Tensor Analysis Applied to General Relativity

TAM 611 Class Notes by Professor Richard Rand

CORRECTED VERSION

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1. Geodesics

Let the line element in some space be given by

$$ds^2 = g_{ij} dx^i dx^j \tag{1}$$

The geodesic curve $x^i = x^i(s)$ is defined by the calculus of variations problem:

$$\delta \int_1^2 ds = 0 \tag{2}$$

To simplify the computation we use the trick of noting that, from (1),

$$1 = \frac{ds^2}{ds^2} = g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} \tag{3}$$

so that (2) can be written

$$\delta \int_1^2 1 \cdot ds = \delta \int_1^2 g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} ds = 0 \tag{4}$$

The Euler equations corresponding to eq.(4) are:

$$\frac{d}{ds} \frac{\partial F}{\partial \dot{x}^k} - \frac{\partial F}{\partial x^k} = 0 \tag{5}$$

where $F = g_{ij} \dot{x}^i \dot{x}^j$ and where dots represent differentiation with respect to s. Eq.(5) becomes

$$\frac{d}{ds} \left(2 g_{ik} \dot{x}^i \right) - \frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j = 0 \tag{6}$$

Expanding the derivative term and dividing by 2 we get

$$g_{ik} \ddot{x}^i + \frac{\partial g_{ik}}{\partial x^j} \dot{x}^j \dot{x}^i - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j = 0 \tag{7}$$

If we interchange the i and j dummy indices in the middle term we get

$$\frac{\partial g_{ik}}{\partial x^j} \dot{x}^j \dot{x}^i = \frac{\partial g_{jk}}{\partial x^i} \dot{x}^i \dot{x}^j \tag{8}$$

so that eq.(7) becomes

$$g_{ik} \ddot{x}^i + \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} \right) \dot{x}^i \dot{x}^j - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j = 0 \tag{9}$$

Now multiply by g^{km} to get

$$\ddot{x}^m + \frac{1}{2} g^{km} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \dot{x}^i \dot{x}^j = 0 \tag{10}$$

which may be written

$$\ddot{x}^m + \left\{ \begin{matrix} m \\ i\ j \end{matrix} \right\} \dot{x}^i \dot{x}^j = 0 \tag{11}$$

2. <u>General Relativity</u>

Reference: "The Meaning of Relativity" by Albert Einstein

We consider the problem of a particle moving in a $1/r^2$ gravity field, e.g. a planet moving around the sun. Newton's equations can be written

EQUATION A: $$F = -\nabla V = ma \tag{12}$$

where the potential energy $V = -k/r$ satisfies Laplace's equation:

EQUATION B: $$\nabla^2 V = 0 \tag{13}$$

Einstein replaced equations A and B by other statements which were entirely geometrical. Equation A was replaced by the condition that the particle move on a geodesic in the 4 dimensional space-time continuum. Equation B was replaced by conditions which specified the metric tensor g_{ij} of the space-time continuum. We will look at both of these conditions next.

In special relativity the line element ds of the space-time continuum is given by:

$$ds^2 = dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2} \tag{14}$$

which may be written

$$ds^2 = (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{15}$$

where $x^4 = t$, $x^1 = x/c$, $x^2 = y/c$, $x^3 = z/c$, and where c is the speed of light. This metric is related to the Lorentz transformation of special relativity and will not be discussed here. It represents "flat" (or "Minkowskian") space-time. For the curved space-time continuum of general relativity, eq.(15) is generalized to

$$ds^2 = g_{ij} dx^i dx^j \tag{16}$$

where i and j go from 1 to 4. Due to the symmetry of the metric tensor, there are 10 independent g_{ij}'s.

Einstein replaces equation A (12) with the statement that in free space a particle moves on a geodesic in space-time:

$$\ddot{x}^m + \left\{ \begin{matrix} m \\ i\ j \end{matrix} \right\} \dot{x}^i \dot{x}^j = 0 \tag{17}$$

Einstein replaces equation B (13) with 10 nonlinear PDE's on the 10 g_{ij}'s. His PDE's involve the Riemann-Christoffel tensor R^i_{jkm}, to be discussed next.

3. <u>The Riemann-Christoffel tensor</u>

Before we begin the calculation which leads to the definition of this tensor, here is why it is important:
Theorem: A necessary and sufficient condition for a space to be flat is that $R^i_{jkm} = 0$.

The Riemann-Christoffel tensor is related to the second covariant derivative of a vector field. We begin by considering the first covariant derivative of a vector field A:

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \left\{ \begin{array}{c} m \\ i\ j \end{array} \right\} A_m \tag{18}$$

Now we may take the covariant derivative of (18) to get the second covariant derivative of the vector field A:

$$A_{i,jk} = \frac{\partial A_{i,j}}{\partial x^k} - \left\{ \begin{array}{c} m \\ j\ k \end{array} \right\} A_{i,m} - \left\{ \begin{array}{c} m \\ i\ k \end{array} \right\} A_{m,j} \tag{19}$$

Now we substitute (18) into (19) and ignore the second term on the right hand side of (19) because it is symmetric in j and k. The result is:

$$\begin{aligned} A_{i,jk} - A_{i,kj} = & \frac{\partial}{\partial x^k}\left[\frac{\partial A_i}{\partial x^j} - \left\{ \begin{array}{c} m \\ i\ j \end{array} \right\} A_m\right] - \left\{ \begin{array}{c} m \\ i\ k \end{array} \right\}\left[\frac{\partial A_m}{\partial x^j} - \left\{ \begin{array}{c} r \\ m\ j \end{array} \right\} A_r\right] \\ & -\frac{\partial}{\partial x^j}\left[\frac{\partial A_i}{\partial x^k} - \left\{ \begin{array}{c} m \\ i\ k \end{array} \right\} A_m\right] + \left\{ \begin{array}{c} m \\ i\ j \end{array} \right\}\left[\frac{\partial A_m}{\partial x^k} - \left\{ \begin{array}{c} r \\ m\ k \end{array} \right\} A_r\right] \end{aligned} \tag{20}$$

The Riemann-Christoffel tensor R^m_{ijk} is defined in terms of the second covariant derivative (20) as follows:

$$A_{i,jk} - A_{i,kj} = R^m_{ijk} A_m \tag{21}$$

which turns out to give

$$R^m_{ijk} = \frac{\partial}{\partial x^j}\left\{ \begin{array}{c} m \\ i\ k \end{array} \right\} - \frac{\partial}{\partial x^k}\left\{ \begin{array}{c} m \\ i\ j \end{array} \right\} + \left\{ \begin{array}{c} r \\ i\ k \end{array} \right\}\left\{ \begin{array}{c} m \\ r\ j \end{array} \right\} - \left\{ \begin{array}{c} r \\ i\ j \end{array} \right\}\left\{ \begin{array}{c} m \\ r\ k \end{array} \right\} \tag{22}$$

Einstein's 10 field equations on the 10 $g'_{ij}s$ (which replace equation B (13) in Newton's equations) are obtained by contraction of the Riemann-Christoffel tensor:

$$R^m_{ijm} = 0 \tag{23}$$

4. Newton's equations as a first approximation

In order to see the relation between Newton's equations and general relativity, we need to examine the general relativistic equations (11) and (23) in the limit that:
i) $c >> 1$, and
ii) g_{ij} is nearly Minkowskian.
Assumption ii) gives us that

$$ds^2 \approx dt^2 - \frac{dx^2}{c^2} - \frac{dy^2}{c^2} - \frac{dz^2}{c^2} = (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

whereupon assumption i) gives us that

$$ds \approx dt = dx^4$$

The geodesic equation (11) becomes simplified because, by assumption i), $\dot{x}^1$, $\dot{x}^2$ and $\dot{x}^3$ are all small compared to $\dot{x}^4 \approx 1$:

$$\ddot{x}^m + \left\{ \begin{array}{c} m \\ 4\ 4 \end{array} \right\} (1)(1) = 0 \tag{24}$$

where (compare eqs.(10) and (11))

$$\left\{ \begin{array}{c} m \\ 4\ 4 \end{array} \right\} = \frac{1}{2} g^{km} \left(\frac{\partial g_{4k}}{\partial x^4} + \frac{\partial g_{4k}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^k} \right) \tag{25}$$

Now we make the further assumption that
iii) the field is static, so that the g_{ij} do not depend on time $t = x^4$. This makes the partial derivatives with respect to x^4 vanish in eq.(25), giving

$$\left\{ \begin{array}{c} m \\ 4\ 4 \end{array} \right\} = -\frac{1}{2} g^{km} \frac{\partial g_{44}}{\partial x^k} \approx \frac{1}{2} \frac{\partial g_{44}}{\partial x^m} \tag{26}$$

where the last approximation is due to assumption ii). Thus Einstein's geodesic equation of motion (11) becomes

$$\ddot{x}^m + \frac{1}{2} \frac{\partial g_{44}}{\partial x^m} = 0 \tag{27}$$

which agrees with Newton's equation A (12) if we identify

$$g_{44} = \frac{2V}{m} \tag{28}$$

Next let us consider Einstein's field equations (23):

$$R^m_{ijm} = \frac{\partial}{\partial x^j} \left\{ \begin{array}{c} m \\ i\ m \end{array} \right\} - \frac{\partial}{\partial x^m} \left\{ \begin{array}{c} m \\ i\ j \end{array} \right\} + \left\{ \begin{array}{c} r \\ i\ m \end{array} \right\} \left\{ \begin{array}{c} m \\ r\ j \end{array} \right\} - \left\{ \begin{array}{c} r \\ i\ j \end{array} \right\} \left\{ \begin{array}{c} m \\ r\ m \end{array} \right\} = 0 \tag{29}$$

Assumption ii) leads to the conclusion that the last two terms are small of second order:

$$R^m_{ijm} \approx \frac{\partial}{\partial x^j} \left\{ \begin{array}{c} m \\ i\ m \end{array} \right\} - \frac{\partial}{\partial x^m} \left\{ \begin{array}{c} m \\ i\ j \end{array} \right\} = 0 \tag{30}$$

Consider in particular eq.(30) for $i = j = 4$:

$$R^m_{44m} \approx \frac{\partial}{\partial x^4} \left\{ \begin{array}{c} m \\ 4\ m \end{array} \right\} - \frac{\partial}{\partial x^m} \left\{ \begin{array}{c} m \\ 4\ 4 \end{array} \right\} = 0 \tag{31}$$

The first term in eq.(31) is zero by assumption iii) (static field), giving

$$\frac{\partial}{\partial x^1} \left\{ \begin{array}{c} 1 \\ 4\ 4 \end{array} \right\} + \frac{\partial}{\partial x^2} \left\{ \begin{array}{c} 2 \\ 4\ 4 \end{array} \right\} + \frac{\partial}{\partial x^3} \left\{ \begin{array}{c} 3 \\ 4\ 4 \end{array} \right\} = 0 \tag{32}$$

Using eq.(26) in eq.(32), we obtain

$$\nabla^2 g_{44} = 0 \tag{33}$$

which with eq.(28) gives Newton's equation B (13), $\nabla^2 V = 0$.

5. Precession of Mercury

In this section we omit the assumptions i),ii) and iii) of the previous section. Schwarzschild in 1916 found an exact solution of the field equations in spherical coordinates r, ϕ (=longitude), and θ (=colatitude):

$$ds^2 = \left(1 - \frac{R}{r}\right) dt^2 - \frac{1}{1 - \frac{R}{r}} \frac{dr^2}{c^2} - r^2 \frac{d\theta^2}{c^2} - r^2 \sin^2\theta \frac{d\phi^2}{c^2} \tag{34}$$

where R is a constant of integration. (Incidentally, the singularity at $r = R$ in eq.(34) is associated with "black holes".)

Using the g_{ij} of eq.(34) in the geodesic equation (11), and setting $u = 1/r$ and θ=$\pi/2$ (for planar motion), there results the following equation on the orbit of a particle moving around the sun:

$$\frac{d^2u}{d\phi^2} + u = \frac{Rc^2}{2h^2} + \frac{3}{2} R u^2 \tag{35}$$

where h is a constant of integration. The first three terms of eq.(35) are the usual Newtonian equations. The corresponding linear ODE is a simple harmonic oscillator and has the solution

$$u = A + B\cos\phi \quad \Rightarrow \quad r = \frac{1}{A + B\cos\phi} \tag{36}$$

which is the equation of an ellipse. The effect of general relativity is to add the fourth term of eq.(35). The ODE is now nonlinear but can be solved. The resulting solution predicts that the ellipse (36) will precess, in agreement with observations made on the planet Mercury.