What Limits the Oscillations Amplitude of the Single-Branch Pulsating Heat Pipe

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ABSTRACT

In this paper, we analytically describe the self-oscillations in the Single-Branch Pulsating Heat Pipe (SB-PHP) and show that nonlinearities in the system are responsible for the saturation of its oscillations amplitude. The SB-PHP is a device exhibiting oscillations of a single vapor bubble and a single liquid plug. Increasing the heater temperature above a critical value leads to the startup where the oscillation amplitude increases in time and eventually saturates. In this paper, we explain what leads to the amplitude saturation. We start from a known theoretical SB-PHP model, which includes two nonlinearities: the pressure nonlinearity (perfect gas law) and the nonlinearity of the wall temperature profile (sigmoid function in the axial direction). We apply the center manifold reduction technique followed by the normal form technique and end up with an approximate analytical solution (describing the position and velocity of the liquid plug, and the mass of vapor, as a function of time). This solution shows that a small perturbation of the equilibrium leads to oscillations growing over time (the equilibrium is unstable) if the phase-change coefficient is greater than the friction coefficient. The amplitude then saturates and the system reaches a steady-state regime (a stable limit cycle, as proven via the Poincaré-Andronov-Hopf bifurcation theorem). We show that saturation of the amplitude is due to the pressure and wall temperature profile nonlinearities of the system. We conclude that one can increase the oscillation amplitude by increasing the equilibrium instability and by decreasing these nonlinearities.

KEYWORDS: Hopf bifurcation, limit cycle, self-oscillations, nonlinear dynamics, cooling

1. INTRODUCTION

The Single-Branch Pulsating Heat Pipe (SB-PHP) is a tube closed at one end and open at the other which is filled with a liquid (see Figure 1). The closed-end is heated and a vapor bubble eventually forms. Increasing the temperature of the heater above a threshold leads to oscillations of the liquid plug (start-up shown in Figure 2).

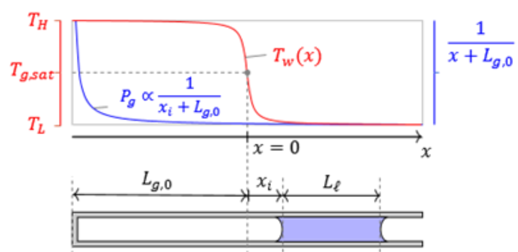

Figure 1 SB-PHP tube with the temperature of the wall $T_w(x)$ imposed (Fig. adapted from [1]).

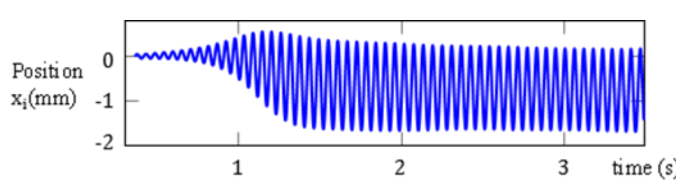

Figure 2 Oscillation start-up from experiment from [2].

Das et al. [3] have shown, assuming no phase-change, that the vapor in the SB-PHP acts as a spring and, coupled with the inertia of the liquid plug, leads to a spring-mass system. Later on, Tessier-Poirier et al. revealed the instability mechanism leading to the oscillation start-up both theoretically and experimentally [2]. The evaporation-condensation leads to a change of pressure which acts as a positive feedback. When the positive feedback is greater than the friction, a perturbation of the equilibrium leads to oscillations growing exponentially, in the linear approximation (up to 1s in

Figure 2). Experimentally, we however observe that the amplitude eventually saturates. To understand the saturation mechanism, one must take the nonlinearities into account. In this paper, we do so by obtaining an approximate analytical solution of a known nonlinear theoretical model [2]. We show that both the pressure and the wall temperature profile nonlinearities included in the model limit the oscillation amplitude. Our analytical solution shows that one can increase the oscillation amplitude by increasing the instability and by decreasing these nonlinearities.

2. MODELLING

To describe the oscillations, we use a known model of the SB-PHP [2] with some simplifications. The momentum balance is applied on the liquid plug as: $m_\ell \, \ddot{x}_i = \left(P_g - \hat{P}_e\right)A + F_f$, with x_i the meniscus position (see Figure 1) and $\ddot{x}_i$ the liquid plug acceleration. The forces are due to the gas pressure of the vapor P_g, the external pressure P_e, the gravity if the tube is tilted (the external pressure augmented by gravity is $\hat{P}_e = P_e + \rho_\ell L_\ell \cdot g \sin\theta$) and the friction force from the walls F_f, with $F_f \propto \dot{x}_i$. The vapor's pressure can be described by the perfect gas law: $P_g = m_g R_g T_g / \left(x_i + L_{g,0}\right)A$, where m_g is the mass of vapor, R_g is the gas constant for vapor, T_g is the vapor temperature (which we will assume constant), $L_{g,0}$ is the length of vapor at equilibrium (see Figure 1) and A is the inner section area. The mass of vapor m_g varies through evaporation and condensation. We assume local phase-change at the meniscus so $\dot{m}_g = \left(T_w(x_i) - T_{g,sat}\right)/H_v R_{th}$, where $T_w(x_i)$ is the wall temperature at the meniscus position, $T_{g,sat}$ is the temperature of the meniscus (at saturation), H_v is the enthalpy of vaporization and R_{th} is the phase-change thermal resistance. We assume $T_w(x)$ follows an arctangent profile (Figure 1). We now consider perturbations of the equilibrium, make the equations dimensionless and rewrite them in a phase-space representation (set of first order differential equations), leading to Eq.(1). The new variables q_1, q_2 and q_3 are the dimensionless perturbations of the equilibrium for the position, velocity and mass of vapor respectively. We have $q_1 = x_i / L_{g,0}$ and $q_3 = \left(m_g - m_{g,0}\right)/m_{g,0}$. The time is made dimensionless using the natural angular frequency $\omega_n = \sqrt{P_{g,0}/\rho_\ell \, L_\ell \, L_{g,0}}$. The parameters are the phase-change coefficient σ, the friction coefficient ζ_f and the dimensionless difference of temperature between the hot end and the cold end T_{HL}. Note that dT_w/dx is the axial wall thermal gradient.

$$\dot{q}_1 = q_2, \qquad \dot{q}_2 = -\left(\frac{1}{1+q_1}\right)q_1 + \left(\frac{1}{1+q_1}\right)q_3 - 2\zeta_f \, q_2, \qquad \dot{q}_3 = T_{HL} \operatorname{atan}\left[-\frac{2\sigma}{T_{HL}} \cdot q_1\right] \tag{1}$$

$$\sigma = \frac{L_{g,0}}{2 \, m_{g,0} \, \omega_n H_v R_{th}} \left(\frac{-dT_w}{dx}\right)_{x=0}, \qquad \zeta_f = \frac{8\pi\mu L_\ell}{2 \, m_\ell \, \omega_n}, \qquad T_{HL} = \frac{T_H - T_L}{\pi \, m_{g,0} \, \omega_n \, H_v \, R_{th}} \tag{2}$$

Eq.(1) are nonlinear because of the pressure and the wall temperature profile. The $\dot{q}_2$ equation is the momentum balance; the first two terms of the right-hand side are the dimensionless pressure while the last term is the dimensionless friction force. The $1/(1+q_1)$ prefactor appears because the pressure is nonlinear with the position, it increases to infinity as the liquid plug approaches the closed end (Figure 1). The $\dot{q}_3$ equation is the evaporation rate; it is nonlinear because the wall temperature saturates on both sides of the equilibrium (Figure 1).

3. ANALYTICAL STUDY - NORMAL FORM

3.1. Obtaining the normal form

First, we introduce nonlinear coefficients to control and keep track of the nonlinearities. From Eq.(1), we rewrite the $\dot{q}_2$ equation as $\dot{q}_2 = \dot{q}_{2L} + c_P(\dot{q}_{2NL} - \dot{q}_{2L})$ where $\dot{q}_{2L}$ is the $\dot{q}_2$ equation linearized and $\dot{q}_{2NL}$ is the nonlinear one. The coefficient c_P controls the pressure nonlinearity: for $c_P = 0$, we have $\dot{q}_2$ linearized and for $c_P = 1$, we have the full nonlinear equation. We do the same for the $\dot{q}_3$ equation, introducing the coefficient c_T which controls the temperature nonlinearity. Solving the linearized Eq.(1) (see [2]), we find that the stability of the equilibrium depends on the bifurcation parameter $\delta = \sigma - \zeta_f$. We now apply successive nonlinear techniques to simplify the equations (see [4] for reference). We first reduce the system to only two equations using the center manifold reduction technique. We then make the equations simpler by transforming them into their normal forms. The resulting equations are topologically equivalent to the original ones (conserve the qualitative features) near the point $q_1 = q_2 = q_3 = \delta = 0$. Details of the procedure are beyond the scope of this article. The new system is stated in terms of the phase-space variables $z_1(t)$ and $z_2(t)$ which have the solutions $z_1 = r(t)\cos\theta(t)$ and $z_2 = r(t)\sin\theta(t)$, where the amplitude $r(t)$ and the phase $\theta(t)$ are functions of time and are described by the following uncoupled differential equations:

$$\dot{r} = d\delta r + a_0 r^3 + O(\delta^2 r, \delta r^3, r^5) \quad (3)$$
$$\dot{\theta} = \omega_0 + c\delta + b_0 r^2 + O(\delta^2, \delta r^2, r^4) \quad (4)$$

$$d = \frac{1}{1+4\zeta_f^2}, \quad a_0 = \left(\frac{-\zeta_f^3}{T_{HL}^2(1+4\zeta_f^2)}\right)c_T + \left(\frac{-\zeta_f}{8(1+\zeta_f^2)}\right)c_P, \qquad \omega_0 = 1, \qquad c = \frac{2\zeta_f}{1+4\zeta_f^2} \quad (5)$$

We will discuss what the amplitude $r(t)$ actually looks like in a moment. At this point however, the reader might rightfully wonder how r is related to the amplitude in the original variables q_i, the ones we really care about. Making the inverse transformations, we end up with the following solution for the original phase-space variables q_i:

$$q_i = \tfrac{1}{2}A_{i0} + A_{i1}\sin(\theta + \varphi_{i1}) + A_{i2}\sin(2\theta + \varphi_{i2}) + O(\delta^3, \delta^2 r, \delta r^2, r^3) \quad (6)$$

We have the following coefficients (for small ζ_f) for the dimensionless position q_1: $A_{10} \approx c_P\, r(t)^2$, $A_{11} \approx r(t)$ and $A_{12} \approx \frac{1}{2}\, c_P\, \sigma\, r(t)^2$. The fundamental $A_{11}\sin(\theta + \varphi_{11}) \approx r(t)\sin(\theta + \varphi_{11})$ is the dominant oscillating term, and $r(t)$ is a good approximation for the oscillation amplitude in q_1.

3.2. Analyzing the vector field

Now, let's study $r(t)$. Plotting $\dot{r}(t)$ directly as a function of $r(t)$ from Eq.(3) reveals the qualitative properties of the system (Figure 3). Figure 4 shows the resulting amplitude $r(t)$ as function of time.

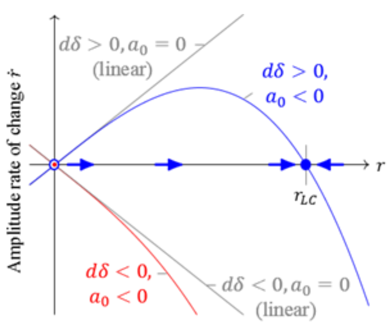

Figure 3 Vector field $\dot{r}(r)$ from Eq.(3). The amplitude r increases (decreases) over time for $\dot{r} > 0$ ($\dot{r} < 0$). Here, the equilibrium $r = 0$ is stable (unstable) for $d\delta < 0$ ($d\delta > 0$) and a steady-stade oscillating regime (stable limit cycle) exists at $r = r_{LC}$ for $d\delta > 0$ and $a_0 < 0$.

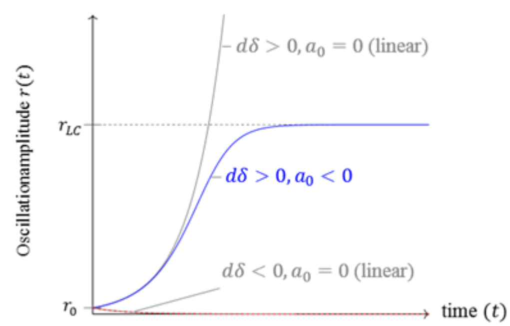

Figure 4 Evolution of the amplitude $r(t)$ after a small perturbation r_0 of the equilibrium $= 0$; the unstable nonlinear solution saturates at the amplitude r_{LC}.

Linearized vector field. Let's consider the linearized system first, by taking $a_0 = 0$ in Eq.(3). We get $\dot{r}(t) = d\delta \cdot r$ which are straight lines in the graph $\dot{r}(r)$ (Figure 3). A small perturbation of the equilibrium $r = 0$ leads to oscillations decreasing back to the equilibrium for $d\delta < 0$ (stable equilibrium) or growing over time (start-up) for $d\delta > 0$ (unstable equilibrium). Solving $\dot{r}(t) = d\delta \cdot r$ leads to $r(t) = e^{d\delta\, t} + r_0$, the amplitude evolves exponentially (Figure 4). Given $d > 0$, the stability of the equilibrium depends on $\delta = \sigma - \zeta_f$ only. **The start-up occurs when the phase-change coefficient σ is greater than the dissipation coefficient ζ_f.** As a start-up criteria, we may use **the instability number $\Pi = \sigma/\zeta_f$**, in which case start-up occurs when Π is increased above 1, see Eq.(7).

$$\delta = \zeta_f(\Pi - 1)\,, \qquad with: \;\; \Pi = \frac{\rho_\ell R_g T_{g,0}}{8\pi H_v R_{th} \hat{P}_e}\left(\frac{-dT_w}{dx}\right)_{x=0} \quad (7)$$

In accordance with experimental observations, the instability number Π predicts that the start-up can be triggered by increasing the temperature of the heater (which increases both $T_{g,0}$ and the thermal gradient), by decreasing the external pressure or by decreasing the thermal resistance R_{th}. The stability analysis of the equilibrium $r = 0$ remains valid when nonlinear terms are taken into account, according to the Hartman-Grobman theorem [4,5] (except for the special case with $\delta = 0$ exactly, the stability then depends on a_0).

Nonlinear vector field. Let's now consider the nonlinear equation (3), which includes the term $-a_0 r^3$. Let's note that for $c_P = c_T = 0$, we have $a_0 = 0$, confirming that the a_0 term expresses the effect of the pressure and wall temperature nonlinearities. Also, we have $a_0 \leq 0$ Eq.(5) since $z_f > 0$ and $T_{HL} > 0$. Because of the a_0 term, the function $\dot{r}(r)$ is no longer a straight line in Figure 3, but curves downwards. When the equilibrium is unstable ($\delta > 0$), a perturbation of the equilibrium leads to oscillations growing over time. As the amplitude grows, the term $-a_0 r^3$ becomes significant relative to the term $d\delta \cdot r$, and the amplitude rate of change $\dot{r}$ eventually decreases until it reaches 0, whereupon the amplitude r then reaches a constant value $r = r_{LC}$. This steady-state regime (a limit cycle in nonlinear dynamics terms [5]) is attractive: for $r < r_{LC}$, the amplitude increases toward r_{LC} and for $r > r_{LC}$, it decreases towards r_{LC}. As δ is increased and crosses 0, the limit cycle is created; a supercritical Poincaré-Andronov-Hopf bifurcation occurs. We can obtain the solution for the limit cycle by solving Eq.(3,4) with $\dot{r} = 0$. We get the expressions of the amplitude r_{LC} and of the angular frequency Ω Eq.(8) which can be substituted in Eq.(6). Analyzing r_{LC}, we clearly see that one may increase the amplitude by increasing the instability $d\delta$ (either by increasing the phase-change coefficient σ or by decreasing the friction coefficient ζ_f) or by decreasing the nonlinearities through a_0.

$$r_{LC} = \sqrt{-d\delta/a_0} \quad and \quad \theta(t) = \Omega t + \theta_0 \quad with \quad \Omega = \omega_0 + (c - b_0 d/a_0)\delta \tag{8}$$

What about the higher order terms we neglected in Eq.(3,4), do they affect our conclusions ? The short answer is no. The Poincaré-Andronov-Hopf bifurcation theorem [4] ensures that for δ sufficiently small, the following two cases hold: 1) $a_0 < 0, d\delta < 0$: the origin is an asymptotically stable fixed point, 2) $a_0 < 0, d\delta > 0$: the origin is an unstable fixed point and there exist an asymptotically stable limit cycle. This theorem enables us to prove (under the model hypotheses) that the pressure and wall temperature nonlinearities are both saturating mechanisms.

4. CONCLUSION

In this paper, we have obtained an approximate analytical solution of a SB-PHP theoretical model using center manifold reduction and normal form techniques. We have discussed the instability mechanism where the phase-change acts as a positive feedback and the friction as dissipation. Two nonlinearities were included in the model, the pressure nonlinearity (perfect gas law) and the wall temperature profile nonlinearity (follows an arctangent profile in the axial direction). The solution shows that, after the oscillation start-up, the amplitude saturates because of both of those nonlinearities. The system reaches a stable limit cycle. We have shown from the analytical solution that one can increase the final amplitude by increasing the instability or by decreasing the nonlinearities. This work provides a theoretical basis for understanding the physics of pulsating heat pipes and improving their performance.

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