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On the Stability of the Vibrations of Two Coupled Particles in the Plane

The stability of the vibrations of two identical particles constrained to a plane and restrained by three identical linear springs with initial stress is studied by uncoupling the first variational equations and applying Floquet theory and perturbations. A new stability parameter is introduced which indicates stability by virtue of its sign.

Introduction

THE STABILITY of the vibrations of a single particle constrained to a plane and restrained by two linear springs with initial stress has been studied by Yang and Rosenberg [1, 2],[1] and by Rand and Tseng [3, 4]. Rosenberg and Atkinson [5] and Rosenberg [6] have investigated the stability of the vibrations of two particles constrained to a line and restrained by three nonlinear springs.

This paper extends the foregoing work by considering the stability of the vibrations of two particles constrained to a plane and restrained by three linear springs with initial stress.

Formulation

Consider two identical particles constrained to move in the same plane and restrained by three identical linear springs with initial stress, as in Fig. 1. Choose the unit of mass such that the mass of each of the particles is unity, the unit of length such that the distance between the particles is unity when the system is in equilibrium, and the unit of time such that the spring constant of each spring is unity.

Let each spring have the original length L. Clearly,

$$L > 0 \tag{1}$$

Let T be the initial tension in the springs when the system is at rest. Then

$$T = K(1 - L) = 1 - L \tag{2}$$

and from (1)

$$T < 1 \tag{3}$$

Let the coordinates of the first particle be (x_1,y_1), measured from the equilibrium position as in Fig. 1. Similarly, let the coordinates of the other particle be (x_2,y_2). Then the equations of motion of the particles become

$$\ddot{x}_1 = -[(1 + x_1)(d_1 - L)/d_1] + [(x_2 - x_1 + 1)(d_2 - L)/d_2] \tag{4}$$

$$\ddot{y}_1 = -[y_1(d_1 - L)/d_1] + [(y_2 - y_1)(d_2 - L)/d_2] \tag{5}$$

$$\ddot{x}_2 = [(1 - x_2)(d_3 - L)/d_3] - [(x_2 - x_1 + 1)(d_2 - L)/d_2] \tag{6}$$

$$\ddot{y}_2 = -[y_2(d_3 - L)/d_3] - [(y_2 - y_1)(d_2 - L)/d_2] \tag{7}$$

where

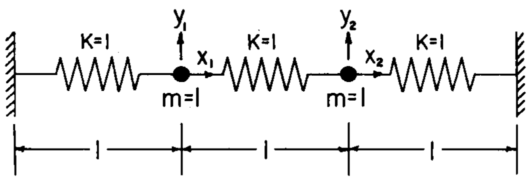

Fig. 1 Two coupled particles in the plane

$$d_1^2 = (1 + x_1)^2 + y_1^2$$
$$d_2^2 = (x_2 - x_1 + 1)^2 + (y_2 - y_1)^2$$
$$d_3^2 = (1 - x_2)^2 + y_2^2$$

and where dots represent differentiation with respect to time t.

If $y_1 \equiv y_2 \equiv 0$, two of the equations of motion are satisfied identically, while the other two become

$$\ddot{x}_1 = -2x_1 + x_2 \tag{8}$$

$$\ddot{x}_2 = -2x_2 + x_1 \tag{9}$$

A possible motion of the system, the first x-mode, is then

$$x_1(t) = x_2(t) = \xi(t) = e \cos t \tag{10}$$

$$y_1(t) \equiv y_2(t) \equiv 0 \tag{11}$$

where, to prevent collisions of the particles with the supports,

$$0 \leq e < 1 \tag{12}$$

Another possible motion, the second x-mode, is

$$x_1(t) = -x_2(t) = \eta(t) = e \cos \sqrt{3}\, t \tag{13}$$

$$y_1(t) \equiv y_2(t) \equiv 0 \tag{14}$$

where, to prevent collisions of the particles with each other,

$$0 \leq e < 1/2 \tag{15}$$

It is desired to investigate the stability of the x-modes.

Stability Analysis

The first variational equations corresponding to either x-mode are

$$\delta\ddot{x}_1 = -2\delta x_1 + \delta x_2 \tag{16}$$

$$\delta\ddot{x}_2 = -2\delta x_2 + \delta x_1 \tag{17}$$

$$\delta\ddot{y}_1 = -[\delta y_1(1 + x_1 - L)/(1 + x_1)] + [(\delta y_2 - \delta y_1)(x_2 - x_1 + 1 - L)/(x_2 - x_1 + 1)] \tag{18}$$

$$\delta\ddot{y}_2 = -[\delta y_2(1 - x_2 - L)/(1 - x_2)] - [(\delta y_2 - \delta y_1)(x_2 - x_1 + 1 - L)/(x_2 - x_1 + 1)] \tag{19}$$

In order for an x-mode to be stable, the differential equations

[1] Numbers in brackets designate References at end of paper.

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(16)–(19) must be stable. (An equation is said to be stable if all of its solutions are bounded for all $t > 0$, and unstable if an unbounded solution exists.) Equations (16) and (17) are of the same form as the equations governing the x-modes, equations (8) and (9). Hence all solutions to (16) and (17) are bounded, and (16) and (17) are stable.

Equations (18) and (19) may be written in the matrix form

$$\ddot{v} + Av = 0 \tag{20}$$

where

$$v = \begin{bmatrix} \delta y_1 \\ \delta y_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a(t) & b(t) \\ b(t) & c(t) \end{bmatrix}$$

and where, for the first x-mode,

$$a(t) = T + [(T + \xi)/(1 + \xi)]$$
$$b(t) = -T$$
$$c(t) = T + [(T - \xi)/(1 - \xi)]$$

while for the second x-mode,

$$a(t) = c(t) = [(T + \eta)/(1 + \eta)] + [(T - 2\eta)/(1 - 2\eta)]$$
$$b(t) = -(T - 2\eta)/(1 - 2\eta)$$

Equation (20) represents two coupled Hill's equations. Hsu [7] has discussed the uncoupling of such systems of equations.

Let the matrix B be defined by, for the first x-mode

$$B = \begin{bmatrix} 1 & 1 \\ \gamma + (\gamma^2 + 1)^{1/2} & \gamma - (\gamma^2 + 1)^{1/2} \end{bmatrix}$$

where

$$\gamma = (c - a)/2b$$

and for the second x-mode

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be defined by

$$v = Bu \tag{21}$$

Then it is found that

$$\ddot{u}_1 + \lambda_1 u_1 = 0 \tag{22}$$

and

$$\ddot{u}_2 + \lambda_2 u_2 = 0 \tag{23}$$

where

$$2\lambda_{1,2} = a + c \pm [(a - c)^2 + 4b^2]^{1/2}$$

For both x-modes the elements of B are bounded functions of t. Therefore, from (21), the stability of u_1 and u_2 is equivalent to the stability of δy_1 and δy_2.

The equations governing stability are, then, for the first x-mode,

$$(1 - \xi^2)\ddot{u}_1 + \{2T - (T + 1)\xi^2 + [(T - 1)^2\xi^2 + T^2(1 - \xi^2)^2]^{1/2}\}u_1 = 0 \tag{24}$$

and

$$(1 - \xi^2)\ddot{u}_2 + \{2T - (T + 1)\xi^2 - [(T - 1)^2\xi^2 + T^2(1 - \xi^2)^2]^{1/2}\}u_2 = 0 \tag{25}$$

while for the second x-mode,

$$(1 - \eta - 2\eta^2)\ddot{u}_1 + 3(T - \eta - 2\eta^2)u_1 = 0 \tag{26}$$

and

$$(1 + \eta)\ddot{u}_2 + (T + \eta)u_2 = 0 \tag{27}$$

(The leading coefficients in (24)–(27) cannot vanish because of (12) and (15).)

Equations (24)–(27) are all particular cases of Hill's equation,

$$\ddot{u} + f(t)u = 0 \tag{28}$$

where $f(t)$ is periodic, the stability of which has been studied by using Floquet theory ([8, Chapter 6]). It is desired to find those regions in the e-T plane which are stable (within the physical restrictions (3) and (12) or (15)). Corresponding to transition values of T and e from stability to instability, there must exist at least one periodic solution to (28) of period Ω or 2Ω, where Ω is the smallest period of $f(t)$ ([8, p. 201]). For the first x-mode, $\Omega = \pi$, while for the second x-mode, $\Omega = 2\pi/\sqrt{3}$. Therefore, in order to obtain all transition values of T and e, it is sufficient to examine solutions of period $2\Omega/N$, all of which have period 2Ω. (Here and in what follows, $N = 0, 1, 2, \ldots$)

Now for $e = 0$ and $T > 0$, the solutions to (24) are of the form $\sin \sqrt{3T}\, t$ and $\cos \sqrt{3T}\, t$, which have period $2\pi/\sqrt{3T}$. Thus, for $e = 0$, transition points can occur in (24) only if

$$2\pi/\sqrt{3T} = 2\Omega/N = 2\pi/N$$

or

$$T = N^2/3$$

Note that $N = 0$ corresponds to a constant, which is a solution to each of (24)–(27) when $T = e = 0$, and which may be thought of as a periodic function of period 2Ω.

For $e = 0$ and $T \leq 0$, each of (24)–(27) have unbounded solutions, and hence for each the entire negative T-axis is unstable.

Thus, for (24), one expects two transition curves to intersect each of the foregoing transition points on the T-axis, the solution along one behaving like $\sin Nt$, the solution along the other like $\cos Nt$ for $e = 0$. ($N = 0$ is an exception, and only a single transition curve intersects the T-axis at $T = 0$. On this curve the solution behaves like a constant for $e = 0$.) This is similar to the situation for Mathieu's equation, for example, [9, p. 40].

Hence, in the region of physical interest for the first x-mode, (3) and (12), transition points along the T-axis for (24) can only occur at $T = 0, {}^1/_3$. Similar arguments reveal that transition points along the T-axis within the corresponding region of physical interest can only occur at $T = 0, 1$ for (25); at $T = 0, {}^1/_4, 1$ for (26); and at $T = 0, {}^3/_4$ for (27).

The form of the transition curves is surprisingly simple for (24) and (25). The transition curve which intersects the T-axis at $T = 0$ is, in fact, the curve $T \equiv 0$. This follows from the fact that an exact solution for $T \equiv 0$ to (24) is $u_1 = 1 + \xi$, and to (25) is $u_2 = 1 - \xi$, both of which are periodic of period 2π.

Moreover, for $T \equiv {}^1/_3$, equation (24) becomes

$$\ddot{u}_1 + u_1 = 0.$$

Thus, on the curve $T \equiv {}^1/_3$, there exist two periodic solutions to (24) which are linearly independent. Since all other solutions for points on this curve can be expressed as a linear combination of these two solutions, the curve itself is stable. Therefore only one curve intersects the point $T = {}^1/_3$, $e = 0$, and no region of instability occurs at this point. It is as if the two expected transition curves have coalesced. This phenomenon was first observed in the case of a single particle in the plane [3].

In a similar fashion, $T \equiv 1$ is a stable curve for both (25) and (26).

The regions of stability in the e-T plane for the first x-mode are equivalent to the intersection of the regions of stability of (24) and (25). Similarly, the second x-mode will be stable if and only if both (26) and (27) are stable.

Hence, for the first x-mode, the only transition curve which occurs is $T \equiv 0$. The stability diagram is shown in Fig. 2.

To obtain explicit expressions for the transition curves of (26) and (27), a perturbation method is used [8, p. 209].

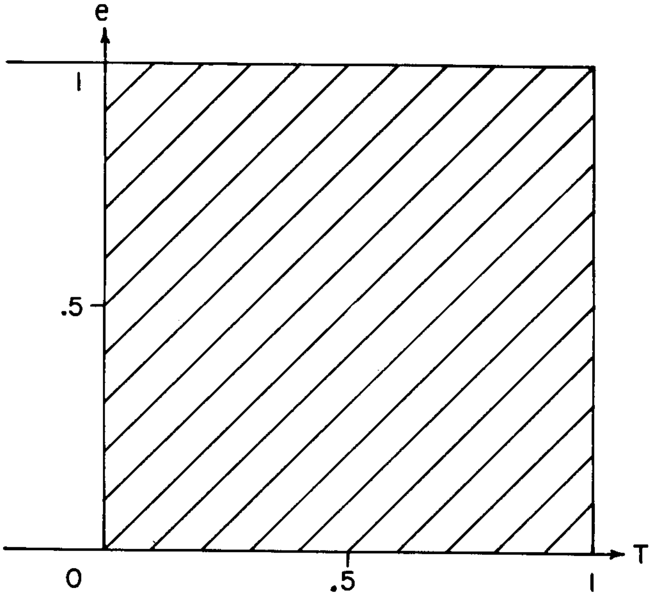

Fig. 2 Stability diagram for first x-mode; shaded region represents stability

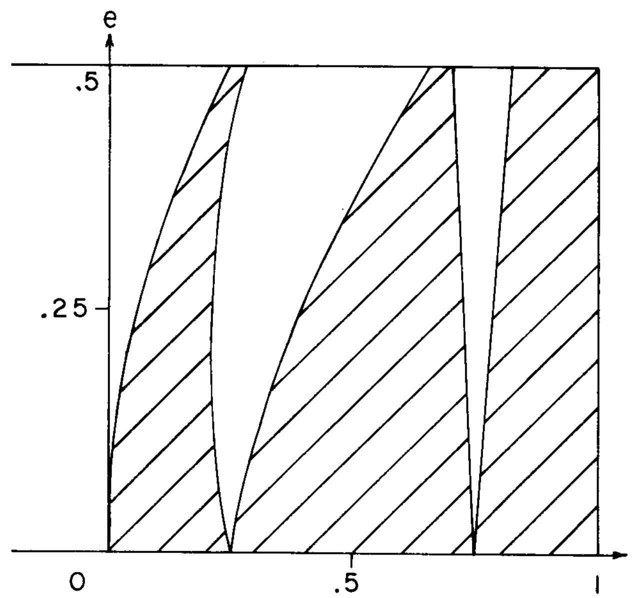

Fig. 3 Stability diagram for second x-mode; shaded regions represent stability

Expand

$$u_i(t) = u_{i0}(t) + eu_{i1}(t) + e^2u_{i2}(t) + \ldots, \qquad i = 1, 2 \tag{29}$$

$$T = T_0 + eT_1 + e^2T_2 + \ldots \tag{30}$$

and substitute (29) and (30) into (26) and (27), taking T_0 as 0, $^1/_4$ for (26) and as 0, $^3/_4$ for (27). By equating the coefficients of like powers of e, obtain a linear differential equation with constant coefficients on $u_{iN}(t)$. Requiring $u_{iN}(t)$ to be periodic gives a value for T_N. For $T_0 > 0$, u_{i0} must be taken separately as sin $(\sqrt{3}t/2)$ and cos $(\sqrt{3}t/2)$, since each choice gives a separate transition curve.

In this manner the following transition curves were found. For (26),

$$T = e^2 + 0(e^3) \tag{31}$$

$$T = (^1/_4) \pm (^3/_8)e + (111/128)e^2 + 0(e^3) \tag{32}$$

while for (27),

$$T = (^1/_3)e^2 + 0(e^3) \tag{33}$$

$$T = (^3/_4) \pm (^1/_8)e + (^{23}/_{384})e^2 + 0(e^3) \tag{34}$$

Hence, for the second x-mode, the only transition curves which occur are (31), (32), and (34). The stability diagram is shown in Fig. 3.

Stability Indicator

The stability indicator $\sigma(e,T)$ maps the region of physical interest, R, i.e., (3) and (12) or (15), into the line $-\infty < \sigma < \infty$ such that all transition curves in R go to $\sigma = 0$, all stable regions in R go to $\sigma > 0$, and all unstable regions in R go to $\sigma < 0$.

For the first x-mode, σ is defined as

$$\sigma = T$$

while for the second x-mode

$$\sigma = (T\text{-}e^2)\left(T - \frac{1}{4} - \frac{3}{8}\,e - \frac{111}{128}\,e^2\right) \times \left(T - \frac{1}{4} + \frac{3}{8}\,e - \frac{111}{128}\,e^2\right)$$
$$\times \left(T - \frac{3}{4} - \frac{1}{8}\,e - \frac{23}{384}\,e^2\right)\left(T - \frac{3}{4} + \frac{1}{8}\,e - \frac{23}{384}\,e^2\right)$$

The stability indicator allows the determination of stability without specific reference to the stability diagram.

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