Proceedings of ASME 2009 International Design Engineering Technical Conferences &
Computers and Information in Engineering Conference
IDETC/CIE 2009
August 30 - September 2, 2009, San Diego, California, USA

DETC2009-86197

DYNAMICS OF COUPLED BUBBLE OSCILLATORS WITH DELAY

Richard H. Rand
Dept. Theoretical and Applied Mechanics
Cornell University
Ithaca NY 14853
Email: rhr2@cornell.edu

Christoffer R. Heckman
Dept. Theoretical and Applied Mechanics
Cornell University
Ithaca NY 14853
Email: crh94@cornell.edu

ABSTRACT

We investigate the stability of the in-phase mode in a system of two delay-coupled bubble oscillators. The bubble oscillator model is based on a 1956 paper by Keller and Kolodner. Delay coupling is due to the time it takes for a signal to travel from one bubble to another through the liquid medium that surrounds them. Using techniques from the theory of delay-differential equations as well as the method of averaging, we show that the equilibrium of the in-phase mode can be made unstable if the delay is long enough and if the coupling strength is large enough, resulting in a Hopf bifurcation. This work is motivated by medical applications involving noninvasive localized drug delivery via microbubbles.

INTRODUCTION

Delay in dynamical systems is exhibited whenever the system's behavior is dependent at least in part on its history. Many technological and biological systems are known to exhibit such behavior; coupled laser systems, high-speed milling, population dynamics and gene expression are some examples of delayed systems. This paper treats a new application of delay-differential equations, that of a microbubble cloud under acoustic forcing. This system is of particular interest in biomedicine, where microbubbles are key to several contexts. For example, microbubbles are used in the noninvasive, localized delivery of drugs. In this process, microbubble surfaces are coated with drugs that work best locally. The microbubbles are propagated to the target site and collapsed by a much stronger ultrasound wave [1]. Full understanding of the behavior of these systems of coupled microbubbles involves taking into account the speed of sound in the liquid, which will lead to a delay in induced pressure waves between the bubbles in the cloud.

Differential equation models of bubble dynamics in a compressible fluid were first studied by Joseph Keller and his associates [2], [3]. In this paper we extend their work by including delay in a system of coupled bubble oscillators.

THE BUBBLE EQUATION

We begin with the equation of Keller and Kolodner [2] that relates the time-dependent radius of a gas and vapor bubble $a(t)$ with the hydrostatic pressure it experiences in a compressible liquid:

this should be a +

$$(\dot{a}-c)\left(a\ddot{a}+\frac{3}{2}\dot{a}^2-\Delta\right)-\dot{a}^3-a^{-1}\left(a^2\Delta\right)^{\cdot}=0 \qquad (1)$$

Here, $\Delta=\rho^{-1}(p(a)-p_0)$, where ρ is the density of the liquid, and p_0 is the far-field liquid pressure. The pressure $p(a)$ inside the bubble is calculated using the adiabatic relation $p(a)=k\left(\frac{4\pi}{3}a^3\right)^{-\gamma}$, where k is determined by the quantity and type of gas in the bubble and γ is the adiabatic exponent of the gas. Next, we substitute dimensionless speed $c'=c(\rho p_0^{-1})^{\frac{1}{2}}$ into eq.(1), and obtain the dimensionless equation [2]:

$$(\dot{a}-c)(a\ddot{a}+\frac{3}{2}\dot{a}^2-a^{-3\gamma}+1)-\dot{a}^3-(3\gamma-2)a^{-3\gamma}\dot{a}-2\dot{a}=0 \quad (2)$$

where we have dropped the prime on c for convenience. Eq.(2) has an equilibrium solution at

$$a=a_e=1 \quad (3)$$

To determine its stability, we set $a=a_e+x=1+x$ and linearize about $x=0$, giving:

$$c\ddot{x}+3\gamma\dot{x}+3c\gamma x=0 \quad (4)$$

Since c and γ are positive-valued parameters, eq.(4) corresponds to a damped linear oscillator, which tells us that the equilibrium (3) is stable.

TWO COUPLED BUBBLE OSCILLATORS

In this work we consider the dynamics of a system of two coupled bubble oscillators, each of the form of eq.(2), with delay coupling. Manasseh et al. [4] have studied coupled bubble oscillators without delay. The source of the delay comes from the time it takes for the signal to travel from one bubble to the other through the liquid medium which surrounds them. Adding the coupling terms used in [4], the governing eqs. of the bubble system are:

$$(\dot{a}-c)(a\ddot{a}+\frac{3}{2}\dot{a}^2-a^{-3\gamma}+1)-\dot{a}^3-(3\gamma-2)a^{-3\gamma}\dot{a}-2\dot{a} = P\dot{b}(t-T) \quad (5)$$

$$(\dot{b}-c)(b\ddot{b}+\frac{3}{2}\dot{b}^2-b^{-3\gamma}+1)-\dot{b}^3-(3\gamma-2)b^{-3\gamma}\dot{b}-2\dot{b} = P\dot{a}(t-T) \quad (6)$$

where T is the delay and P is a coupling coefficient. Here we have omitted coupling terms of the form $P_1 b(t-T)$ and $P_1 a(t-T)$ from eqs.(5) and (6), respectively, where P_1 is a coupling coefficient [4].

THE IN-PHASE MODE

The system (5),(6) possesses an invariant manifold called the *in-phase mode* given by $a=b$, $\dot{a}=\dot{b}$. The dynamics of the in-phase mode is governed by the equation:

$$(\dot{a}-c)(a\ddot{a}+\frac{3}{2}\dot{a}^2-a^{-3\gamma}+1)-\dot{a}^3-(3\gamma-2)a^{-3\gamma}\dot{a}-2\dot{a} = P\dot{a}(t-T) \quad (7)$$

This equation has the same equilibrium (3) as eq.(2), namely $a=a_e=1$. In the case that $P=0$, we have seen in eqs.(2)-(4) that this equilibrium is stable. To determine stability for $P>0$, we again set $a=a_e+x=1+x$ and linearize about $x=0$, giving:

$$c\ddot{x}+3\gamma\dot{x}+3c\gamma x=-P\dot{x}(t-T) \quad (8)$$

Before proceeding with an analytical treatment of eq.(8), we use the MATLAB function DDE23 to numerically integrate eqs.(7) and (8). We choose the following dimensionless parameters based on the papers by Keller et al.:

$$c=94, \quad \gamma=1.25, \quad P=10 \quad (9)$$

Results of the numerical integration for the nonlinear eq.(7) are shown in Figs.1,2, whereas comparable results for the linearized eq.(8) are shown in Figs.3,4.

Inspection of Figs.1-4 reveals that the equilibrium $a=1$ loses its stability as the delay T is increased through a critical value T_{cr}. This corresponds to the birth of a limit cycle via a Hopf bifurcation in the nonlinear eq.(7). Associated with this periodic motion is its frequency ω_{cr}. From Figs.1-4 we obtain the following approximate values for T_{cr} and ω_{cr}:

$$T_{cr}\approx 1, \qquad \omega_{cr}\approx 2 \quad (10)$$

Eq.(8) is a linear differential-delay equation. To solve it, we set $x=\exp\lambda t$ (see [5]), giving

$$c\lambda^2+3\gamma\lambda+3c\gamma=-P\lambda\exp-\lambda T \quad (11)$$

We seek the smallest value of delay $T=T_{cr}$ which causes instability. This will correspond to imaginary values of λ. Thus we substitute $\lambda = i\omega$ in eq.(11) giving two real equations for the real-valued parameters ω and T:

$$P\omega \sin \omega T = c(\omega^2 - 3\gamma) \tag{12}$$
$$P\omega \cos \omega T = -3\gamma\omega \tag{13}$$

Eq.(13) gives

$$\omega T = \arccos\left(\frac{-3\gamma}{P}\right) \tag{14}$$

whereupon eq.(12) becomes

$$\omega^2 - \frac{\sqrt{P^2 - 9\gamma^2}\,\omega}{c} - 3\gamma = 0 \tag{15}$$

from which we obtain

$$\omega_{cr} = \frac{\sqrt{P^2 - 9\gamma^2 + 12c^2\gamma} + \sqrt{P^2 - 9\gamma^2}}{2c} \tag{16}$$

which, when combined with (14), gives

$$T_{cr} = \frac{2c \arccos\left(-\frac{3\gamma}{P}\right)}{\sqrt{P^2 - 9\gamma^2 + 12c^2\gamma} + \sqrt{P^2 - 9\gamma^2}} \tag{17}$$

For the parameters of eq.(9), eqs.(16),(17) give

$$T_{cr} = 0.9842, \quad \omega_{cr} = 1.9864 \tag{18}$$

which agree with the simulations in Figs.1-4, cf. eq.(18).

Eq.(17) shows that a necessary condition for instability is that the coupling parameter P must satisfy the inequality:

$$P > 3\gamma \tag{19}$$

Eq.(17) gives that as $P \to 3\gamma$, $T_{cr} \to \frac{\pi}{\sqrt{3\gamma}} = 1.622$ for $\gamma = 1.25$. Fig.5 shows a plot of T_{cr} as a function of P for parameters $c = 94$ and $\gamma = 1.25$, from eq.(17). So for instability we need both $P > 3\gamma$ and $T > T_{cr}$.

SECOND-ORDER AVERAGING

In this section we treat the in-phase mode eq.(7) using the method of averaging [5]. The perturbation scheme is based on assuming the fluid is close to incompressible, in which case sound speed c is very large. We begin by scaling parameters and stretching time:

$$c = \frac{1}{\varepsilon^2} \tag{20}$$
$$a = a_e + \varepsilon x = 1 + \varepsilon x \tag{21}$$
$$\bar{t} = \omega t, \quad \omega = \sqrt{3\gamma} \tag{22}$$

Here c is scaled as $1/\varepsilon^2$ instead of $1/\varepsilon$ in order to put the equation into an appropriate form for averaging. Specifically, we must eliminate the appearance of a linear damping term $\dot{x}$ at $O(\varepsilon)$, see [5] pp.23-24. Substituting (20)-(22) into (7) and expanding for small ε, we get:

$$\ddot{x} + x = f(x, \dot{x})\varepsilon + g(x, \dot{x}, \dot{x}(t - \omega T))\varepsilon^2 + O(\varepsilon^3) \tag{23}$$

where we have dropped the bars on t for convenience, and where:

$$f(x, \dot{x}) = k_1 x^2 + k_2 \dot{x}^2 \tag{24}$$

$$g(x, \dot{x}, \dot{x}(t - \omega T)) = k_3 x + k_4 \dot{x} + k_5 x^3 + k_6 \dot{x}^2 x + k_7 \dot{x}(t - \omega T) \tag{25}$$

where

$$\mathrm{k}_1 = \frac{3}{2}(\gamma + 1) \tag{26}$$
$$\mathrm{k}_2 = -\frac{3}{2} \tag{27}$$
$$\mathrm{k}_3 = 0 \tag{28}$$
$$\mathrm{k}_4 = -\omega \tag{29}$$
$$\mathrm{k}_5 = -3\gamma\left(1 + \frac{\gamma}{2}\right) - \frac{11}{6} \tag{30}$$
$$\mathrm{k}_6 = \frac{3}{2} \tag{31}$$
$$\mathrm{k}_7 = -\frac{P}{\omega} \tag{32}$$

Figs.6 and 7 show a simulation of eq.(23) for the parameters of eq.(9). This shows that (23) behaves similarly to (7) and (8) as regards stability of the origin.

Note from eq.(23) that the $O(\varepsilon)$ terms are quadratic. This situation is well-known to require that whichever perturbation method is used, be it Lindstedt's method, averaging, two variable expansion method or multiple scales, it is necessary to include terms of $O(\varepsilon^2)$, i.e., it is necessary to apply the perturbation method to second order [5]. We used second order averaging. Our computations were done in MACSYMA using programs that have been previously published [6], [7]. The method assumes a solution to the $\varepsilon = 0$ equation in the form

$$x = R\cos(t+\psi), \quad \dot{x} = -R\sin(t+\psi) \tag{33}$$

where R and ψ are slowly-varying functions of time. Second order averaging gives the following slow flow eqs. on R and ψ:

$$\frac{dR}{dt} = \frac{\varepsilon^2\,(\mathrm{k}_7\cos\omega T + \mathrm{k}_4)\,R}{2} \tag{34}$$

$$\frac{d\psi}{dt} = \tag{35}$$

$$-\frac{\varepsilon^2\left(R^2(3\mathrm{k}_6+9\mathrm{k}_5+4{\mathrm{k}_2}^2+10\mathrm{k}_1\mathrm{k}_2+10{\mathrm{k}_1}^2)+12\mathrm{k}_7\sin\omega T+12\mathrm{k}_3\right)}{24}$$

It is to be noted that in the process of deriving these equations, $R(t-\omega T)$ and $\psi(t-\omega T)$ have been respectively replaced by $R(t)$ and $\psi(t)$, a step that assumes that ωT is small [8].

Eq.(34) shows that the origin $R=0$, i.e. $x=\dot{x}=0$, is stable if $k_7\cos\omega T + k_4 \le 0$, and unstable otherwise. Eqs.(29),(32) give the following condition for instability:

$$k_7\cos\omega T + k_4 > 0 \Rightarrow -\omega - \frac{P}{\omega}\cos\omega T > 0 \Rightarrow \omega^2 + P\cos\omega T < 0 \tag{36}$$

The last inequality in (36) shows that for $P > 0$ no instability can result for small values of delay, in which case $\cos\omega T \approx 1 > 0$. Although instability can result for larger values of T (as we have seen), the averaging method is restricted to small delays.

Note that all the terms in eq.(23) are conservative except for the damping terms

$$k_4\dot{x} + k_7\dot{x}(t-\omega T) = -\omega\dot{x} - \frac{P}{\omega}\dot{x}(t-\omega T) \tag{37}$$

where we have used (29),(32). In the absence of delay, the damping coefficient would be $\omega + P/\omega$. The method of averaging shows that the effect of small delay is to modify this coefficient to be $\omega + (P/\omega)\cos\omega T$. Since $|\cos\omega T| \le 1$, this shows that small delays decrease the effective damping, i.e., make the equilibrium less stable.

CONCLUSION

In this paper we have begun to explore the dynamics of two delay-coupled bubble oscillators, eqs.(5),(6), and in particular we have studied the dynamics of the in-phase mode, eq.(7). We investigated the stability of equilibrium in the in-phase mode through the use of the linear variational eqs.(8). Analysis of the characteristic eq.(11) yielded closed form expressions for T_{cr} and ω_{cr}, eqs.(16),(17).

We also used the method of averaging to study the in-phase mode. This entailed a rescaling based on perturbing off of the incompressible fluid limit in which sound speed c is infinite. The truncated eq.(23) was subjected to second order averaging, and resulted in the slow flow (34),(35). This method limited conclusions to the case of small delay, where it was shown that the origin is stable.

In a classic paper, Keller and Kolodner [2] showed that the uncoupled bubble oscillator (eq.(7) with $P=0$) is conservative in the incompressible limit, and is damped if c is allowed to take on a finite value. Our study of the in-phase mode adds a delay feedback term to the system studied in [2]. We showed that the equilibrium can be made unstable if the delay is long enough and if the coupling coefficient P is large enough. Although our use of the method of averaging was unable to pick up this instability, it showed that small delays decrease the stability of the equilibrium by decreasing the effective damping.

Future work will include a study of more general dynamics of the coupled system (5),(6).

REFERENCES

[1] Leighton, T.G., “From Sea to Surgeries, from Babbling Brooks to to Baby Scans: Bubble Acoustics at ISVR”, Proceedings of the Institute of Acoustics 26:357–381 (2004).

[2] Keller, J.B. and Kolodner, I.I., “Damping of Underwater Explosion Bubble Oscillations”, J. Applied Physics 27:1152-1161 (1956).

[3] Keller, J.B. and Miksis, M., “Bubble Oscillations of Large Amplitude”, J. Acoust. Soc. Am. 68:628-633 (1980).

[4] Manasseh, R., Nikolovska, A., Ooi, A., and Yoshida, S., “Anisotropy in the Sound Field Generated by a Bubble Chain”, J. Sound Vibration 278:807-82 (2004).

[5] Rand, R.H., *Lecture Notes on Nonlinear Vibrations (version 52)*, available online at http://audiophile.tam.cornell.edu/randdocs (2007).

[6] Rand, R. H., and Armbruster, D., *Perturbation methods, bifurcation theory, and computer algebra,* New York: Springer-Verlag (1987).

[7] Rand, R. H. *Topics in nonlinear dynamics with computer algebra,* Langhorne, PA: Gordon and Breach Science Publishers (1994).

[8] Wirkus, S. and Rand, R., “Dynamics of Two Coupled van der Pol Oscillators with Delay Coupling”, Nonlinear Dynamics 30:205-221 (2002).

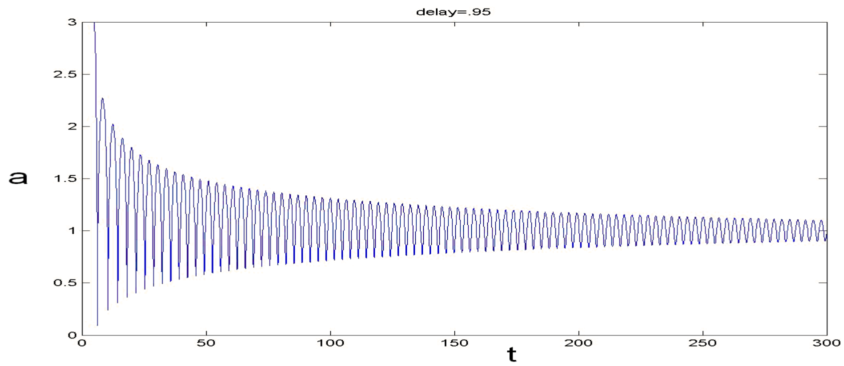

Figure 1: Numerical integration of eq.(7) for the parameters of eq.(9) with delay T=0.95. Note that the equilibrium is stable.

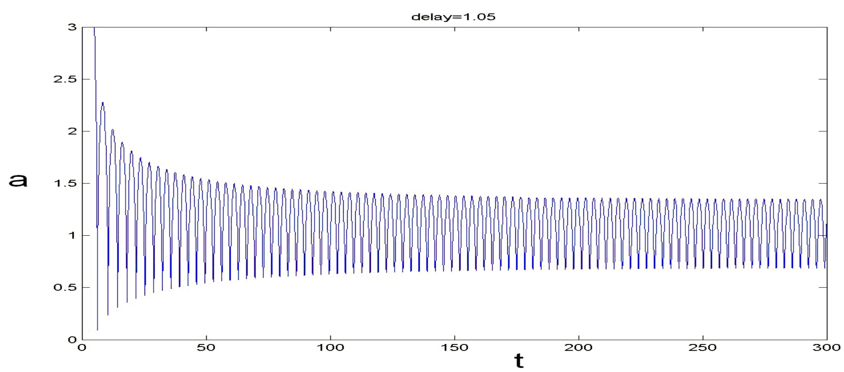

Figure 2: Numerical integration of eq.(7) for the parameters of eq.(9) with delay T=1.05. Note that the equilibrium has become unstable indicating a Hopf bifurcation.

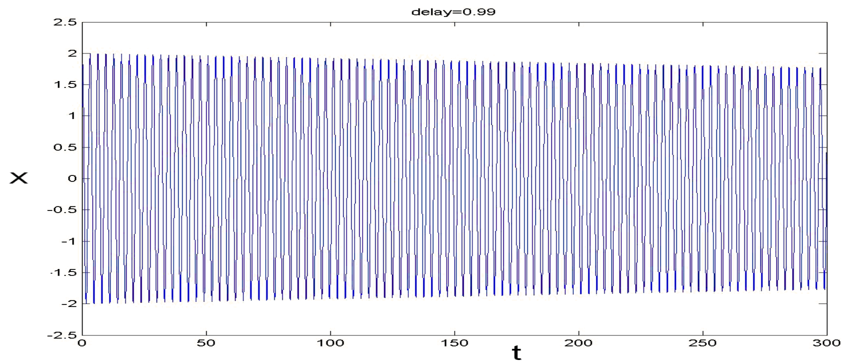

Figure 3: Numerical integration of the linearized eq.(8) for the parameters of eq.(9) with delay T=0.99. Note that the equilibrium is stable.

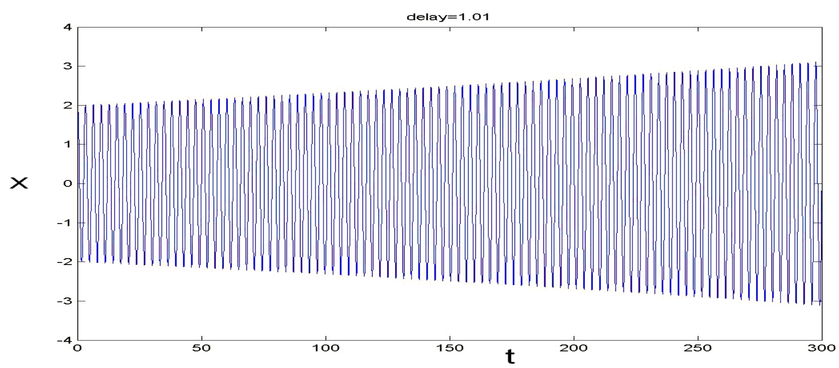

Figure 4: Numerical integration of the linearized eq.(8) for the parameters of eq.(9) with delay T=1.01. Note that the equilibrium is unstable.

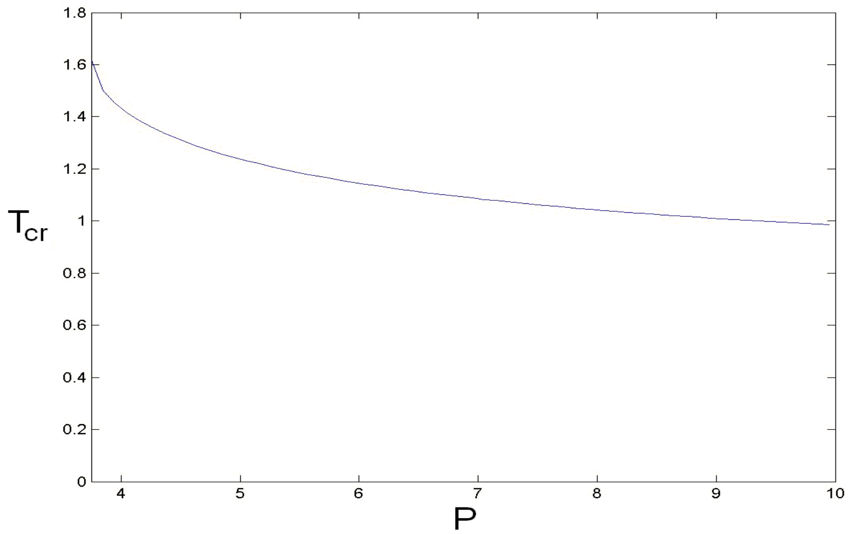

Figure 5: T_{cr} versus P for parameters $c = 94$ and $\gamma = 1.25$, from eq.(17).

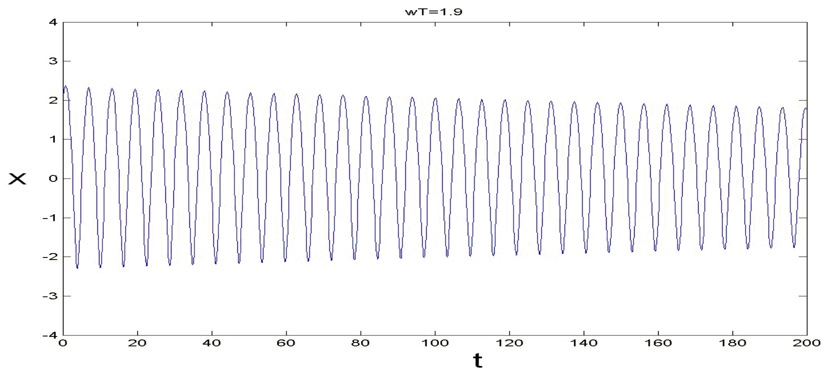

Figure 6: Numerical integration of the truncated eq.(23) for the parameters of eq.(9) with delay ωT=1.9, corresponding to $T_{cr} \approx 1$. Note that the equilibrium is stable.

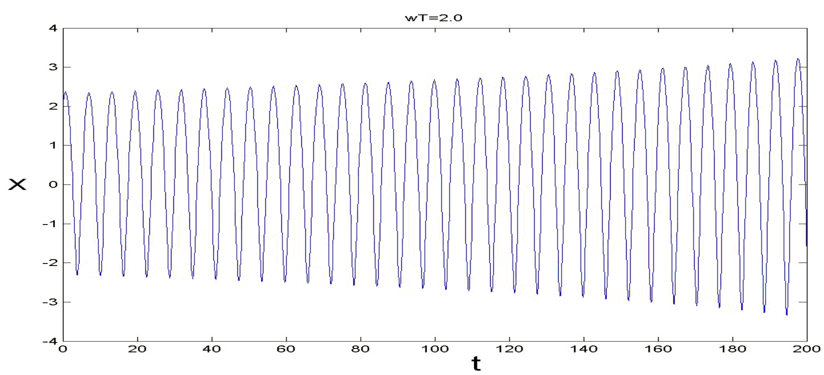

Figure 7: Numerical integration of the linearized eq.(8) for the parameters of eq.(9) with delay ωT=2.0, corresponding to $T_{cr} \approx 1$. Note that the equilibrium is unstable.