Two Variable (or rather Three Variable) Method to $O(e^2)$

R.Rand, June 10, 2012

Example

$$x'' + x = e(x^2 + xx') + e^2 \mu x', \qquad e << 1 \tag{1}$$

Set

$$\xi = t, \quad \eta = et, \quad \zeta = e^2 t \tag{2}$$

Expand

$$x = x_0 + e x_1 + e^2 x_2 + \cdots \tag{3}$$

Then

$$x' = \frac{dx}{d\xi}\frac{d\xi}{dt} + \frac{dx}{d\eta}\frac{d\eta}{dt} + \frac{dx}{d\zeta}\frac{d\zeta}{dt} = \frac{dx}{d\xi} + e\frac{dx}{d\eta} + e^2\frac{dx}{d\zeta} \tag{4}$$

and

$$x'' = \frac{d^2x}{d\xi^2} + 2\,\frac{d^2x}{d\eta d\xi}\,e + \left(2\,\frac{d^2x}{d\xi d\zeta} + \frac{d^2x}{d\eta^2}\right) e^2 + \cdots \tag{5}$$

Plug everything into (1), giving:

$$Lx_0 = \frac{d^2\mathrm{x}_0}{d\xi^2} + \mathrm{x}_0 = 0 \tag{6}$$

$$Lx_1 + 2\,\frac{d^2\mathrm{x}_0}{d\eta d\xi} = {\mathrm{x}_0}^2 + \frac{d\mathrm{x}_0}{d\xi}\,\mathrm{x}_0 \tag{7}$$

$$Lx_2 + 2\,\frac{d^2\mathrm{x}_1}{d\eta d\xi} + 2\,\frac{d^2\mathrm{x}_0}{d\xi d\zeta} + \frac{d^2\mathrm{x}_0}{d\eta^2} = \mathrm{x}_0\,\frac{d\mathrm{x}_1}{d\xi} + \frac{d\mathrm{x}_0}{d\xi}\,\mathrm{x}_1 + 2\,\mathrm{x}_0\,\mathrm{x}_1 + \mu\,\frac{d\mathrm{x}_0}{d\xi} + \mathrm{x}_0\,\frac{d\mathrm{x}_0}{d\eta} \tag{8}$$

(6) gives

$$x_0 = a\,\cos\xi + b\,\sin\xi, \qquad a = a(\eta,\zeta),\ b = b(\eta,\zeta) \tag{9}$$

Substitution into (7) and removal of secular terms gives the slow flow

$$\frac{da}{d\eta} = 0, \qquad \frac{db}{d\eta} = 0 \quad \Rightarrow \quad a = a(\zeta), \quad b = b(\zeta) \tag{10}$$

and we obtain:

$$\mathrm{x}_1 = -\frac{(b^2 + 2\,a\,b - a^2)\,\sin(2\,\xi) + (-b^2 + 2\,a\,b + a^2)\,\cos(2\,\xi) - 3\,b^2 - 3\,a^2}{6} + d\,\sin\xi + c\,\cos\xi \tag{11}$$

where $c = c(\eta, \zeta), \quad d = d(\eta, \zeta)$. Substitution into (8) and removal of secular terms gives the slow flow:

$$\frac{da}{d\zeta} = \frac{a\,\mu}{2} - \frac{dc}{d\eta} - \frac{11\,b^3}{24} + \frac{a\,b^2}{8} - \frac{11\,a^2\,b}{24} + \frac{a^3}{8} \tag{12}$$

$$\frac{db}{d\zeta} = \frac{b\,\mu}{2} - \frac{dd}{d\eta} + \frac{b^3}{8} + \frac{11\,a\,b^2}{24} + \frac{a^2\,b}{8} + \frac{11\,a^3}{24} \tag{13}$$

From eq.(10), every term in (12),(13) is independent of η except for $dd/d\eta$ and $dc/d\eta$. These two terms must therefore be constants, and we take them as zero for convenience. Transforming to polar coordinates,

$$a = r\cos\theta, \qquad b = r\sin\theta \tag{14}$$

we obtain:

$$\frac{dr}{d\zeta} = \frac{r^3}{8} + \frac{\mu\,r}{2} \tag{15}$$

$$\frac{d\theta}{d\zeta} = \frac{11\,r^2}{24} \tag{16}$$

At steady state,

$$r = 2\sqrt{-\mu} \tag{17}$$

$$\theta = \frac{11 r^2}{24}\zeta = -\frac{11}{6}\mu e^2 t \tag{18}$$

which gives

$$x_0 = r\cos(\xi - \theta) = 2\sqrt{-\mu}\cos(t + \frac{11}{6}\mu e^2 t)) \tag{19}$$

which agrees with the result obtained by using Lindstedt's method, see my online Notes, p.24.

Lindstedt's method sets $\tau = (1 + k_2 e^2)t$ and obtains $x_0 = A\cos\tau$ where $A = 2\sqrt{-\mu}$ and $k_2 = \frac{11}{6}\mu$, giving the result (19).

IMPORTANT NOTE: Although the three variable method gives the same steady state as Lindstedt's method, the latter uses a time stretch parameter k_2 whereas the former effectively sets k_2 to zero.