrt[n]{H}$ is infinite.

Some of the rules have already been proved. All the other proofs are similar and will be skipped.

By combining the rules for reciprocals and products we readily obtain the following rules for quotients.

RULES FOR QUOTIENTS

$\frac{\varepsilon}{b}, \frac{\varepsilon}{H}$, and $\frac{b}{H}$ are infinitesimal.

$\frac{b}{c}$ is finite but not infinitesimal.

$\frac{b}{\varepsilon}, \frac{H}{\varepsilon}$, and $\frac{H}{b}$ are infinite, provided $\varepsilon \neq 0$.

For instance, b/H can be written as a product

$$\frac{b}{H} = b \cdot \frac{1}{H}.$$

Since b is finite and $1/H$ is infinitesimal, the product is infinitesimal.

Similarly, b is finite but not infinitesimal and $1/\varepsilon$ is infinite, so the product

$$\frac{b}{\varepsilon} = b \cdot \frac{1}{\varepsilon}$$

is infinite.

Notice that we have given no rule for the following combinations.

$\frac{\varepsilon}{\delta}$, the quotient of two infinitesimals.

$\frac{H}{K}$, the quotient of two infinite numbers.

$H\varepsilon$, the product of an infinite number and an infinitesimal.

$H + K$, the sum of two infinite numbers.

Each of these can be either infinitesimal, finite but not infinitesimal, or infinite, depending on what ε, δ, H and K are. For this reason they are called *indeterminate forms.*

Here are three very different quotients of infinitesimals.

$\frac{\varepsilon^2}{\varepsilon}$ is infinitesimal (equal to ε).

$\frac{\varepsilon}{\varepsilon}$ is finite but not infinitesimal (equal to 1).

$\frac{\varepsilon}{\varepsilon^2}$ is infinite $\left(\text{equal to } \frac{1}{\varepsilon}\right)$.

Table 1.5.1 shows the three possibilities for each indeterminate form.

Table 1.5.1

indeterminate form	Examples: infinitesimal	Examples: finite (equal to 1)	Examples: infinite
$\frac{\varepsilon}{\delta}$	$\frac{\varepsilon^2}{\varepsilon}$	$\frac{\varepsilon}{\varepsilon}$	$\frac{\varepsilon}{\varepsilon^2}$
$\frac{H}{K}$	$\frac{H}{H^2}$	$\frac{H}{H}$	$\frac{H^2}{H}$
$H\varepsilon$	$H \cdot \frac{1}{H^2}$	$H \cdot \frac{1}{H}$	$H^2 \cdot \frac{1}{H}$
$H + K$	$H + (-H)$	$(H + 1) + (-H)$	$H + H$

Here are some examples which show how to use our rules.

EXAMPLE 1 Consider $(b - 3\varepsilon)/(c + 2\delta)$. ε is infinitesimal, so -3ε is infinitesimal, and $b - 3\varepsilon$ is finite but not infinitesimal. Similarly, $c + 2\delta$ is finite but not infinitesimal. Therefore the quotient

$$\frac{b - 3\varepsilon}{c + 2\delta}$$

is finite but not infinitesimal.

The next three examples are quotients of infinitesimals.

EXAMPLE 2 The quotient

$$\frac{5\varepsilon^4 - 8\varepsilon^3 + \varepsilon^2}{3\varepsilon}$$

is infinitesimal, provided $\varepsilon \neq 0$.

The given number is equal to

(1) $$\tfrac{5}{3}\varepsilon^3 - \tfrac{8}{3}\varepsilon^2 + \tfrac{1}{3}\varepsilon.$$

We see in turn that $\varepsilon, \varepsilon^2, \varepsilon^3, \frac{1}{3}\varepsilon, -\frac{8}{3}\varepsilon^2, \frac{5}{3}\varepsilon^3$ are infinitesimal; hence the sum (1) is infinitesimal.

EXAMPLE 3 If $\varepsilon \neq 0$, the quotient

$$\frac{3\varepsilon^3 + \varepsilon^2 - 6\varepsilon}{2\varepsilon^2 + \varepsilon}$$

is finite but not infinitesimal.

Cancelling an ε from numerator and denominator, we get

$$\frac{3\varepsilon^2 + \varepsilon - 6}{2\varepsilon + 1}. \tag{2}$$

Since $3\varepsilon^2 + \varepsilon$ is infinitesimal while -6 is finite but not infinitesimal, the numerator

$$3\varepsilon^2 + \varepsilon - 6$$

is finite but not infinitesimal. Similarly, the denominator $2\varepsilon + 1$, and hence the quotient (2) is finite but not infinitesimal.

EXAMPLE 4 If $\varepsilon \neq 0$, the quotient

$$\frac{\varepsilon^4 - \varepsilon^3 + 2\varepsilon^2}{5\varepsilon^4 + \varepsilon^3}$$

is infinite.

We first note that the denominator $5\varepsilon^4 + \varepsilon^3$ is not zero because it can be written as a product of nonzero factors,

$$5\varepsilon^4 + \varepsilon^3 = \varepsilon \cdot \varepsilon \cdot \varepsilon \cdot (5\varepsilon + 1).$$

When we cancel ε^2 from the numerator and denominator we get

$$\frac{\varepsilon^2 - \varepsilon + 2}{5\varepsilon^2 + \varepsilon}.$$

We see in turn that:

$\varepsilon^2 - \varepsilon + 2$ is finite but not infinitesimal,

$5\varepsilon^2 + \varepsilon$ is infinitesimal,

$\dfrac{\varepsilon^2 - \varepsilon + 2}{5\varepsilon^2 + \varepsilon}$ is infinite.

EXAMPLE 5 $\dfrac{2H^2 + H}{H^2 - H + 2}$ is finite but not infinitesimal.

In this example the trick is to multiply both numerator and denominator by $1/H^2$. We get

$$\frac{2 + 1/H}{1 - 1/H + 2/H^2}.$$

Now $1/H$ and $1/H^2$ are infinitesimal. Therefore both the numerator and denominator are finite but not infinitesimal, and so is the quotient.

In the next theorem we list facts about the ordering of the hyperreals.

THEOREM 2

(i) *Every hyperreal number which is between two infinitesimals is infinitesimal.*

(ii) *Every hyperreal number which is between two finite hyperreal numbers is finite.*

(iii) *Every hyperreal number which is greater than some positive infinite number is positive infinite.*

(iv) *Every hyperreal number which is less than some negative infinite number is negative infinite.*

All the proofs are easy. We prove (iii), which is especially useful. Assume H is positive infinite and $H < K$. Then for any real number r, $r < H < K$. Therefore, $r < K$ and K is positive infinite. ⊣

EXAMPLE 6 If H and K are positive infinite hyperreal numbers, then $H + K$ is positive infinite. This is true because $H + K$ is greater than H.

Our last example concerns square roots.

EXAMPLE 7 If H is positive infinite then, surprisingly,

$$\sqrt{H+1} - \sqrt{H-1}$$

is infinitesimal.

This is shown using an algebraic trick.

$$\sqrt{H+1} - \sqrt{H-1} = \frac{(\sqrt{H+1} - \sqrt{H-1})(\sqrt{H+1} + \sqrt{H-1})}{\sqrt{H+1} + \sqrt{H-1}}$$

$$= \frac{(H+1) - (H-1)}{\sqrt{H+1} + \sqrt{H-1}} = \frac{2}{\sqrt{H+1} + \sqrt{H-1}}.$$

The numbers $H + 1$, $H - 1$, and their square roots are positive infinite, and thus the sum $\sqrt{H+1} + \sqrt{H-1}$ is positive infinite. Therefore the quotient

$$\sqrt{H+1} - \sqrt{H-1} = \frac{2}{\sqrt{H+1} + \sqrt{H-1}},$$

a finite number divided by an infinite number, is infinitesimal.

PROBLEMS FOR SECTION 1.5

In Problems 1–40, assume that: ε, δ are positive infinitesimal, H, K are positive infinite. Determine whether the given expression is infinitesimal, finite but not infinitesimal, or infinite.

1	$76{,}000{,}000\varepsilon$	**2**	$3\varepsilon + 4\delta$
3	$1 + 1/\varepsilon$	**4**	$3\varepsilon^3 - 2\varepsilon^2 + \varepsilon + 1$
5	$1/\sqrt{\varepsilon}$	**6**	ε/H
7	$H/1{,}000{,}000$	**8**	$(3 + \varepsilon)^2 - 9$

9 $(3+\varepsilon)(4+\delta)-12$

10 $\dfrac{1+\varepsilon+3\varepsilon^2}{2-\varepsilon-8\varepsilon^3}$

11 $\dfrac{2\varepsilon^3-\varepsilon^4}{4\varepsilon-\varepsilon^2+\varepsilon^3}$

12 $\dfrac{2\varepsilon^3-\varepsilon^4}{4\varepsilon^3+\varepsilon^4}$

13 $\dfrac{3\varepsilon-4\varepsilon^2}{\varepsilon^2+5\varepsilon^3}$

14 $\dfrac{\sqrt{\varepsilon}+\varepsilon}{\sqrt{\varepsilon}+1}$

15 $\dfrac{1}{\sqrt{\varepsilon}-\varepsilon}$

16 $\dfrac{1}{\varepsilon}\cdot\sqrt{\varepsilon}$

17 $\dfrac{1}{\varepsilon}\cdot 5\varepsilon$

18 $\dfrac{1}{\varepsilon}\cdot\varepsilon^3$

19 $\dfrac{1}{\varepsilon}\left(\dfrac{1}{3+\varepsilon}-\dfrac{1}{3}\right)$

20 $\dfrac{2H+1}{3H+2}$

21 $\dfrac{2H^4+3H-6}{4H^3+5}$

22 $\dfrac{H+4+\varepsilon}{H^2+2\varepsilon}$

23 $\dfrac{H+K}{HK}$

24 $\dfrac{H-K}{H^2+K^2}$

25 H^2-H

26 $(H+\varepsilon)-(H-\varepsilon)$

27 $\left(H+\dfrac{1}{H}\right)^2-\left(H-\dfrac{1}{H}\right)^2$

28 $\left(H+\dfrac{\varepsilon}{H}\right)^2-\left(H-\dfrac{\varepsilon}{H}\right)^2$

29 $\dfrac{\sqrt{4+\varepsilon}-2}{\varepsilon}$

30 $\dfrac{1}{\varepsilon}\left(1-\dfrac{1}{\sqrt{1+\varepsilon}}\right)$

31 $H\left(\sqrt{3+\dfrac{1}{H}}-\sqrt{3}\right)$

32 $\dfrac{\sqrt{H}}{\sqrt{H+1}+\sqrt{H+2}}$

33 $H(\sqrt{H+2}-\sqrt{H})$

34 $\dfrac{1-\sqrt[3]{1+\varepsilon}}{\varepsilon}$

35 $\sqrt[3]{H}-\sqrt[3]{H+1}$

36 $H-\sqrt{H+1}\sqrt{H+2}$

37 $\dfrac{(3+\varepsilon)(4+\delta)-12}{\varepsilon\delta}$

38 $\dfrac{5+\varepsilon}{7+\delta}-\dfrac{5}{7}$

39 $\dfrac{\varepsilon+\delta}{\sqrt{\varepsilon^2+\delta^2}}$
(*Hint:* Assume $\varepsilon \geq \delta$ and divide through by ε.)

40 $\dfrac{H+K}{\sqrt{H^2+K^2}}$

41 In (a)–(f) below, determine which of the two numbers is greater.

(a) ε or ε^2 (b) $\dfrac{1}{\varepsilon^3}$ or $\dfrac{1}{\varepsilon^4}$ (c) H or H^2

(d) ε or $\sqrt{\varepsilon}$ (e) H or $\sqrt{H}$ (f) $\sqrt{H}$ or $\sqrt[3]{H}$

□ 42 Let x, y be positive hyperreal numbers. Can $\dfrac{x}{y}+\dfrac{y}{x}$ be infinite? Finite? Infinitesimal?

□ 43 Let a and b be real. When is $(3\varepsilon^2-\varepsilon+a)/(4\varepsilon^2+2\varepsilon+b)$
(a) infinitesimal?
(b) finite but not infinitesimal?
(c) infinite?

□ 44 Let a and b be real. When is $(aH^2-2H+5)/(bH^2+H-2)$
(a) infinitesimal?
(b) finite but not infinitesimal?
(c) infinite?

1.6 STANDARD PARTS

This section centers on the concept of two infinitely close hyperreal numbers.

DEFINITION

Two hyperreal numbers b and c are said to be ***infinitely close*** *to each other, in symbols $b \approx c$, if their difference $b - c$ is infinitesimal. $b \not\approx c$ means that b is not infinitely close to c.*

Here are three simple remarks.

(1) *If ε is infinitesimal, then $b \approx b + \varepsilon$.* This is true because the difference, $b - (b + \varepsilon) = -\varepsilon$, is infinitesimal.

(2) *b is infinitesimal if and only if $b \approx 0$.* The formula $b \approx 0$ will be used as a short way of writing "b is infinitesimal".

(3) *If b and c are real and b is infinitely close to c, then b equals c.* $b - c$ is real and infinitesimal, hence zero; so $b = c$.

The relation $\approx$ between hyperreal numbers behaves somewhat like equality, but, of course, is not the same as equality. Here are three basic properties of $\approx$.

THEOREM 1

Let a, b and c be hyperreal numbers.

(i) $a \approx a$
(ii) *If $a \approx b$, then $b \approx a$.*
(iii) *If $a \approx b$ and $b \approx c$, then $a \approx c$.*

These properties are useful when we wish to show that two numbers are infinitely close to each other.

The reason for (i) is that $a - a$ is an infinitesimal, namely zero. For (ii), we note that if $a - b$ is an infinitesimal ε, then $b - a = -\varepsilon$, which is also infinitesimal. Finally, (iii) is true because $a - c$ is the sum of two infinitesimals, namely $a - b$ and $b - c$.

THEOREM 2

Assume $a \approx b$. Then

(i) *If a is infinitesimal, so is b.*
(ii) *If a is finite, so is b.*
(iii) *If a is infinite, so is b.*

This can be seen using the rules for sums, because b is equal to $a + \varepsilon$ where ε is infinitesimal; $\varepsilon = b - a$.

The real numbers are sometimes called "standard" numbers, while the hyperreal numbers which are not real are called "nonstandard" numbers. For this reason the real number which is infinitely close to b is called the "standard part" of b. An infinite number cannot have a standard part because it can't be infinitely close to a finite number (Theorem 2). Our fourth axiom for hyperreal numbers states that every finite number has a standard part.

IV*. STANDARD PART AXIOM

Every finite hyperreal number is infinitely close to exactly one real number.

DEFINITION

Let b be a finite hyperreal number. The ***standard part*** *of b, denoted by $st(b)$, is the real number which is infinitely close to b. Infinite hyperreal numbers do not have standard parts.*

Here are some facts which follow at once from the definition.

Let b be a finite hyperreal number.

(1) *$st(b)$ is a real number.*
(2) *$b = st(b) + \varepsilon$ for some infinitesimal ε.*
(3) *If b is real, then $b = st(b)$.*

Our next aim is to develop some skill in computing standard parts. This will be one of the basic methods throughout the Calculus course. The next theorem is the principal tool.

THEOREM 3

Let a and b be finite hyperreal numbers. Then

(i) $st(-a) = -st(a)$.
(ii) $st(a + b) = st(a) + st(b)$.
(iii) $st(a - b) = st(a) - st(b)$.
(iv) $st(ab) = st(a) \cdot st(b)$.
(v) *If* $st(b) \neq 0$, *then* $st(a/b) = st(a)/st(b)$.
(vi) $st(a^n) = (st(a))^n$.
(vii) *If* $a \geq 0$, *then* $st(\sqrt[n]{a}) = \sqrt[n]{st(a)}$.
(viii) *If* $a \leq b$, *then* $st(a) \leq st(b)$.

This theorem gives formulas for the standard parts of the simplest expressions. The proof of the theorem will be a first illustration of how standard parts are computed and will be given in full.

Let r be the standard part of a and s the standard part of b,

$$r = st(a), \qquad s = st(b).$$

Then we can write $\qquad a = r + \varepsilon, \qquad b = s + \delta$

for some infinitesimals ε and δ.

Proof of (i) $st(-a) = -st(a)$.

We show that $-a$ is infinitely close to $-r$, so that $-r$ is the standard part of $-a$. Here is the computation in detail:

$$-a = -(r + \varepsilon) = -r - \varepsilon \approx -r.$$
$$st(-a) = -r.$$

Proof of (ii) $st(a + b) = st(a) + st(b)$.

We compute the standard part of $a + b$ by showing that $a + b$ is infinitely close to $r + s$.

$$\begin{aligned} a + b &= (r + \varepsilon) + (s + \delta) \\ &= (r + s) + (\varepsilon + \delta) \approx r + s. \end{aligned}$$
$$st(a + b) = r + s.$$

The proof of (iii) is just like that of (ii).

Proof of (iv) $st(ab) = st(a) \cdot st(b)$.

We compute the standard part of ab as follows.

$$\begin{aligned} ab &= (r + \varepsilon)(s + \delta) \\ &= rs + r\delta + s\varepsilon + \varepsilon\delta \approx rs. \end{aligned}$$
$$st(ab) = rs.$$

Proof of (v) *If* $st(b) \neq 0$, *then* $st\left(\frac{a}{b}\right) = \frac{st(a)}{st(b)}$.

Our plan is to show that the standard part of a/b is r/s by showing that the difference

$$\frac{a}{b} - \frac{r}{s}$$

is infinitesimal. In the last section we developed rules for showing that certain numbers are infinitesimal. Now we can use what we have learned. First we use algebra to make the computation

$$\begin{aligned} \frac{a}{b} - \frac{r}{s} &= \frac{as - br}{bs} = \frac{(r + \varepsilon)s - (s + \delta)r}{bs} \\ &= \frac{rs + \varepsilon s - sr - \delta r}{bs} = (\varepsilon s - \delta r) \cdot \frac{1}{b} \cdot \frac{1}{s}. \end{aligned}$$

Now using our rules for infinitesimals, we see that $(\varepsilon s - \delta r)$ is infinitesimal. Since $s = st(b) \neq 0$, b is not infinitesimal, and therefore $1/b$ and $1/s$ are finite. Therefore the product

$$\frac{a}{b} - \frac{r}{s} = (\varepsilon s - \delta r) \cdot \frac{1}{b} \cdot \frac{1}{s}$$

is infinitesimal. We have shown that $a/b \approx r/s$, whence $st(a/b) = r/s$.

Proof of (vi) $st(a^n) = (st(a))^n$.

This is proved by using the product rule n times.

$$st(a^n) = st(\underbrace{a \cdot a \cdots a}_{n \text{ times}}) = \underbrace{st(a) \cdot st(a) \cdots st(a)}_{n \text{ times}} = (st(a))^n.$$

Proof of (vii) *If* $a \geq 0$, *then* $st(\sqrt[n]{a}) = \sqrt[n]{st(a)}$.

Remember that for $a > 0$, $\sqrt[n]{a}$ is the positive nth root. Let $b = \sqrt[n]{a}$. Then $b^n = a$, and $b \geq 0$, so $s \geq 0$.

Now by the power formula (vi),

$$r = st(a) = st(b^n) = [st(b)]^n = s^n.$$

Thus s is a nonnegative real number whose nth power is r, and so $s = \sqrt[n]{r}$.

Proof of (viii) *If* $a \leq b$, *then* $st(a) \leq st(b)$.

Let $a \leq b$. Then

$$r + \varepsilon \leq s + \delta, \qquad r \leq s + (\delta - \varepsilon).$$

For any positive real number t, we have

$$\delta - \varepsilon < t, \qquad r \leq s + (\delta - \varepsilon) < s + t,$$

and therefore $r \leq s$.

Often the symbols Δx, Δy, etc. are used for infinitesimals. In the following examples we use the rules in Theorem 3 as a starting point for computing standard parts of more complicated expressions.

EXAMPLE 1 When Δx is an infinitesimal and x is real, compute the standard part of

$$3x^2 + 3x\,\Delta x + (\Delta x)^2.$$

Using the rules in Theorem 3, we can write

$$\begin{aligned} st(3x^2 + 3x\,\Delta x + (\Delta x)^2) &= st(3x^2) + st(3x\,\Delta x) + st((\Delta x)^2) \\ &= 3x^2 + st(3x) \cdot st(\Delta x) + st(\Delta x)^2 \\ &= 3x^2 + 3x \cdot 0 + 0^2 = 3x^2. \end{aligned}$$

EXAMPLE 2 If $st(c) = 4$ and $c \neq 4$, find

$$st\left(\frac{c^2 + 2c - 24}{c^2 - 16}\right).$$

We note that the denominator has standard part 0,

$$st(c^2 - 16) = st(c)^2 - 16 = 4^2 - 16 = 0.$$

However, since $c \neq 4$ the fraction is defined, and it can be simplified by factoring the numerator and denominator,

$$\frac{c^2 + 2c - 24}{c^2 - 16} = \frac{(c + 6)(c - 4)}{(c + 4)(c - 4)} = \frac{c + 6}{c + 4}.$$

Then $$st\left(\frac{c^2 + 2c - 24}{c^2 - 16}\right) = st\left(\frac{c + 6}{c + 4}\right) = \frac{st(c + 6)}{st(c + 4)}$$
$$= \frac{st(c) + 6}{st(c) + 4} = \frac{4 + 6}{4 + 4} = \frac{10}{8}.$$

We now have three kinds of computation available to us. First, there are computations involving hyperreal numbers. In Example 2, the two steps giving

$$\frac{c^2 + 2c - 24}{c^2 - 16} = \frac{c + 6}{c + 4}$$

are computations of this kind. The computations of this first kind are justified by the Algebraic Axioms for hyperreal numbers.

Second, we have computations which involve standard parts. In Example 2, the three steps giving

$$st\frac{c^2 + 2c - 24}{c^2 - 16} = \frac{st(c) + 6}{st(c) + 4}$$

are of this kind. This second kind of computation depends on Theorem 3.

Third there are computations with ordinary real numbers. Sometimes the real numbers will appear as standard parts. In Example 2, the last two steps which give

$$\frac{st(c) + 6}{st(c) + 4} = \frac{10}{8}$$

are computations with ordinary real numbers.

Usually, in computing the standard part of a hyperreal number, we use the first kind of computation, then the second kind, and then the third kind, in that order. We shall give two more somewhat different examples and pick out these three stages in the computations.

EXAMPLE 3 If H is a positive infinite hyperreal number, compute the standard part of

$$c = \frac{2H^3 + 5H^2 - 3H}{7H^3 - 2H^2 + 4H}.$$

In this example both the numerator and denominator are infinite, and we have to use the first type of computation to get the equation into a different form before we can take standard parts.

First stage

$$c = \frac{2H^3 + 5H^2 - 3H}{7H^3 - 2H^2 + 4H} = \frac{H^{-3}\cdot(2H^3 + 5H^2 - 3H)}{H^{-3}\cdot(7H^3 - 2H^2 + 4H)} = \frac{2 + 5H^{-1} - 3H^{-2}}{7 - 2H^{-1} + 4H^{-2}}.$$

Second stage H^{-1} and H^{-2} are infinitesimal, so

$$st(c) = st\left(\frac{2 + 5H^{-1} - 3H^{-2}}{7 - 2H^{-1} + 4H^{-2}}\right) = \frac{st(2 + 5H^{-1} - 3H^{-2})}{st(7 - 2H^{-1} + 4H^{-2})}$$
$$= \frac{st(2) + st(5H^{-1}) - st(3H^{-2})}{st(7) - st(2H^{-1}) + st(4H^{-2})} = \frac{2 + 0 - 0}{7 - 0 + 0}.$$

Third stage $$st(c) = \frac{2+0-0}{7-0+0} = \frac{2}{7}.$$

EXAMPLE 4 If ε is infinitesimal but not zero, find the standard part of

$$b = \frac{\varepsilon}{5-\sqrt{25+\varepsilon}}.$$

Both the numerator and denominator are nonzero infinitesimals.

First stage We multiply both numerator and denominator by $5+\sqrt{25+\varepsilon}$.

$$b = \frac{\varepsilon}{5-\sqrt{25+\varepsilon}} = \frac{\varepsilon(5+\sqrt{25+\varepsilon})}{(5-\sqrt{25+\varepsilon})(5+\sqrt{25+\varepsilon})}$$
$$= \frac{\varepsilon(5+\sqrt{25+\varepsilon})}{25-(25+\varepsilon)} = \frac{\varepsilon(5+\sqrt{25+\varepsilon})}{-\varepsilon}$$
$$= -5-\sqrt{25+\varepsilon}.$$

Second stage $$st(b) = st(-5-\sqrt{25+\varepsilon}) = st(-5) - st(\sqrt{25+\varepsilon})$$
$$= -5-\sqrt{st(25+\varepsilon)} = -5-\sqrt{25}.$$
$$st(b) = -5-\sqrt{25} = -10.$$

EXAMPLE 5 Remember that infinite hyperreal numbers do not have standard parts. Consider the infinite hyperreal number

$$\frac{3+\varepsilon}{4\varepsilon+\varepsilon^2},$$

where ε is a nonzero infinitesimal. The numerator and denominator have standard parts

$$st(3+\varepsilon) = 3, \qquad st(4\varepsilon+\varepsilon^2) = 0.$$

However, the quotient has no standard part. In other words,

$$st\left(\frac{3+\varepsilon}{4\varepsilon+\varepsilon^2}\right) \text{ is undefined.}$$

PROBLEMS FOR SECTION 1.6

Compute the standard parts of the following.

1	$2+\varepsilon+3\varepsilon^2$,	ε infinitesimal
2	$b+2\varepsilon-\varepsilon^2$,	$st(b)=5$, ε infinitesimal
3	$\frac{2-3\varepsilon}{5+4\varepsilon}$,	ε infinitesimal
4	$y^4+2y^2\,\Delta y+\Delta y^3$,	y real, Δy infinitesimal
5	$(x^2+3x\,\Delta x+\Delta x^2)^6$,	x real, Δx infinitesimal
6	$\sqrt{x+\Delta x}+\sqrt{x-\Delta x}$,	x positive real, Δx infinitesimal
7	$\frac{\varepsilon^3-\varepsilon^2+4\varepsilon}{3\varepsilon^2+2\varepsilon-3}$,	ε infinitesimal

8 $\dfrac{\varepsilon^4 - \varepsilon^3 + \varepsilon^2}{2\varepsilon^2}$, $\varepsilon \neq 0$ infinitesimal

9 $\dfrac{4\varepsilon^4 - 3\varepsilon^3 + 2\varepsilon^2}{3\varepsilon^4 - 2\varepsilon^3 + \varepsilon^2}$, $\varepsilon \neq 0$ infinitesimal

10 $(2 + \varepsilon + \delta)(3 - \varepsilon\delta)$, ε, δ infinitesimal

11 $\sqrt{a + \varepsilon}\sqrt{a + \delta}$, $st(a) = 3$, ε, δ infinitesimal

12 $\dfrac{2H + 4}{3H - 6}$, H infinite

13 $\dfrac{6H - 7}{H^2 + 2}$, H infinite

14 $\dfrac{3H^2 - 5H + 2}{H^2 + 1}$, H infinite

15 $\dfrac{H + 1 + \varepsilon}{2H - 1 + 3\varepsilon}$, H infinite, ε infinitesimal

16 $\dfrac{H^4 + 3H^2 + 1}{4H^4 + 2H^2 - 1}$, H infinite

17 $\dfrac{2b^2 + c + 1}{3c^2 + 6b + 1}$, $st(b) = 2$, $st(c) = -1$

18 $\sqrt{b^2 + bc + b - c}$, $st(b) = 3$, $st(c) = 2$

19 $\dfrac{(x + \varepsilon)(y + \varepsilon) - xy}{\varepsilon}$, x, y real, $\varepsilon \neq 0$ infinitesimal

20 $\dfrac{(x + \Delta x)^2 - x^2}{\Delta x}$, x real, $\Delta x \neq 0$ infinitesimal

21 $\dfrac{(x + \Delta x)^3 - x^3}{\Delta x}$, x real, $\Delta x \neq 0$ infinitesimal

22 $\dfrac{1/(a + \varepsilon) - 1/a}{\varepsilon}$, $a \neq 0$ real, $\varepsilon \neq 0$ infinitesimal

23 $\dfrac{b^2 - 25}{b - 5}$, $b \neq 5$ and $st(b) = 5$

24 $\dfrac{4 - a}{2 - \sqrt{a}}$, $a \neq 4$ and $st(a) = 4$

25 $\dfrac{3 - \sqrt{c + 2}}{c - 7}$, $c \neq 7$ and $st(c) = 7$

26 $\dfrac{3 - \sqrt{c + 2}}{c - 7}$, $st(c) = 5$

27 $\dfrac{a^2 - 5a + 6}{a - 3}$, $a \neq 3$ and $st(a) = 3$

28 $\dfrac{2b^2 - b - 6}{b^2 - 3b + 2}$, $b \neq 2$ and $st(b) = 2$

29 $\dfrac{c^2 + 5c + 6}{c^2 + 4c + 3}$, $c \neq -3$ and $st(c) = -3$

30 $\dfrac{\sqrt{25 - \varepsilon} - 5}{\varepsilon}$, $\varepsilon \neq 0$ and ε infinitesimal

31 $\dfrac{1}{\varepsilon}\left(\dfrac{1}{\sqrt{4 + \varepsilon}} - \dfrac{1}{2}\right)$, $\varepsilon \neq 0$ and ε infinitesimal

32 $2H\left(\sqrt{1 + \dfrac{1}{H}} - 1\right)$, H positive infinite

33 $\dfrac{\sqrt{H+1}}{\sqrt{2H}+\sqrt{H-1}}$, H positive infinite

34 $\sqrt{H^2+H+1}-H$, H positive infinite

In the following problems let a, b, a_1, b_1 be hyperreal numbers with $a \approx a_1, b \approx b_1$.

☐ 35 Show that $a + b \approx a_1 + b_1$.
Hint: Put $a_1 = a + \varepsilon, b_1 = b + \delta$, and compute the difference $(a_1 + b_1) - (a + b)$.

☐ 36 Show that if a, b are finite, then $ab \approx a_1 b_1$.

☐ 37 Show that if $a = b = H, a_1 = b_1 = H + 1/H$, then $ab \not\approx a_1 b_1$. (H positive infinite).

1.7 FUNCTIONS OF REAL NUMBERS

The next two sections are about real numbers only. The calculus deals with problems in which one quantity depends on one or more others. For example, the area of a circle depends on its radius. The length of a day depends on both the latitude and the date. The price of an object depends on the supply and the demand. The way in which one quantity depends on one or more others can be described mathematically by a function of one or more variables.

DEFINITION

*A **real function of one variable** is a set f of ordered pairs of real numbers such that for every real number a one of the following two things happens:*

(i) *There is exactly one real number b for which the ordered pair (a, b) is a member of f. In this case we say that $f(a)$ is defined and we write $f(a) = b$. The number b is called the value of f at a.*

(ii) *There is no real number b for which the ordered pair (a, b) is a member of f. In this case we say that $f(a)$ is undefined.*

Thus $f(a) = b$ means that the ordered pair (a, b) is an element of f.

Here is one way to visualize a function. Imagine a black box labeled f as in Figure 1.7.1. Inside the box there is some apparatus which we can't see. On both the left and right sides of the box there is a copy of the real line, called the input line and

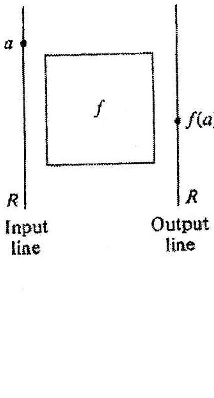


Figure 1.7.1

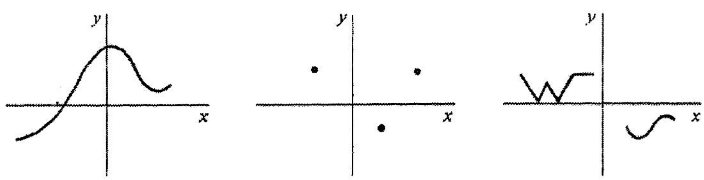
output line, respectively. Whenever we point to a number a on the input line, either one point b will light up on the output line to tell us that $f(a) = b$, or else nothing will happen, in which case $f(a)$ is undefined.

A second way to visualize a function is by drawing its graph. The *graph* of a real function f of one variable is the set of all points $P(x, y)$ in the plane such that $y = f(x)$. To draw the graph, we plot the value of x on the horizontal, or x-axis and the value of $f(x)$ on the vertical, or y-axis. How can we tell whether a set of points in the plane is the graph of some function? By reading the definition of a function again, we have an answer.

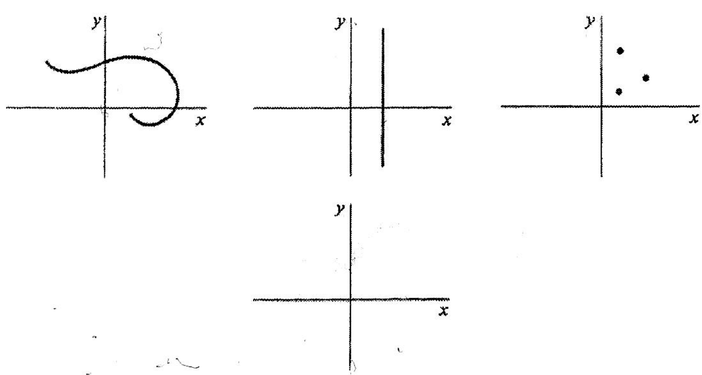
A set of points in the plane is the graph of some function f if and only if for each vertical line one of the following happens:

(1) Exactly one point on the line belongs to the set.
(2) No point on the line belongs to the set.

A vertical line crossing the x-axis at a point a will meet the set in exactly one point (a, b) if $f(a)$ is defined and $f(a) = b$, and the line will not meet the set at all if $f(a)$ is undefined. Try this rule out on the sets of points shown in Figure 1.7.2.



Graphs of functions



Not graphs of functions

Figure 1.7.2

Here are two examples of real functions of one variable. Each function will be described in two ways: the black box approach, where a rule is given for finding the value of the function at each real number, and the graph method, where an equation is given for the graph of the function.